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LOCALLY COHERENT RATES OF EXCHANGE

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A theory of coherence is formulated for rates of exchange between events. The theory can be viewed as a generalization of de Finetti's theory of coherence as well as the theory of conditional coherence. Coherent rates of exchange on a fixed Boolean algebra are in one-to-one correspondence with finitely additive conditional probability measures on the algebra. Results of Rényi and Krauss on conditional probability spaces are used to show that coherent rates of exchange are generated by ordered families of finitely additive measures, possibly infinite measures. This provides an interpretation of improper prior distributions in terms of coherence. An extension theorem is proved and gives a generalization of extension theorems for finitely additive probability measures.

1. Heuristics. Suppose the sample space Ω for some chance experiment is the set of points on the real line. A statistician believes that sets having the same finite, positive Lebesgue measure are equally likely; so Lebesgue measure μ_1 might be used as an improper prior. However, the statistician also feels that finite sets of the same cardinality are equally likely. Now Lebesgue measure gives all such sets measure zero and so counting measure μ_0 seems more appropriate for finite sets. Finally the statistician feels that sets having the same positive density are equally likely, where the density of a set A is the limit

$$\mu_2(A) = \lim_{x \to \infty} \frac{1}{2x} \int_{-x}^{x} 1_A(t)\mu_1(dt)$$

when the limit exists. Now if $\mu_2(A) > 0$, both $\mu_0(A)$ and $\mu_1(A)$ are infinite. In the past statisticians wishing to express vague prior information have often chosen an improper distribution such as μ_1, which assesses all "large" sets as having infinite mass. Some have used finitely additive proper priors like μ_2 which give all "small" sets mass zero.

Is there a way of expressing these opinions simultaneously and of assessing their coherence? To answer these questions, we propose a theory of exchange rates. The idea is that, if two sets are believed to be equally likely, the statistician should be willing to trade a prospective payoff on the one for an equal payoff on the other. The usual theory of coherence involves comparing a payoff on each event to a payoff on the whole sample space (a sure thing). This theory is inadequate for comparing two events both of which are infinitesimally small in relation to the whole space. The theory of exchange rates makes such comparisons quite natural.

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The appropriate notion of coherence for exchanges involving a finite number of "small" sets cannot be the usual one of avoiding a sure loss. The union of all the sets involved in any such exchange can again be "small." We will call a rate locally coherent if no exchange involving a finite number of sets results in a loss on all of their union. (Formal definitions are in the next section.)

Every measure μ determines a natural exchange rate between sets of finite positive measure; $\mu(A)$ one-dollar payoffs on A are worth $\mu(B)$ one-dollar payoffs on B. Thus the theory of coherence for rates of exchange will also apply to measures including improper ones like Lebesgue measure. (This idea that $\mu(A)/\mu(B)$ is the relative value of a ticket on A to one on B is mentioned by Hartigan [9], page 15.)

2. Definitions and summary of results. Let Ω be the sample space for a chance experiment and let $\mathbf{D}$ be a collection of pairs (A, B) of subsets of Ω such that the second element B is not empty. A *rate of exchange* r on $\mathbf{D}$ is a mapping from $\mathbf{D}$ to $[0, \infty]$. Associated to each pair $(A, B) \in \mathbf{D}$ is the *simple exchange*

$$\begin{aligned} S_{A,B}(w) &= r(A, B)B(w) - A(w), \quad && \text{if } w \in B, \\ &= -A(w), && \text{if } w \in B. \end{aligned}$$

(In this expression and in the sequel, events and their indicator functions are identified.) We imagine that a bookie offers such simple exchanges to a gambler. If B occurs, the bookie pays $\$r(A, B)$ to the gambler, and if A occurs, the gambler pays the bookie \$1. If neither A nor B occurs, no money changes hands. [Some readers may wish to interpret $r(A, B)$ as the bookie's odds on A against B.]

An *exchange* e is any well-defined linear combination of simple exchanges. [The usual conventions are made about arithmetic operations with ∞ and $-\infty$. In particular, $\infty - \infty$ and $0 \cdot \infty$ are not defined. However, we also adopt the usual convention that $\infty \cdot B = \infty$ on B and $\infty \cdot B = 0$ on B^c. By the way, we could avoid the use of infinite numbers by interpreting a rate $r(A, B) = \infty$ as meaning that the bookie will accept any exchange $rB - A$ where $r > 0$.] Let λ be a real-valued function defined on $\mathbf{D}$ which is zero except for a finite number of pairs (A, B) and let

$$e(\lambda) = \sum_{(A,B)} \lambda(A, B) S_{A,B}, \tag{2.1}$$

assuming the sum is well-defined. Every exchange e is of this form for some λ. For each λ, let the *support* of λ be the set $\operatorname{supp}(\lambda) = \bigcup\{A \cup B\colon \lambda(A, B) \neq 0\}$. Notice that, for an exchange $e(\lambda)$, no money changes hands if $\operatorname{supp}(\lambda)$ does not occur. This suggests the following notion of coherence for the bookie.

DEFINITION. The rate of exchange r is *locally coherent* if there is no exchange $e(\lambda)$ which is strictly positive on $\operatorname{supp}(\lambda)$.

Notice that if $e(\lambda) > 0$ on $\text{supp}(\lambda)$, then $e(\lambda)$ has a positive infimum on $\text{supp}(\lambda)$. This is because exchanges have only finitely many possible values.

We use the term "local coherence" rather than "coherence" because the bookie is required to avoid losses on certain proper subsets of the outcome space. The usual theory of de Finetti [5, 6] only requires the bookie to avoid sure losses on the whole space. In an interesting paper Smith [20] develops a notion of consistency which is related to local coherence.

There is a simple relationship between the de Finetti theory and that presented here. Let **C** be the collection of sets A such that $(A, \Omega) \in \mathbf{D}$ and set $p(A) = r(A, \Omega)$. Then

$$S_{A,\Omega} = p(A) - A$$

and we can regard $\$p(A)$ as the bookie's price for a ticket worth \$1 if A occurs. The support of any exchange involving Ω will, of course, be Ω. Thus if r is a coherent rate of exchange, then p will be coherent in the sense of de Finetti (i.e., no linear combination of exchanges $S_{A,\Omega}$ is everywhere positive). However, the converse is easy to disprove. For example, r could be incoherent when **C** is empty or r could be incoherent because of bad behavior on p-null sets.

A number of authors, including de Finetti, Holzer [11] and Regazzini [16], have studied notions of conditional coherence which generalize de Finetti's theory of coherence. We believe the theory of local coherence extends the notion of conditional coherence. The precise relationship between the two theories will be examined in Section 7.

A stronger requirement than local coherence is that a bookie avoid exchanges which are positive somewhere and nonnegative everywhere.

DEFINITION. The rate of exchange r is *strictly coherent* if there is no exchange e with $e \geq 0$ on all of Ω with strict inequality holding somewhere.

The notion of strict coherence was studied by Kemeny [12] in the context of betting odds rather than rates of exchange.

The next section establishes some of the basic properties of locally coherent exchange rates. It is shown in Section 4 that a locally coherent rate defined on an arbitrary domain can always be extended to the algebra of all subsets. In Section 5 it is shown that locally coherent rates of exchange on an algebra of sets are in one-to-one correspondence with conditional probability measures. This correspondence together with Rényi's characterization of conditional probabilities in terms of linearly ordered families of measures leads to an analogous characterization of locally coherent rates in Section 6. This characterization is useful for the interpretation of improper priors and also results in a simple characterization of strictly coherent rates of exchange.

No attempt is made here to develop a theory of local coherence for statistical models comparable to the coherence theories of Heath, Lane and Sudderth [10, 14, 15]. The interesting papers of Brunk [2] and Regazzini [17] are related to this problem.

3. Elementary properties of locally coherent rates.

THEOREM 3.1. *Let r be a locally coherent rate of exchange. Then the following are true whenever the quantities are well-defined*:

(i) $r(A, A) = 1$.
(ii) $r(A_1 \cup A_2, B) = r(A_1, B) + r(A_2, B)$ *if* $A_1 \cap A_2 = \varnothing$.
(iii) $r(A_1, B) \le r(A_2, B)$ *if* $A_1 \subset A_2$.
(iv) $r(A, B)r(B, C) = r(A, C)$.
(v) $r(A, B) = r(B, A)^{-1}$.
(vi) $r(A, B) = r(A, C)r(B, C)^{-1}$.
(vii) $r(A, B_1) \ge r(A, B_2)$ *if* $B_1 \subset B_2$.

[The assertions above are vacuous if any of the quantities occurring in them are undefined. For instance, if (A_1, B) and (A_2, B) are in the domain of r and $(A_1 \cup A_2, B)$ is not in the domain of r, then (ii) is vacuous. Also, if $r(A, C) = r(B, C) = 0$, then $r(B, C)^{-1} = \infty$ and the right-hand side of (vi) is undefined so that (vi) is vacuous.]

PROOF. (i) If $r(A, A) > 1$, then $r(A, A)A - A > 0$ on A. If $r(A, A) < 1$, then $-[r(A, A)A - A] > 0$ on A.

(ii) Suppose the left-hand side is larger than the right so that $r(A_1, B)$ and $r(A_2, B)$ are less than ∞. Then there are $\varepsilon > 0$ and $\delta > 0$ such that

$$r(A_1 \cup A_2, B) - (1 + \varepsilon)[r(A_1, B) + r(A_2, B)] > \delta > 0.$$

Consider the exchange

$$\begin{aligned} e = {} & [r(A_1 \cup A_2, B)B - (A_1 \cup A_2)] - (1 + \varepsilon)[r(A_1, B)B - A_1] \\ & -(1 + \varepsilon)[r(A_2, B)B - A_2] \\ \ge {} & \delta B + \varepsilon(A_1 \cup A_2). \end{aligned}$$

Then $e > 0$ on $B \cup A_1 \cup A_2$, a contradiction.

A contradiction is reached by a similar argument if the right-hand side is assumed larger than the left.

(iii) If (iii) does not hold, then $r(A_2, B) < r(A_1, B)$ so that $r(A_2, B) < \infty$. Also, for some $\varepsilon > 0$,

$$r(A_2, B) \le (1 + \varepsilon)r(A_2, B) < r(A_1, B).$$

Consequently,

$$e = [r(A_1, B)B - A_1] - (1 + \varepsilon)[r(A_2, B)B - A_2] > 0$$

on $\operatorname{supp}(e) = B \cup A_2$, a contradiction.

(iv) Suppose the left-hand side is larger than the right. So $r(A, C) < \infty$. Choose ε in $(0, 1)$ so that

$$\delta = (1 - \varepsilon)r(A, B)r(B, C) - (1 + \varepsilon)r(A, C) > 0.$$

Because $r(A, B)r(B, C)$ can be assumed to be well-defined, the following exchange is too:

$$\begin{aligned} e &= [r(A,B)B - A] + (1-\varepsilon)r(A,B)[r(B,C)C - B] \\ &\quad -(1+\varepsilon)[r,(A,C)C - A] \\ &= \delta C + \varepsilon r(A,B)B + \varepsilon A. \end{aligned}$$

Then $e > 0$ on $A \cup B \cup C$, a contradiction. [Notice $r(A, B) > 0$ if the left side of (iv) is larger than the right.]

Next suppose the right-hand side of (iv) is larger than the left. So $r(A, B) < \infty$, $r(B, C) < \infty$ and $r(A, C) > 0$. There is an ε in $(0, 1)$ and $\delta > 0$ so that

$$(1-\varepsilon)r(A,C) - (1+\varepsilon)r(A,B)r(B,C) > \delta > 0.$$

[If $r(A, B) = 0$, replace it by a positive number so that the inequality still holds.] Consider the exchange

$$\begin{aligned} e &= -[r(A,B)B - A] - (1+\varepsilon)r(A,B)[r(B,C)C - B] \\ &\quad +(1+\varepsilon)[r(A,C)C - A] \\ &\geq \delta C + \varepsilon r(A,B)B + \varepsilon A. \end{aligned}$$

Then $e > 0$ on $A \cup B \cup C$, a contradiction. [In the case where $r(A, B) = 0$, replace its second occurrence in the definition of e by the positive number used to replace it in the definition of δ.]

(v) If $0 < r(A, B) < \infty$, the desired equality follows from (i) and (iv).

If $r(A, B) = 0$ and $r(B, A) < \infty$, then

$$1 = r(A,A) = r(A,B)r(B,A) = 0,$$

a contradiction.

Similarly, if $r(A, B) = \infty$, we must have $r(B, A) = 0$ to avoid a contradiction.

(vi) By (v), $r(B, C)^{-1} = r(C, B)$. Now use (iv).

(vii) Use (iii) and (v). □

4. An extension theorem. Let r be a rate of exchange defined on a domain $\mathbf{D}$ consisting of pairs (A, B) in $\mathbf{B} \times \mathbf{B}^0$ where $\mathbf{B}$ is an algebra of subsets of Ω and $\mathbf{B}^0 = \mathbf{B} \setminus \{\phi\}$.

THEOREM 4.1. *If r is locally coherent, then r has a locally coherent extension to all of* $\mathbf{B} \times \mathbf{B}^0$.

The proof, which was suggested to us by an anonymous referee, is in two steps. The first step is to extend r to one additional pair (A_0, B_0). The second step uses Zorn's lemma to complete the proof.

This technique has been used previously by de Finetti [6] to extend coherent probabilities. It was also used by several authors including Dubins [7], Regazzini [16] and Holzer [11] to obtain extensions of conditional probabilities and previsions.

The first step of the argument will be given in a lemma which is similar to and generalizes de Finetti's techniques which are explained, for example, in [6], pages 77–78. Some additional notation is needed for the statement of the lemma.

Let $(A_0, B_0) \in \mathbf{B} \times \mathbf{B}^0 \setminus \mathbf{D}$. For each exchange $e(\lambda)$ as in (2.1) and every $\rho \in [0, \infty]$, define a new exchange

$$e(\lambda, \rho) = \rho B_0 - A_0 + e(\lambda).$$

The exchange $e(\lambda, \rho)$ will be well-defined if $e(\rho)$ is well-defined and either $\rho < \infty$ or $\rho = \infty$ and $e(\lambda) > -\infty$ on B_0. All exchanges written below are assumed to be well-defined.

Define the *upper exchange rate* for (A_0, B_0) as

$$r^* = r^*(A_0, B_0) = \inf\{\rho \colon e(\lambda, \rho) > 0 \text{ on } A_0 \cup B_0 \cup \operatorname{supp}(\lambda) \text{ for some } e(\lambda)\}$$

and define the *lower exchange rate* as

$$r_* = r_*(A_0, B_0) = \sup\{\rho \colon e(\lambda, \rho) < 0 \text{ on } A_0 \cup B_0 \cup \operatorname{supp}(\lambda) \text{ for some } e(\lambda)\}.$$

Notice that the set of ρ's occurring in the definition of r^* is an interval containing $+\infty$ because $e(\lambda, \rho) > 0$ on the set $D = A_0 \cup B_0 \cup \operatorname{supp}(\lambda)$ and $\sigma \geq \rho$ implies $e(\lambda, \sigma) > 0$ on D. Similarly, the set of ρ's occurring in the definition of r_* is an interval containing 0.

LEMMA 4.1. *An extension of r to $\mathbf{D} \cup \{A_0, B_0\}$ is locally coherent if and only if*

$$r_* \leq r(A_0, B_0) \leq r^*. \tag{4.1}$$

Furthermore, $r_* \leq r^*$ so that a locally coherent extension is possible.

PROOF. It is clear from the definition of local coherence that any locally coherent extension of r to (A_0, B_0) must satisfy (4.1), and it is trivial to verify that any value of $r(A_0, B_0)$ satisfying (4.1) gives a locally coherent extension.

It only remains to be shown that $r_* \leq r^*$. Suppose to the contrary that $r_* > r^*$. Let ρ be a number in the open interval (r^*, r_*). By definition of r^* and r_*, there exist exchanges $e(\lambda_1)$ and $e(\lambda_2)$ such that

$$e(\lambda_1, \rho) > 0 \quad \text{on supp } (\lambda_1),$$
$$e(\lambda_2, \rho) < 0 \quad \text{on supp } (\lambda_2).$$

Hence,

$$e(\lambda_1) - e(\lambda_2) = e(\lambda_1, \rho) - e(\lambda_2, \rho)$$

is an exchange of the form $e(\lambda)$ which is positive on $\operatorname{supp}(\lambda_1) \cup \operatorname{supp}(\lambda_2)$, a set which contains $\operatorname{supp}(\lambda)$. This contradicts the local coherence of r on $\mathbf{D}$. □

The second step in the proof of the theorem uses some form of the axiom of choice. To apply Zorn's lemma by analogy with Holzer's argument for Theorem 3.4 in [11], consider the collection of pairs $(s, \mathbf{E})$ where $\mathbf{D} \subset \mathbf{E} \subset \mathbf{B} \times \mathbf{B}^0$ and s is a locally coherent rate on $\mathbf{E}$ whose restriction to $\mathbf{D}$ is r. Partially order the collection by defining

$$(s_1, \mathbf{E}_1) \le (s_2, \mathbf{E}_2)$$

if $\mathbf{E}_1 \subset \mathbf{E}_2$ and s_2 restricted to $\mathbf{E}_1$ is s_1. It is easy to see that every chain has an upper bound. Hence, Zorn's lemma applies to give a maximal element $(s, \mathbf{E})$. By the lemma, $\mathbf{E}$ must be all of $\mathbf{B} \times \mathbf{B}^0$.

A proof could also be based on the finite intersection property as in Lemma 8 of Dubins [7].

COROLLARY 4.1. *If* A_0, B_0 *belong to* $\mathbf{B}$, *then for each* ρ *in the interval* $[r_*, r^*]$, *there is a fully defined locally coherent extension of* r *with* $r(A_0, B_0) = \rho$.

5. Conditional probability and rates of exchange. Let $\mathbf{B}$ be an algebra of subsets of Ω and let $\mathbf{B}^0$ be the collection of nonempty sets in $\mathbf{B}$.

DEFINITION 5.1. A *conditional probability* P *on* $\mathbf{B}$ is a mapping $P = P(\cdot \mid \cdot)$ from $\mathbf{B} \times \mathbf{B}^0$ to the real numbers satisfying:

(a) $P(\cdot|B)$ is a finitely additive probability measure on $\mathbf{B}$ for every $B \in \mathbf{B}^0$, with $P(B|B) = 1$.

(b) $P(A \cap B|C) = P(A|C)P(B|A \cap C)$ for A, B in $\mathbf{B}$, C, $A \cap C$ in $\mathbf{B}^0$.

This definition is from Krauss [13] and is essentially that of Rényi [18] except that countable additivity of the conditional measures is not required here.

A rate of exchange r with domain $\mathbf{B} \times \mathbf{B}^0$ is said to be a *rate of exchange on* $\mathbf{B}$.

The result of this section is that locally coherent rates of exchange and conditional probabilities on an algebra can be viewed as different aspects of the same objects. Together with the equivalence property of Holzer and Regazzini ([11], Theorem 5.3; see also [16]), it also shows the equivalence of these notions with coherent conditional probabilities on an algebra.

THEOREM 5.1. (i) *If* r *is a locally coherent rate of exchange on* $\mathbf{B}$ *and* P *is defined by*

$$\begin{aligned} P(A|B) &= r(A \cap B, B), && \text{if } A \cap B \neq \varnothing, \\ &= 0, && \text{if } A \cap B = \varnothing, \end{aligned}$$

for $A \in \mathbf{B}$, $B \in \mathbf{B}^0$, *then* P *is a conditional probability on* $\mathbf{B}$.

(ii) *If* P *is a conditional probability on* $\mathbf{B}$ *and* r *is defined by*

$$\begin{aligned} r(A, B) &= \frac{P(A|A \cup B)}{P(B|A \cup B)}, && \text{if } P(B|A \cup B) > 0, \\ &= \infty, && \text{if } P(B|A \cup B) = 0, \end{aligned}$$

for $A \in \mathbf{B}$, $B \in \mathbf{B}^0$, *then* r *is a locally coherent rate of exchange on* $\mathbf{B}$.

(iii) *The mappings* $r \to P$ *and* $P \to r$ *defined in* (i) *and* (ii) *are inverses of each other and therefore define a one-to-one correspondence.*

PROOF. (i) Use (i), (ii) and (iv) of Theorem 3.1.

(ii) Let $e = e(\lambda)$ be an exchange with $C = \text{supp}(\lambda)$. Write

$$e = \sum_{i=1}^{n} \lambda_i S_i,$$

where $\lambda_i = \lambda(A_i, B_i) \neq 0$, $S_i = r(A_i, B_i)B_i - A_i$, $A_i \in \mathbf{B}$, $B_i \in \mathbf{B}^0$ for $i = 1, \ldots, n$, and $C = \bigcup_{i=1}^{n}(A_i \cup B_i)$.

In order to reach a contradiction, assume

$$\inf_C e > 0.$$

An immediate consequence is that, if $r(A_i, B_i) = \infty$, then $\lambda_i > 0$.

Let $E(\cdot|C)$ be the operator corresponding to integration with respect to the measure $P(\cdot|C)$, where the finitely additive integral is defined, for example, as in Dunford and Schwartz [8], Section 3.2.2. To reach a contradiction, it suffices to show

$$E(\lambda_i S_i|C) \leq 0 \tag{5.1}$$

for $i = 1, \ldots, n$, for then $E(e|C) \leq 0$. To prove (5.1), we will consider three cases and, to simplify notation, we will omit the subscript i.

CASE 1. $0 < r(A, B) < \infty$. In this case,

$$E(\lambda S|C) = \lambda[r(A, B)P(B|C) - P(A|C)] = 0$$

because

$$r(A, B) = \frac{P(A|A \cup B)}{P(B|A \cup B)} = \frac{P(A|C)}{P(B|C)} \quad \text{if } P(A \cup B|C) > 0.$$

[The case where $P(A \cup B|C) = 0$ is trivial.] To verify the last equality, use Definition 5.1(b) (conditional probability) to calculate

$$\begin{aligned} P(A|C) &= P((A \cup B) \cap A|C) \\ &= P(A \cup B|C)P(A|(A \cup B) \cap C) \\ &= P(A \cup B|C)P(A|A \cup B) \end{aligned} \tag{5.2}$$

and similarly

$$P(B|C) = P(A \cup B|C)P(B|A \cup B).$$

CASE 2. $r(A, B) = 0$. By the definition of r in (ii), $P(A|A \cup B) = 0$, and then by the calculation in (5.2), $P(A|C) = 0$. Hence, $E(\lambda S|C) = 0$.

CASE 3. $r(A, B) = \infty$. As was remarked above, $\lambda > 0$ in this case. Also, $P(B|A \cup B) = 0$ and hence $P(B|C) = 0$. It follows from the definition of the integral in [8] that integrals over sets of measure zero are also zero and hence that

$$E(\lambda S|C) = -\lambda P(A|C) \leq 0. \qquad \square$$

It follows from Theorems 4.1 and 5.1 that a locally coherent rate of exchange on an arbitrary domain **D** is consistent with some conditional probability on the algebra of all subsets. However, the correspondence will not in general be one-to-one, as is explained in Section 7.

6. Linearly ordered families of measures, Carlson's construction and strict coherence. For a conditional probability P on an algebra **B**, there is a natural ordering of nonempty events: $A \leq B$ if and only if $P(B|A \cup B) > 0$ and $A < B$ if and only if $P(A|A \cup B) = 0$. This is a linear ordering with associated equivalence relation $A \sim B$ if and only if both $P(A|A \cup B)$ and $P(B|A \cup B)$ are positive. This ordering was introduced by de Finetti [4], used by Rényi [19] in his study of countably additive conditional probabilities and by Krauss [13] in the general finitely additive setting.

Suppose r is a locally coherent rate of exchange on **B**.

LEMMA 6.1. *If A and B are nonempty members of* **B**, *then*:

(i) $A \sim B$ *if and only if* $0 < r(A, B) < \infty$,

if and only if $0 < r(B, A) < \infty$.

(ii) $A < B$ *if and only if* $0 = r(A, B)$,

if and only if $r(B, A) = \infty$.

(iii) $A \leq B$ *if and only if* $r(A, B) < \infty$,

if and only if $0 < r(B, A)$.

PROOF. Use Theorem 5.1 and Theorem 3.1(v). □

Let $[B]$ be the equivalence class of B under $\sim$ and set Γ equal to the collection of all equivalence classes. For $\alpha, \beta \in \Gamma$, write $\alpha \leq \beta$ when $A \leq B$ for some $A \in \alpha$, $B \in \beta$.

THEOREM 6.1 (Rényi, Krauss). *The set Γ of equivalence classes is linearly ordered under $\leq$. For each $\alpha \in \Gamma$, there is a finitely additive measure m_α on* **B** *which is unique up to proportionality and such that*:

(i) $0 < m_\alpha(B) < \infty$ *for* $B \in \alpha$.

(ii) $m_\alpha(B) = 0$ *for* $[B] < \alpha$.

(iii) $m_\alpha(B) = \infty$ *for* $\alpha < [B]$.

(iv) $r(A, B) = m_\alpha(A)/m_\alpha(B)$ *if* $B \in \alpha$, $A \in$ **B**.

(v) *If* $\alpha < \beta$, *then* $m_\alpha(B) < \infty \Rightarrow m_\beta(B) = 0$.

Conversely, suppose Γ is a linearly ordered set and $\{m_\alpha, \alpha \in \Gamma\}$ is a family of measures on **B** *satisfying* (v). *Suppose also that, for every nonempty $B \in$* **B**, *there is an $\alpha \in \Gamma$ such that $0 < m_\alpha(B) < \infty$. For that α, which is unique by* (v), *define*

$$r(A, B) = m_\alpha(A)/m_\alpha(B) \tag{6.1}$$

for all $A \in$ **B**. *Then r is a locally coherent rate of exchange on* **B**.

The proof of this result can be found in Rényi [19] and Krauss [13] although these authors work with conditional probabilities rather than the equivalent rates. The proof is not difficult and the measure m_α on the equivalence class $[B]$ is just $r(\cdot, B)$ up to a proportionality constant.

REMARK. If any of the measures occurring in the converse half of Theorem 6.1 takes on only the values 0 and ∞, then it may be deleted from the family since it does not contribute to the construction of r in (6.1). After these measures are deleted, each remaining measure takes on a positive finite value and it follows from condition (v) that the mapping $\alpha \to m_\alpha$ is one-to-one.

EXAMPLE 6.1. Let m be a finitely additive measure on an algebra **B** and define

$$r(A, B) = m(A)/m(B)$$

whenever the right-hand side is well-defined. (**B** could be the algebra of Borel sets in R^n and m could be Lebesgue measure on **B**.)

EXAMPLE 6.2. Let **B** be the Borel subsets of the real line, let μ_0 be counting measure, let μ_1 be Lebesgue measure and let μ_2 be any finitely additive extension of the density to **B**. (See Section 1.) It is easily verified that $\mu_i(B) < \infty \Rightarrow \mu_j(B) = 0$ for $i < j$ and $B \in \mathbf{B}$. Define

$$r_i(A, B) = \mu_i(A)/\mu_i(B), \qquad i = 1, 2, 3,$$

whenever the right-hand side is well-defined. The r_i agree on any points which lie in the domains of more than one and so we can let $r(A, B) = r_i(A, B)$ on the domain of r_i.

EXAMPLE 6.3. Let **B** be the Borel subsets of R^n and, for $0 \le \alpha \le n$, let m_α be α-dimensional Hausdorff measure on **B**. Define a rate r by (6.1) whenever the denominator is finite and positive.

The rates defined in all three examples are locally coherent. This follows from the second half of Theorem 6.1 together with the following lemma.

Let $\{m_\alpha, \alpha \in I\}$ be a family of distinct finitely additive measures on **B** with I a linearly ordered set. Say the family is *linearly ordered* if it satisfies condition (v) of Theorem 6.1 and call the family *complete* if, for each $B \in \mathbf{B}^0$, there is an $\alpha \in I$ such that $0 < m_\alpha(B) < \infty$.

LEMMA 6.2. *Every linearly ordered family of measures is contained in a complete, linearly ordered family of measures.*

PROOF. By Zorn's lemma, there is a maximal linearly ordered family $\{m_\alpha, \alpha \in \Gamma\}$ containing the given family. Suppose it is not complete. Then there is a set $B \in \mathbf{B}$ such that $m_\alpha(B)$ is 0 or ∞ for every $\alpha \in \Gamma$. Let $\Gamma_\infty = \{\alpha \in \Gamma: m_\alpha(B) = \infty\}$ and $\Gamma_0 = \{\alpha \in \Gamma: m_\alpha(B) = 0\}$. Then $c = (\Gamma_\infty, \Gamma_0)$ is a Dedekind cut of Γ, i.e., a partition of Γ such that $\alpha \in \Gamma_\infty$, $\beta \in \Gamma_0$ implies $\alpha < \beta$. We can adjoin c to Γ setting $\Gamma' = \Gamma \cup \{c\}$ with the ordering on Γ' to satisfy $\alpha < c < \beta$ for $\alpha \in \Gamma_\infty$, $\beta \in \Gamma_0$. Let $\mathbf{F}$ be the ideal of all sets $A \in \mathbf{B}$ such that $m_\alpha(A) < \infty$ for some $\alpha \in \Gamma_\infty$ and let $\mathbf{N}$ be the ideal of $A \in \mathbf{B}$ such that $m_\beta(A) = 0$ for all $\beta \in \Gamma_0$. Then

$$\mathbf{F} \cup \{B\} \subset \mathbf{N}.$$

Define $\mathbf{A} = \mathbf{B} \cap B$ to be the algebra of sets in $\mathbf{B}$ which are subsets of B. Then $\mathbf{A} \cap \mathbf{F} = \mathbf{F} \cap B$ is a proper ideal in $\mathbf{A}$ and, consequently, there is a finitely additive probability measure m on $\mathbf{A}$ which annihilates $\mathbf{F} \cap B$. Define m_c on $\mathbf{B}$ by setting

$$\begin{aligned} m_c(A) &= m(A \cap B), \quad &&\text{if } A \in \mathbf{N}, \\ &= \infty, &&\text{if } A \notin \mathbf{N}, \end{aligned}$$

for $A \in \mathbf{B}$. Then $\{m_\alpha, \alpha \in \Gamma'\}$ is a linearly ordered family contradicting the maximality of $\{m_\alpha, \alpha \in \Gamma\}$. □

REMARK. An alternative proof of Lemma 6.2 would use Theorem 6.1 after showing that any linearly ordered family of measures induces a locally coherent rate by the formula (6.1).

The converse half of Theorem 6.1 shows how to construct a locally coherent rate r from a complete, linearly ordered family of measures. There is a more recent technique of Carlson [3] which makes it possible to obtain a locally coherent rate from any complete family of finite measures after the index set is well-ordered.

THEOREM 6.2. *Let I be a well-ordered set and let $\{m_\alpha, \alpha \in I\}$ be a complete family of finitely additive, finite measures on* $\mathbf{B}$. *For $A \in \mathbf{B}$, $B \in \mathbf{B}^0$, let $\alpha(B)$ be the least $\alpha \in I$ such that $0 < m_\alpha(B)$ and define*

$$\begin{aligned} P(A|B) &= m_{\alpha(B)}(A \cap B)/m_{\alpha(B)}(B), \\ r(A, B) &= m_{\alpha(A \cup B)}(A)/m_{\alpha(A \cup B)}(B), \quad &&\textit{if } m_{\alpha(A \cup B)}(B) > 0, \\ &= \infty, &&\textit{if not}. \end{aligned}$$

Then (i) *P is a conditional probability on* $\mathbf{B}$ *and* (ii) *r is the locally coherent rate of exchange associated with P.*

PROOF. (i) Part (a) of Definition 5.1 is obvious. To check (b), notice that, if $m_{\alpha(C)}(A \cap C) > 0$, then $\alpha(A \cap C) = \alpha(C)$ and

$$\begin{aligned} P(A|C)P(B|A \cap C) &= \frac{m_{\alpha(C)}(A \cap C)}{m_{\alpha(C)}(C)} \cdot \frac{m_{\alpha(C)}(A \cap B \cap C)}{m_{\alpha(C)}(A \cap C)} \\ &= P(A \cap B|C). \end{aligned}$$

If $m_{\alpha(C)}(A \cap C) = 0$, then $P(A|C) = 0 = P(A \cap B|C)$ and (b) holds.

(ii) This is easily verified using the formula in Theorem 5.1(ii). □

The construction of Theorem 6.2 makes it easy to define countably additive conditional probabilities, a problem found difficult by Krauss [13], page 236.

Apply Lemma 6.2 and Theorem 6.1 to a singleton $\{m\}$ as in Example 6.1 to see that every improper (or proper) prior m is consistent with a locally coherent rate of exchange.

Not every m determines a strictly coherent rate, but it is now easy to characterize those which do.

THEOREM 6.3. *A rate of exchange r on $\mathbf{B}$ is strictly coherent if and only if there is a finitely additive measure m on $\mathbf{B}$ such that, for every $A \in \mathbf{B}$ and every $B \in \mathbf{B}^0$, $0 < m(B) < \infty$ and $r(A, B) = m(A)/m(B)$.*

PROOF. Suppose r is strictly coherent. Then r is certainly locally coherent. Let $\{m_\alpha, \alpha \in \Gamma\}$ be the family given by Theorem 6.1. We need to show that Γ contains only a single element. Suppose to the contrary that $\alpha, \beta \in \Gamma$ with $\alpha < \beta$. Choose sets $A \in \alpha$, $B \in \beta$. Then $r(A, B) = m_\beta(A)/m_\beta(B) = 0/m_\beta(B) = 0$. Thus the exchange

$$\begin{aligned} e &= -(r(A, B)B - A) \\ &= A \end{aligned}$$

is everywhere nonnegative and positive on A, contradicting strict coherence.

For the converse, suppose m is a measure on $\mathbf{B}$ which is everywhere finite and positive on $\mathbf{B}^0$ and that $r(A, B) = m(A)/m(B)$ for $A \in B$, $B \in \mathbf{B}^0$. Then every simple exchange and, hence, every exchange has integral zero with respect to m. Thus no exchange e can be everywhere nonnegative and somewhere positive. (The set where $e > 0$ would belong to $\mathbf{B}^0$ and have positive measure under m.) □

Kemeny [12] argues that strict coherence is a reasonable requirement in his framework. It seems a bit stringent to us, because, in view of Theorem 6.2, it would rule out even proper, countably additive priors on an algebra such as the Borel subsets of the unit interval.

7. Conditional coherence, local coherence and group invariant rates of exchange. As was shown in Theorem 5.1, there is a natural one-to-one correspondence between locally coherent exchange rates defined on $\mathbf{B} \times \mathbf{B}^0$ and conditional probabilities on the same domain. This section treats the relationship between the two concepts when domains may be proper subsets of $\mathbf{B} \times \mathbf{B}^0$. In particular, the relationship between local coherence and conditional coherence as defined by Holzer [11] and Regazzini [16] is discussed.

Suppose first that r is an exchange rate with domain $\mathbf{E} \subset \mathbf{B} \times \mathbf{B}^0$. Define a function $P(\cdot \mid \cdot)$ with domain $\mathbf{D} = \{(A, B): (A \cap B, B) \in \mathbf{E}\}$ by setting

$$P(A|B) = r(A \cap B, B) \tag{7.1}$$

as in Theorem 5.1(i). If r is locally coherent, then it is almost immediate that P is *conditionally coherent* in the sense of Holzer [11] or Regazzini [16] in that it is impossible for any finite linear combination

$$\sum_{i=1}^{n} \lambda_i [P(A_i|B_i) - A_i] B_i$$

with $\lambda_i \neq 0$ to be positive everywhere on $\bigcup_{i=1}^{n} B_i$.

On the other hand, suppose $P(\cdot \mid \cdot)$ is a function defined on a domain $\mathbf{D} \subset \mathbf{B} \times \mathbf{B}^0$. One may attempt to define an exchange rate r by (7.1) for $(A, B) \in \mathbf{D}$. The rate r will be well-defined if, whenever $\{(A_1, B), (A_2, B)\} \subset \mathbf{D}$ with $A_1 \cap B = A_2 \cap B$, then $P(A_1|B) = P(A_2|B)$. This is easily established if P is conditionally coherent as is the local coherence of r.

Consider composing the operations of the two previous paragraphs. If one starts with a conditionally coherent P, defines the associated rate r and continues to define Q from r, it is easy to see that Q is an extension of P. Thus nothing is lost when a conditionaly coherent P is replaced by its associated rate r. However, if one starts with a locally coherent rate r, defines the associated P and defines a rate s from P, it can easily happen that r is a proper extension of s. This is because s is only defined for pairs (A, B) in the domain of r such that $A \subset B$. Thus local coherence can be viewed as a genuine generalization of conditional coherence.

Furthermore there are examples of events A and B for which a rate of exchange is quite naturally defined and for which $P(A|B)$ is not naturally defined. Here is one such example.

Example 7.1. Let $\Omega = N = \{1, 2, \dots\}$. For $n = 1, 2, \dots,$ let m_n be the uniform distribution on $N_n = \{1, 2, \dots, n\}$ and let r_n be the corresponding rate defined as in Example 6.1 by

$$r_n(A, B) = m_n(A)/m_n(B) = |A \cap N_n|/|B \cap N_n|, \tag{7.2}$$

where A and B are subsets of N and $|A|$ denotes the cardinality of a set A. Then define the rate r by

$$r(A, B) = \lim_n r_n(A, B), \tag{7.3}$$

whenever the limit exists. It is easy to verify that the limit of a sequence of locally coherent rates is again locally coherent on its domain. So r is locally coherent.

Notice that r extends the number theoretic density d in the sense that

$$d(A) = \lim_n |A \cap N_n|/n = r(A, N)$$

when the limit is well-defined. It is well-known that there exist subsets A and B of N such that $d(A)$ and $d(B)$ are well-defined and $d(A \cap B)$ is not well-defined.

Our object is to refine this result by constructing A and B so that $d(A) = d(B) = 0$, $r(A, B) = 1$ and $r(A \cap B, A)$ is undefined in the sense that

$$d^*(B|A) = \limsup_n r_n(A \cap B, A) = 1, \tag{7.4}$$
$$d_*(B|A) = \liminf_n r_n(A \cap B, A) = 0.$$

We will take B to be the translate $A + 1 = \{k + 1: k \in A\}$. So it will be clear from (7.2) and (7.3) that $r(A, B) = 1$.

The construction of A is an elaboration of the construction of a set in N with upper density 1 and lower density 0. Define the sequence $\{a_n\}$ inductively by

$$a_1 = 1, \qquad a_{n+1} = (n+1)(a_1 + \cdots + a_n). \tag{7.5}$$

Define further sequences $\{b_n\}$ and $\{s_n\}$ by

$$b_n = na_n, \tag{7.6}$$
$$s_n = b_1 + \cdots + b_n$$

and define sets

$$B_n = \{s_n + k: k = 1, 2, \ldots, a_n\},$$
$$C_n = \{s_n + 2k: k = 1, 2, \ldots, a_n\}$$

for each n. Finally let

$$A = B_1 \cup C_2 \cup B_3 \cup C_4 \cup \ldots.$$

Notice that, for each n, the set of integers from s_n to $s_{n+1} - 1$ has b_n elements and the intersection of A with this set has only $a_n + 1$ elements. It follows from (7.5) and (7.6) that A has density 0.

Also, for $n = 1, 2, \ldots$, $A \cap (A + 1) = B$ contains every member of B_n except the first. Hence, for n odd,

$$r_{s_{n+1}}(A \cap (A+1), A) \geq a_n/(a_1 + \cdots + a_n),$$

which converges to 1 by (7.5). On the other hand, $A \cap (A + 1)$ contains no members of C_n, so that for n even

$$r_{s_{n+1}}(A \cap (A+1), A) \leq (a_1 + \cdots + a_{n-1})/a_n,$$

which converges to 0 by (7.5). The desired (7.4) now follows.

It can also be checked that, in the terminology of Section 4, the upper exchange rate $r^*(A \cap (A + 1), A)$ is 1 and the lower exchange rate $r_*(A \cap (A + 1), A)$ is 0. So, by Lemma 4.1, r will remain locally coherent if it is extended to $(A \cap (A + 1), A)$ so as to assign to this pair any value in the interval $[0, 1]$.

The rate r introduced in Example 7.1 is invariant under translation as was pointed out above. More generally, let T be a group (or semigroup) of symmetries which acts on a sample space Ω. A common assumption is that a probability P is invariant in the sense that $P(A) = P(t(A))$ for events A and $t \in T$. The intuition is that A and $t(A)$ are equally likely. For exchange rates the same intuition leads to the condition $r(t(A), A) = 1$. These invariant exchange rates are studied in a recent paper of Armstrong [1].

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