The Annals of Statistics
1989, Vol. 17, No. 2, 749–763

ON ESTIMATING THE DEPENDENCE BETWEEN TWO POINT PROCESSES[1]

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To assess the dependence structure in a stationary bivariate point process the second-order distribution can be very useful. We prove that the natural estimates of this distribution, based on a realization $A_1 < A_2 < \cdots < A_{n_A}$, $B_1 < B_2 < \cdots < B_{n_B}$ are asymptotically normal and we present a method for constructing approximate confidence intervals for this distribution.

1. Introduction and summary. Let (N_A, N_B) be a stationary bivariate point process on $\mathbb{R}$. This article is concerned with statistical methods for discovering and quantifying an association between the two processes from a realization $A_1 < A_2 < \cdots < A_{n_A}$, $B_1 < B_2 < \cdots < B_{n_B}$ over a long period of time T. The paper is motivated by certain problems that arise in neurophysiology, which are very briefly described as follows [for further details see, e.g., Bryant, Ruiz Marcos and Segundo (1973)].

Two neurons, A and B, are monitored over a period of time T during which each neuron fires a sequence of impulses. The problem is to determine whether or not the impulse times are associated. An association between N_A and N_B may be construed as evidence that either the two neurons are communicating or that they both share input from a third source.

Another problem arises in certain neurophysiological studies of learning and memory. An animal is to be taught (trained) to perform a certain task. Now consider two "connected" neurons, A and B, which are essential in the performance of this task. Record the impulse times during a period before the learning experience, obtaining a realization of $(N_A^{\text{bef.}}, N_B^{\text{bef.}})$, and during a period of time *well after* the learning experience, obtaining a realization of $(N_A^{\text{aft.}}, N_B^{\text{aft.}})$. The processes $N_A^{\text{bef.}}$ and $N_B^{\text{bef.}}$ may be *dependent*. The problem is to determine whether or not this dependence is "stronger" for the processes $N_A^{\text{aft.}}$ and $N_B^{\text{aft.}}$. A neurophysiologist may consider a change in the strength of the dependence as evidence that learning has taken place.

The two problems have very different statistical character. Let S be a statistic that "measures" the dependence between two point processes. The first problem is one of testing the hypothesis that N_A and N_B are independent and requires only knowledge of the distribution of S under the assumption that N_A and N_B are independent. The second problem is much more difficult: To

Received May 1984; revised September 1987.
[1]Research supported by Air Force Office of Scientific Research Grant AFOSR 88-0040 and by Office of Naval Research Contracts N00014-86-K-0156 and N00014-83-K-0238.
AMS 1980 *subject classifications*. Primary 62M09, 62M07; secondary 62G05, 62G10.
Key words and phrases. Bivariate point process, Ripley's K-function, cross-intensity function, stationary point process, stationary sequence.

compare S across two situations we must know the distribution of S when the two point processes are dependent.

In a more general context, Ripley (1976, 1977) introduced a measure K, defined on an appropriate space, that summarizes the second-order properties of the process. Before describing this measure, we need to state some assumptions and introduce some notation. Let $N_i(s, t)$ denote the number of events of type i occurring in the interval $(s, t]$, for $i = A, B$. Assume that each process has no multiple occurrences and that the intensities

$$\lambda_i = \lim_{h \to 0} \frac{1}{h} P\{N_i(t, t+h) > 0\} \quad \text{for } i = A, B \tag{1}$$

are finite. [The existence of these limits was proved by Khintchine (1960).] The λ_i's then have an interpretation as mean number of occurrences per unit of time: For $t_1 < t_2$,

$$EN_i(t_1, t_2) = \lambda_i(t_2 - t_1) \quad \text{for } i = A, B. \tag{2}$$

[This follows from Dobrushin's lemma and Korolyuk's theorem; see Leadbetter (1968).]

We now give an informal description of the measure K, adapted to the present context.

The measure K is defined on the Borel subsets of $\mathbb{R}$, and for $t_1 < t_2$, writing $K(t_1, t_2)$ for $K\{(t_1, t_2)\}$, we have

$$\begin{aligned} K(t_1, t_2) &= \frac{1}{\lambda_A} E\{N_A(t_1, t_2) | \text{ a } B \text{ point at } t = 0\} \\ &\left(= \frac{1}{\lambda_B} E\{N_B(-t_2, -t_1) | \text{ an } A \text{ point at } t = 0\} \right). \end{aligned} \tag{3}$$

Note that if N_A and N_B are independent, then

$$K(t_1, t_2) = t_2 - t_1, \tag{4}$$

regardless of the values λ_A and λ_B. Thus, K represents Lebesgue measure on $\mathbb{R}$.

Ripley proposed the estimate of $K(t_1, t_2)$ given by

$$\hat{K}(t_1, t_2) = \frac{T}{n_A n_B} \sum_{i=1}^{n_B} \sum_{j=1}^{n_A} I\{A_j - B_i \in (t_1, t_2)\}, \tag{5}$$

where $I\{\cdot\}$ denotes the indicator function (actually, the estimate proposed by Ripley has an edge correction for points near the boundary of the period of observation; this edge correction will not concern us).

Previous work on the estimator $\hat{K}$ is concerned with spatial processes. The results center on using $\hat{K}$ to test that a single process is Poisson [Ripley (1977), Ripley (1981), Chapter 8 and Silverman (1976)] and on using $\hat{K}$ to test for independence of two processes [Lotwick and Silverman (1982) and Diggle and Milne (1983)].

In this paper we study the asymptotic properties of $\hat{K}(t_1, t_2)$. The contributions of this paper are twofold:

(6) first, a proof that under certain regularity conditions, as $n_B \to \infty$,
$$\sqrt{n_B}\left(\hat{K}(t_1, t_2) - K(t_1, t_2)\right) \to_d \mathscr{N}\left(0, \sigma^2(t_1, t_2)\right);$$

(7) second, a method for constructing consistent estimates of $\sigma^2(t_1, t_2)$.

Besides providing the basis for a test of independence between N_A and N_B, (6) and (7) enable one to test whether or not $K(t_1, t_2)$ has changed in the experimental situation described earlier.

The cross-intensity function defined by

$$\lambda_{AB}(u) = \lim_{h_1, h_2 \to 0} \frac{1}{h_1 h_2} P\{N_A(u+t, u+t+h_1) > 0; N_B(t, t+h_2) > 0\} \tag{8}$$

is related to K by

$$K(t_1, t_2) = \frac{1}{\lambda_A \lambda_B} \int_{t_1}^{t_2} \lambda_{AB}(u)\, du. \tag{9}$$

Under the independence hypothesis, $\lambda_{AB}(u) = \lambda_A \lambda_B$ for all u. Brillinger (1976) considered the random function

$$J_{AB}^T(u) = \sum_{i=1}^{n_B} \sum_{j=1}^{n_A} I\{A_j - B_i \in (u-h, u+h)\} \tag{10}$$

and showed that under suitable regularity, if $h \to 0$ and $T \to \infty$ in such a way that hT remains constant, then for $u_k^T \to u_k$, $|u_k^T - u_{k'}^T| > 2h$, $1 \le k < k' \le M$, $J_{AB}^T(u_k^T)$ are asymptotically independent Poisson random variables with means $2hT\lambda_{AB}(u_k)$, for $k = 1, \ldots, M$. Thus, $\hat{\lambda}_{AB}(u) = (J_{AB}^T(u))/2hT$ can be used to estimate $\lambda_{AB}(u)$ at a finite number of points.

In practice one would graph the two functions $\hat{\lambda}_{AB}$ and $\hat{K}$ over a finite range, say $[-L, L]$ [i.e., graph $\hat{K}(-L, t)$ for $-L \le t \le L$]. Although from a mathematical viewpoint λ_{AB} and K contain essentially the same information, the statistical properties of their estimates are quite different: Estimation of λ_{AB} is akin to estimating a density, and from Brillinger's result the variance of $\hat{\lambda}_{AB}$ is of the order $1/hT$. On the other hand, estimation of K is akin to estimating a distribution function, and from (6), the variance of $\hat{K}$ is of the smaller order $1/n_B$. A graph of $\hat{\lambda}_{AB}$ may, however, indicate features (spikes, location of maxima and minima, etc.) that cannot be easily seen in the graph of $\hat{K}$. Clearly the two approaches are complementary.

2. Asymptotic distribution of the $\hat{K}$-function.

Let

$$U_{AB}(t_1, t_2) = E\{N_A(t_1, t_2) \mid \text{a } B \text{ point at } t = 0\}. \tag{11}$$

We may estimate $U_{AB}(t_1, t_2)$ by

$$\hat{U}_{AB}(t_1, t_2) = \frac{1}{n_B} \sum_{i=1}^{n_B} \sum_{j=1}^{n_A} I\{A_j - B_i \in (t_1, t_2)\}. \tag{12}$$

Letting

$$\hat{\lambda}_i = \frac{n_i}{T} \quad \text{for } i = A, B, \tag{13}$$

we note that

$$K(t_1, t_2) = \frac{1}{\lambda_A} U_{AB}(t_1, t_2) \quad \text{and} \quad \hat{K}(t_1, t_2) = \frac{1}{\hat{\lambda}_A} \hat{U}_{AB}(t_1, t_2). \tag{14}$$

To prove asymptotic normality of $\hat{K}$ (Theorem 2) we will prove joint asymptotic normality of $(\hat{U}_{AB}(t_1, t_2), \hat{\lambda}_A)$. We will in fact find it necessary to first prove joint asymptotic normality of $(\hat{U}_{AB}(t_1, t_2), \hat{\lambda}_A/\hat{\lambda}_B, 1/\hat{\lambda}_B)$. The delta-method (i.e., a first term Taylor expansion) applied to the function $f(x, y, z) = xz/y$ then yields the asymptotic normality of $\hat{K}(t_1, t_2)$. We also obtain the joint asymptotic normality of $(\hat{U}_{AB}(t_1, t_2), \hat{\lambda}_A, \hat{\lambda}_B)$ by applying the delta-method to the function $g(x, y, z) = (x, y/z, 1/z)$.

We now need to give the statistical setting of our asymptotic investigation. The functions $U_{AB}(t_1, t_2)$ and $K(t_1, t_2)$ involve the notion of the Palm measure. That is for $\varepsilon > 0$, we consider the conditional distribution of the process (N_A, N_B) given that there is a B point in the interval $(0, \varepsilon)$ and take the limiting distribution of (N_A, N_B) as $\varepsilon \to 0$. Intuitively, this corresponds to selecting a B point "arbitrarily" and considering the process with that point labeled the origin. This notion is discussed for univariate processes by Leadbetter (1972) and for bivariate processes by Wisniewski (1972). We will assume that the process is observed during a period of length T starting immediately after the occurrence of an "arbitrary" B point, say B_0 (thus, we will be working with the Palm measure). This mode of sampling is called semisynchronous sampling by Cox and Lewis (1972); see Wisniewski (1972) for some fundamental properties related to it. Also, for the sake of convenience, we will assume that the period of observation ends with a B point.

We now consider two subfields of the σ-field on which the Palm measure is defined. Let $\mathscr{F}_{-\infty}^{B_0}$ and $\mathscr{F}_{B_0+u}^{\infty}$ denote the σ-fields generated by all events occurring before B_0 and after $B_0 + u$, respectively. Let

$$\begin{aligned} \alpha(u) = \sup\Big\{ &|P(E_1 \cap E_2) - P(E_1)P(E_2)|; \\ &E_1 \in \mathscr{F}_{-\infty}^{B_0},\ E_2 \in \mathscr{F}_{B_0+u}^{\infty} \vee \sigma\big(N_B(0, u)\big)\Big\}. \end{aligned} \tag{15}$$

If $\alpha(u) \to 0$ as $u \to \infty$, then the distant future is virtually independent of the past. We will actually need stronger conditions on $\alpha(\cdot)$.

Let $\beta > 0$, $\eta > 1$, $0 < \tau < 1$ be any constants satisfying

$$\left(\eta - \frac{\eta + 1}{\beta + 1}\right)\tau > 1. \tag{16}$$

Assumptions:

A1. $\int_0^\infty [\alpha(t)]^\tau t^\beta \, dt < \infty$.

A2. $\sup_{-\infty < j < \infty} E\{[N_B(j, j+1)]^\eta \mid \text{a } B \text{ point at } t = 0\} = D < \infty$.

A3. $E\{[N_A(t_1, t_2)^{4(1+\tau/(1-\tau))} \mid \text{a } B \text{ point at } t = 0\} < \infty$.

A4. $E[(B_1 - B_0)^{4(1+\tau/(1-\tau))}] < \infty$.

A5. $E[N_A(B_0, B_1)]^{4(1+\tau/(1-\tau))} < \infty$.

Before stating our theorems we discuss our assumptions, and compare them with those of Brillinger (1976). As mentioned earlier, our proof of asymptotic normality of $\hat{K}(t_1, t_2)$ requires first a proof of joint asymptotic normality of $(\hat{U}_{AB}(t_1, t_2), \hat{\lambda}_A/\hat{\lambda}_B, 1/\hat{\lambda}_B)$. We will accomplish this by representing $(\hat{U}_{AB}(t_1, t_2), \hat{\lambda}_A/\hat{\lambda}_B, 1/\hat{\lambda}_B)$ as a sum of a trivariate stationary sequence $\{(T_i^{(1)}, T_i^{(2)}, T_i^{(3)})\}$, to which we will apply a central limit theorem for stationary sequences. Now any such theorem must assume a moment condition on $(T_1^{(1)}, T_1^{(2)}, T_1^{(3)})$ and also a mixing condition on $\{(T_i^{(1)}, T_i^{(2)}, T_i^{(3)})\}$. In general, weakening of the moment condition must be compensated by strengthening of the mixing condition and vice versa. Assumptions 3–5 provide moment conditions on $(T_1^{(1)}, T_1^{(2)}, T_1^{(3)})$. Assumption 2 ensures that the B process "moves along" rapidly enough so that A1, the mixing condition imposed on the point process, translates into a mixing condition for the sequence $\{(T_i^{(1)}, T_i^{(2)}, T_i^{(3)})\}$. Relationship (16) describes in a technical way the interplay between the mixing rate on the point process and the moment condition on $(T_1^{(1)}, T_1^{(2)}, T_1^{(3)})$.

The conditions assumed by Brillinger (1976) neither imply nor are implied by A1–A5 of the present paper. Brillinger assumes a mixing condition on the bivariate point process and also that the "second-order moments" $\lambda_{ij}(\cdot)$ $(i, j = A, B)$ exist and are continuous [he also assumes existence and continuity of the "third- and fourth-order moments"; see equation (2.2) of his paper]. This condition on $\lambda_{AB}(\cdot)$ is not satisfied by the following process: N_B is a Poisson process and N_A is N_B shifted to the right by one unit. In this case, $\lambda_{AB}(1) = \infty$. This process does however satisfy A1–A5. Conversely, it is easy to find processes (N_A, N_B) satisfying all of Brillinger's conditions, but not those of the present paper. Perhaps the simplest example is the following. Let N_A and N_B be independent, N_A being a Poisson process and N_B being an equilibrium renewal process on $(-\infty, \infty)$ [for a definition and a construction see Karlin and Taylor (1975), pages 517–519] with interarrival distribution having a first moment but no second moment. Then A4 is violated and it is not difficult to check that this process satisfies all of Brillinger's conditions.

THEOREM 1. *Assume* A1 *and* A2. *Let* $U_{AB}(t_1, t_2)$, $\hat{U}_{AB}(t_1, t_2)$ *and* $\hat{\lambda}_i$ *for* $i = A, B$ *be defined by* (11), (12), *and* (13), *respectively*.

(i) *Under* A3, *we have as* $n_B \to \infty$,

$$\sqrt{n_B}\left(\hat{U}_{AB}(t_1, t_2) - U_{AB}(t_1, t_2)\right) \to_d \mathscr{N}\left(0, \gamma^2(t_1, t_2)\right).$$

Furthermore, any estimate $\hat{\gamma}^2(t_1, t_2)$ *of the form* (25), *satisfying* (27) *and* (28), *is a consistent estimate of* $\gamma^2(t_1, t_2)$.

(ii) *Under* A4 *and* A5 *we have as* $n_B \to \infty$,

$$\sqrt{n_B}\left(\hat{\lambda}_A - \lambda_A, \hat{\lambda}_B - \lambda_B\right)' \to_d \mathscr{N}(0, \Lambda).$$

Furthermore, any estimate $\hat{\Lambda}$ *of the form* (57) [*refer to* (49) *and* (51)–(56)], *satisfying* (27) *and* (28) *is a consistent estimate of* Λ.

(iii) *Under* A3–A5 *we have as* $n_B \to \infty$,

$$\sqrt{n_B}\left(\hat{U}_{AB}(t_1, t_2) - U_{AB}(t_1, t_2), \hat{\lambda}_A - \lambda_A, \hat{\lambda}_B - \lambda_B\right)' \to_d \mathscr{N}(0, \Sigma(t_1, t_2)).$$

Furthermore, any estimate $\hat{\Sigma}(t_1, t_2)$ *of the form* (56) [*refer to* (49) *and* (51)–(55)], *satisfying* (27) *and* (28), *is a consistent estimate of* $\Sigma(t_1, t_2)$.

PROOF OF (i). We begin by showing asymptotic normality. Let U_i' and U_i be defined by

$$U_i' = \sum_{j=-\infty}^{\infty} I\{A_j \in (B_i + t_1, B_i + t_2)\} I\{B_0 \le A_j \le B_{n_B}\}$$

and

$$U_i = \sum_{j=-\infty}^{\infty} I\{A_j \in (B_i + t_1, B_i + t_2)\}.$$

Note that

$$\hat{U}_{AB}(t_1, t_2) = \frac{1}{n_B} \sum_{i=1}^{n_B} U_i'. \tag{17}$$

It is clear that $\sum_{i=1}^{n_B} U_i - \sum_{i=1}^{n_B} U_i' = O_p(1)$. Thus, it suffices to prove the result with U_i's instead of U_i''s in (17). Observe that the sequence $\{U_i\}_{i=-\infty}^{\infty}$ is stationary, with mean $U_{AB}(t_1, t_2)$ and finite variance (by A3). The U_i's may be far from independent: For small k, U_i and U_{i+k} may be nearly identical. If, however, U_i and U_{i+k} are "nearly independent" for large k, then one can still hope to have a central limit theorem effect. The proof consists of translating A1, the mixing condition on the point process, into a mixing condition on $\{U_i\}$ that allows the application of an appropriate central limit theorem for stationary sequences.

Let $\mu(k)$ be defined for $k = 1, 2, \ldots$ by

$$\begin{aligned} \mu(k) = \sup\{&|P(E_1 \cap E_2) - P(E_1)P(E_2)|; \\ &E_1 \in \sigma(\ldots, U_{-1}, U_0),\ E_2 \in \sigma(U_k, U_{k+1}, \ldots)\}. \end{aligned} \tag{18}$$

[Here, $\sigma(\ldots, U_{-1}, U_0)$ denotes the σ-field generated by $\{\ldots, U_{-1}, U_0\}$, and similarly for $\sigma(U_k, U_{k+1}, \dots)$.] The function $\mu(\cdot)$ is called the mixing coefficient of the sequence $\{U_i\}$. Our goal is to prove that $\sum_{k=1}^{\infty}[\mu(k)]^{\tau} < \infty$. It will be more convenient, however, to show instead that $\sum_{k=1}^{\infty}[\mu(2k)]^{\tau} < \infty$. The two conditions are equivalent since $\mu(\cdot)$ is nonincreasing.

Let $k \geq 1$ be fixed, let $E_1 \in \sigma(\dots, U_{-k})$, $E_2 \in \sigma(U_k, \dots)$ and consider $P(E_1 \cap E_2)$. Let

$$C_{-k} = \{B_{-k} - B_0 \leq -[\![k^{1/(\beta+1)}]\!]\} \quad \text{and} \quad C_k = \{B_k - B_0 \geq [\![k^{1/(\beta+1)}]\!]\},$$

where $[\![\ ]\!]$ denotes the integer part. We may write

$$P(E_1 \cap E_2) = P\{(E_1 \cap C_{-k}) \cap (E_2 \cap C_k)\} + P\{(E_1 \cap E_2) \cap (C_{-k}^c \cup C_k^c)\}, \tag{19}$$

where the superscript c denotes complementation.

Consider the first term on the right side of (19). For all large k, since $B_{-k} - B_0 \leq -[\![k^{1/(\beta+1)}]\!]$ implies that $B_{-k} + t_2 \leq B_0$, we have $E_1 \cap C_{-k} \in \mathscr{F}_{-\infty}^{B_0}$. Furthermore, $E_2 \cap C_k \in \mathscr{F}_{B_0+[\![k^{1/(\beta+1)}]\!]+t_1}^{\infty} \vee \sigma(N_B(0, [\![k^{1/(\beta+1)}]\!] + t_1))$. Therefore

$$P\{(E_1 \cap C_{-k}) \cap (E_2 \cap C_k)\} \leq P(E_1)P(E_2) + \alpha([\![k^{1/(\beta+1)}]\!] + t_1). \tag{20}$$

The second term on the right side of (19) is obviously less than or equal to $P(C_{-k}^c) + P(C_k^c)$. These last two probabilities are dealt with in the same way. Consider $P(C_k^c)$. Observe that

$$P(C_k^c) \leq P\{\text{one of the intervals } (B_0 + j, B_0 + j + 1),\ j = 0, 1, \dots, [\![k^{1/(\beta+1)}]\!] - 1 \text{ has at least } [\![k^{\beta/(\beta+1)}]\!] \text{ points}\}. \tag{21}$$

By A2, Chebyshev's inequality and Boole's inequality, the right side of (21) is less than or equal to $[\![k^{1/(\beta+1)}]\!]D[\![k^{\beta/(\beta+1)}]\!]^{-\eta}$. Combining this with (20) and handling the opposite inequality in a similar way, we obtain

$$\mu(2k) \leq \alpha([\![k^{1/(\beta+1)}]\!] + t_1) + k^{1/(\beta+1)}D[\![k^{\beta/(\beta+1)}]\!]^{-\eta}.$$

Assumption A1 implies that

$$\sum_{k=1}^{\infty} \{\alpha([\![k^{1/(\beta+1)}]\!] + t_1)\}^{\tau} < \infty. \tag{22}$$

Combining (22) and (16) we obtain that $\sum_{k=1}^{\infty}[\mu(2k)]^{\tau} < \infty$ and hence that

$$\sum_{k=1}^{\infty} [\mu(k)]^{\tau} < \infty. \tag{23}$$

Assumption A3 implies that

$$E[U_1^{2(1+\tau/(1-\tau))}] < \infty.$$

This, together with (23) allows us to apply Theorem 1.7 of Ibragimov (1962) to conclude that the series $\sum_{h=1}^{\infty}\text{Cov}(U_0, U_h)$ converges absolutely and that as $n_B \to \infty$,

$$\sqrt{n_B}(\hat{U}_{AB}(t_1, t_2) - U_{AB}(t_1, t_2)) \to_d \mathscr{N}(0, \gamma^2(t_1, t_2)),$$

where

$$\gamma^2(t_1, t_2) = \operatorname{Var} U_0 + 2 \sum_{h=1}^{\infty} \operatorname{Cov}(U_0, U_h). \tag{24}$$

Consider next the estimation of $\gamma^2(t_1, t_2)$. Let $\nu_h = \operatorname{Cov}(U_0, U_h)$ for $h = 0, 1, 2, \dots$, so that $\gamma^2(t_1, t_2) = \nu_0 + 2\sum_{h=1}^{\infty}\nu_h$. Let

$$\hat{\gamma}^2(t_1, t_2) = \hat{\nu}_0 + 2 \sum_{h=1}^{M} c_h \hat{\nu}_h, \tag{25}$$

where $\hat{\nu}_h$ is the sample covariance at lag h,

$$\hat{\nu}_h = \frac{1}{n_B - h} \sum_{i=0}^{n_B - h - 1} (U_i - \overline{U})(U_{i+h} - \overline{U}). \tag{26}$$

[Here $\overline{U} = (1/n_b)\sum_{i=1}^{n_B} U_i$.] We will assume that

$$M = M_{n_B} \text{ satisfies } \frac{M_{n_B}}{(n_B)^{1/3}} \to 0 \text{ as } n_B \to \infty \tag{27}$$

and that the constants $c_h = c_h^{(n_B)}$ satisfy

$$\begin{gathered} \text{for each } n_B = 4, 5, 6, \dots,\ 1 \geq c_1^{(n_B)} \geq c_2^{(n_B)} \geq \cdots \geq c_M^{(n_B)} = 0, \\ \text{and for fixed } h,\ c_h^{(n_B)} \to 1 \text{ as } n_B \to \infty. \end{gathered} \tag{28}$$

The choice of constants M and $c_1, c_2, \dots, c_M$ is discussed at the end of this section. We will show under (27) and (28) that

$$E\left|\hat{\gamma}^2(t_1, t_2) - \gamma^2(t_1, t_2)\right| \to 0. \tag{29}$$

This will imply that $\hat{\gamma}^2(t_1, t_2)$ converges to $\gamma^2(t_1, t_2)$ in probability.

Since $E\hat{\nu}_h$ is not in general equal to ν_h, it is more convenient to first work with

$$\tilde{\nu}_h = \frac{1}{n_B - h} \sum_{i=0}^{n_B - h - 1} (U_i - U_{AB}(t_1, t_2))(U_{i+h} - U_{AB}(t_1, t_2)) \tag{30}$$

and

$$\tilde{\gamma}^2(t_1, t_2) = \tilde{\nu}_0 + 2 \sum_{h=1}^{M} c_h \tilde{\nu}_h. \tag{31}$$

Since $E\tilde{\nu}_h = \nu_h$, if we define

$$\gamma_{n_B}^2(t_1, t_2) = \nu_0 + 2 \sum_{h=1}^{M} c_h \nu_h, \tag{32}$$

we have

$$E\tilde{\gamma}^2(t_1, t_2) = \gamma_{n_B}^2(t_1, t_2). \tag{33}$$

From now on we drop the arguments t_1 and t_2 whenever convenient. The triangle inequality gives

$$E|\hat{\gamma}^2 - \gamma^2| \le E|\hat{\gamma}^2 - \tilde{\gamma}^2| + E|\tilde{\gamma}^2 - \gamma_{n_B}^2| + |\gamma_{n_B}^2 - \gamma^2|. \tag{34}$$

Thus, our objective is to show that each of the three terms on the right side of (34) converges to 0 as $n_B \to \infty$. It is easy to see that under (27) and (28), $|\gamma_{n_B}^2 - \gamma^2| \to 0$ as $n_B \to \infty$.

We now consider $E|\tilde{\gamma}^2 - \gamma_{n_B}^2|$. Let $c_0 = \frac{1}{2}$. We have

$$\begin{aligned} E|\tilde{\gamma}^2 - \gamma_{n_B}^2| &= E\left|2\sum_{h=0}^{M} c_h(\tilde{\nu}_h - \nu_h)\right| \le 2\sum_{h=0}^{M} E|\tilde{\nu}_h - \nu_h| \\ &\le 2\sum_{h=0}^{M}\left\{E(\tilde{\nu}_h - \nu_h)^2\right\}^{1/2} = 2\sum_{h=0}^{M}\{\operatorname{Var}\tilde{\nu}_h\}^{1/2}. \end{aligned} \tag{35}$$

We now examine the variance terms. It is well known (and easy to see) that if $\{Y_i\}$ is a stationary sequence, then

$$\operatorname{Var}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n}\sum_{|l|<n}\left(1 - \frac{|l|}{n}\right)\operatorname{Cov}(Y_1, Y_{1+l}). \tag{36}$$

Equation (30) shows that $\tilde{\nu}_h$ is of the form $\tilde{\nu}_h = (1/n)\sum_{i=1}^{n} Y_i$, where $Y_i = (U_i - \overline{U})(U_{i+h} - \overline{U})$, with $\{Y_i\}$ stationary. We can thus obtain the exact variance of $\tilde{\nu}_h$,

$$\operatorname{Var}\tilde{\nu}_h = \frac{1}{n_B - h}\sum_{|l|<n_B-h}\left(1 - \frac{|l|}{n_B - h}\right)\sigma_h(l), \tag{37}$$

where

$$\sigma_h(l) = \operatorname{Cov}\big((U_i - U_{AB})(U_{i+h} - U_{AB}), (U_{i+l} - U_{AB})(U_{i+h+l} - U_{AB})\big). \tag{38}$$

To obtain a useful bound on $\operatorname{Var}\tilde{\nu}_h$ we will show that

$$|\sigma_h(l)| \le [\mu((|l| - h)_+)]^{\tau} C, \tag{39}$$

where $a_+ = \max\{a, 0\}$ and C is a constant not depending on h or l. The key ingredient in the proof of (39) is the use of a lemma of Ibragimov (1962) that gives an explicit bound for the covariance of two random variables ξ and η satisfying $\xi \in \sigma(\dots, U_{-1}, U_0)$ and $\eta \in \sigma(U_k, U_{k+1}, \dots)$, in terms of $\mu(k)$ and certain moments of ξ and η.

For any l such that $|l| > h$, Lemma 1.3 of Ibragimov (1962) implies that

$$\sigma_h(l) \le [\mu(|l| - h)]^{\tau}\left\{4 + 6E|(U_i - U_{AB})(U_{i+h} - U_{AB})|^{2(1+\tau/(1-\tau))}\right\}. \tag{40}$$

The Cauchy–Schwarz inequality implies that

$$E|(U_i - U_{AB})(U_{i+h} - U_{AB})|^{2(1+\tau/(1-\tau))} \le E|(U_i - U_{AB})|^{4(1+\tau/(1-\tau))}. \tag{41}$$

By A3 the right side of (41) is a finite constant not depending on h or l. This proves (39) for $|l| > h$. For $|l| \le h$ the proof is even simpler and is omitted. We now combine (37) and (39) to obtain

$$
\begin{aligned}
\operatorname{Var}\tilde{\nu}_h &\le \frac{1}{n_B - h} \sum_{|l| < n_B - h} |\sigma_h(l)| \le \frac{1}{n_B - h}\left(\sum_{|l| \le h} |\sigma_h(l)| + \sum_{|l| > h} |\sigma_h(l)| \right) \\
&\le \frac{1}{n_B - h}\left((2h+1)C + \sum_{f=1}^{\infty} [\mu(f)]^{\tau} C \right).
\end{aligned} \tag{42}
$$

Next, we use (23) and (27) to see that there exists an integer J, not depending on h or l, such that

$$\operatorname{Var}\tilde{\nu}_h \le n_B^{-2/3} \quad \text{for all } n_B \ge J. \tag{43}$$

We now return to $E|\tilde{\gamma}^2 - \gamma_{n_B}^2|$ [see (35)]. Combining (27) and (43) we arrive at

$$E|\tilde{\gamma}^2 - \gamma_{n_B}^2| \to 0 \quad \text{as } n_B \to \infty. \tag{44}$$

To complete the proof of (29) we need only show that $E|\hat{\gamma}^2 - \tilde{\gamma}^2| \to 0$ as $n_B \to \infty$. We have

$$E|\hat{\gamma}^2 - \tilde{\gamma}^2| = E\left| 2 \sum_{h=0}^{M} c_h(\hat{\nu}_h - \tilde{\nu}_h) \right| \le 2 \sum_{h=0}^{M} E|\hat{\nu}_h - \tilde{\nu}_h|. \tag{45}$$

We will presently show that

$$E|\hat{\nu}_h - \tilde{\nu}_h| \le \frac{C'}{n_B}, \tag{46}$$

where C' is a constant not depending on h.

Referring to (26) and (30), it is easy to see that

$$
\begin{aligned}
\hat{\nu}_h = \tilde{\nu}_h &+ \left(\overline{U}^{(1)} - U_{AB}\right)\left(U_{AB} - \overline{U}\right) \\
&+ \left(\overline{U}^{(2)} - U_{AB}\right)\left(U_{AB} - \overline{U}\right) + \left(\overline{U} - U_{AB}\right)^2,
\end{aligned} \tag{47}
$$

where

$$\overline{U}^{(1)} = \frac{1}{n_B - h} \sum_{i=0}^{n_B - h - 1} U_i, \qquad \overline{U}^{(2)} = \frac{1}{n_B - h} \sum_{i=h}^{n_B - 1} U_i.$$

Thus, $E|\hat{\nu}_h - \tilde{\nu}_h| \le E|(\overline{U}^{(1)} - U_{AB})(U_{AB} - \overline{U})| + E|(\overline{U}^{(2)} - U_{AB})(U_{AB} - \overline{U})| + E(\overline{U} - U_{AB})^2$. Consider first $E(\overline{U} - U_{AB})^2$. From (36) we have

$$E\left(\overline{U} - U_{AB}\right)^2 = \frac{1}{n_B} \sum_{|l| < n_B} \left(1 - \frac{|l|}{n_B}\right) \nu_l \le \frac{C''}{n_B},$$

where $C'' = \nu_0 + 2\sum_{l=1}^{\infty} |\nu_l| < \infty$. Similarly,

$$E\left(\overline{U}^{(1)} - U_{AB}\right)^2 \le \frac{C''}{n_B - h} \quad \text{and} \quad E\left(\overline{U}^{(2)} - U_{AB}\right)^2 \le \frac{C''}{n_B - h}.$$

The Cauchy–Schwarz inequality now yields (46), which together with (27) implies that $E|\hat{\gamma}_h - \tilde{\gamma}_h| \to 0$ as $n_B \to \infty$. This completes the proof of (i). □

We next prove (iii), since the proof of (ii) proceeds in a very similar way.

PROOF OF (iii). For $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$, let $X_i = \xi_1 U_i + \xi_2 N_A(B_i, B_{i+1}) + \xi_3(B_{i+1} - B_i)$. The sequence $\{X_i\}_{i=-\infty}^{\infty}$ is stationary and if $\pi(\cdot)$ denotes its mixing coefficient, it is clear that (23) holds for $\pi(\cdot)$ as well. This gives a central limit theorem for $\{X_i\}$. It is simple to argue that $EN_A(B_0, B_1) = \lambda_A/\lambda_B$ and that $E(B_1 - B_0) = 1/\lambda_B$. By the Cramér–Wold device we now have that

$$\sqrt{n_B}\left(\frac{\sum_{i=1}^{n_B} U_i}{n_B} - U_{AB}, \frac{n_A}{n_B} - \frac{\lambda_A}{\lambda_B}, \frac{T}{n_B} - \frac{1}{\lambda_B}\right)' \tag{48}$$

is asymptotically normal with mean 0 and covariance matrix, say Ψ. To identify Ψ and describe consistent estimates of it, it is convenient to introduce additional notation. Let

$$V_i^{(1)} = U_i, \qquad V_i^{(2)} = N_A(B_{i+1}, B_i), \qquad V_i^{(3)} = B_{i+1} - B_i \quad \text{for } i = \dots, -2, -1, 0, 1, 2, \dots . \tag{49}$$

Let

$$\psi_h^{(pq)} = \operatorname{Cov}\left(V_0^{(p)}, V_h^{(q)}\right) \quad \text{for } p, q = 1, 2, 3,\ h = \dots, -2, -1, 0, 1, 2, \dots . \tag{50}$$

Note that $\psi_h^{(11)} = \nu_h$. It is clear that the asymptotic covariance matrix of (48) is equal to the matrix whose pqth entry is $\Psi^{(pq)} = \sum_{h=-\infty}^{\infty} \psi_h^{(pq)}$ (note that $\psi_h^{(pq)}$ is not necessarily equal to $\psi_{-h}^{(qp)}$, unless $p = q$). Next, for $p, q = 1, 2, 3$, define

$$\hat{\psi}_h^{(pq)} = \begin{cases} \dfrac{1}{n_B - h} \displaystyle\sum_{i=0}^{n_B - h - 1} \left(V_i^{(p)} - \bar{V}^{(p)}\right)\left(V_{i+h}^{(q)} - \bar{V}^{(q)}\right) \\ \text{for } h = 0, 1, 2, \dots, M, \\ \dfrac{1}{n_B - h} \displaystyle\sum_{i=h}^{n_B - 1} \left(V_i^{(p)} - \bar{V}^{(p)}\right)\left(V_{i+h}^{(q)} - \bar{V}^{(q)}\right) \\ \text{for } h = -1, -2, -3, \dots, -M, \end{cases} \tag{51}$$

where

$$\bar{V}^{(p)} = \frac{1}{n_B} \sum_{i=0}^{n_B - 1} V_i^{(p)}. \tag{52}$$

Let

$$\hat{\Psi}^{(pq)} = \sum_{h=-M}^{M} c_{|h|} \hat{\psi}_h^{(pq)} \quad \text{for } p, q = 1, 2, 3, \tag{53}$$

where M and $\{c_h\}$ satisfy (27) and (28), respectively. Also, let

$$\hat{\Psi} = \text{matrix whose } pq\text{th entry is } \hat{\Psi}_h^{(pq)}. \tag{54}$$

The same argument that was used in the proof of (i) now applies and we see that $\hat{\Psi}$ converges to Ψ componentwise in probability as $n_B \to \infty$.

Consider now the function $g(x, y, z) = (x, y/z, 1/z)$, which maps $(\sum_{i=1}^{n_B} U_i/n_B, n_A/n_B, T/n_B)$ into $(\sum_{i=1}^{n_B} U_i/n_B, \hat{\lambda}_A, \hat{\lambda}_B)$. The derivative of g evaluated at (x, y, z) is

$$Dg(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{z} & \frac{-y}{z^2} \\ 0 & 0 & \frac{-1}{z^2} \end{pmatrix}. \tag{55}$$

An application of the delta-method (with the function g) to the vector (48) gives the asymptotic normality result asserted in part (iii) of the theorem, with

$$\sum(t_1, t_2) = \sum = Dg(U_{AB}, \lambda_A/\lambda_B, 1/\lambda_B)\Psi[Dg(U_{AB}, \lambda_A/\lambda_B, 1/\lambda_B)]'.$$

Defining $\hat{\Sigma} = \hat{\Sigma}(t_1, t_2)$ by

$$\hat{\sum} = Dg(\hat{U}_{AB}, \hat{\lambda}_A/\hat{\lambda}_B, 1/\hat{\lambda}_B)\hat{\Psi}[Dg(\hat{U}_{AB}, \hat{\lambda}_A/\hat{\lambda}_B, 1/\hat{\lambda}_B)]', \tag{56}$$

it is clear that under (27) and (28) $\hat{\Sigma}$ converges to Σ componentwise in probability as $n_B \to \infty$. □

PROOF OF (ii). As was mentioned above, the proof of (ii) is very similar to that of (iii). We now consider the variance estimates. Let $\Sigma^{(pq)}$ and $\hat{\Sigma}^{(pq)}$ denote the pqth entries of Σ and $\hat{\Sigma}$, respectively, and let

$$\hat{\Lambda} = \begin{pmatrix} \hat{\Sigma}^{(22)} & \hat{\Sigma}^{(23)} \\ \hat{\Sigma}^{(32)} & \hat{\Sigma}^{(33)} \end{pmatrix}, \tag{57}$$

with a similar definition for Λ. Obviously, this is the same Λ that appears in the statement of part (ii) of Theorem 1. It is clear that under (27) and (28), as $n_B \to \infty$, $\hat{\Lambda} \to \Lambda$ componentwise in probability. This completes the proof of Theorem 1. □

THEOREM 2. *Let* $K(t_1, t_2)$ *and* $\hat{K}(t_1, t_2)$ *be defined by* (3) *and* (5), *respectively, and assume* A1–A5. *Then, as* $n_B \to \infty$,

$$\sqrt{n_B}\left(\hat{K}(t_1, t_2) - K(t_1, t_2)\right) \to_d \mathcal{N}\left(0, \sigma^2(t_1, t_2)\right).$$

Furthermore, any estimate $\hat{\sigma}^2(t_1, t_2)$ *of the form* (58) [*refer to* (49) *and* (51)–(54)], *satisfying* (27) *and* (28), *is a consistent estimate of* $\sigma^2(t_1, t_2)$.

PROOF. We apply the delta-method to the vector (48) with the function $f(x, y, z) = xz/y$. The derivative of f at the point (x, y, z) is $Df(x, y, z) = (z/y, -xz/y^2, x/y)$. Evaluated at $(U_{AB}, \lambda_A/\lambda_B, 1/\lambda_B)$ and $(\hat{U}_{AB}, \hat{\lambda}_A/\hat{\lambda}_B, 1/\hat{\lambda}_B)$, this is $(1/\lambda_A, K(\lambda_B/\lambda_A), K\lambda_B)$ and $(1/\hat{\lambda}_A, \hat{K}(\hat{\lambda}_B/\hat{\lambda}_A), \hat{K}\hat{\lambda}_B)$, respectively. The asymptotic normality asserted in the theorem follows from the asymptotic normality of the vector (48), with

$$\sigma^2(t_1, t_2) = \left(\frac{1}{\lambda_A}, K\frac{\lambda_B}{\lambda_A}, K\lambda_B\right)\Psi\left(\frac{1}{\lambda_A}, K\frac{\lambda_B}{\lambda_A}, K\lambda_B\right)'$$

and it is clear that under (27) and (28), if $\hat{\sigma}^2(t_1, t_2)$ is defined by

$$(58) \qquad \hat{\sigma}^2(t_1, t_2) = \left(\frac{1}{\hat{\lambda}_A}, \hat{K}\frac{\hat{\lambda}_B}{\hat{\lambda}_A}, \hat{K}\hat{\lambda}_B\right)\hat{\Psi}\left(\frac{1}{\hat{\lambda}_A}, \hat{K}\frac{\hat{\lambda}_B}{\hat{\lambda}_A}, \hat{K}\hat{\lambda}_B\right)',$$

then as $n_B \to \infty$, $\hat{\sigma}^2(t_1, t_2)$ converges to $\sigma^2(t_1, t_2)$ in probability. This completes the proof of Theorem 2. □

Results giving the asymptotic normality of estimates of λ_A and λ_B (under varying sets of assumptions) already exist in the literature; see, e.g., Daley and Vere-Jones (1972), Theorem 8.6. It was necessary to establish joint asymptotic normality of $\hat{\lambda}_A$ and $\hat{U}_{AB}(t_1, t_2)$ in order to obtain asymptotic normality of $\hat{K}(t_1, t_2)$.

We now discuss the choice of the constants M and $c_1, c_2, \dots, c_M$, which enter into the estimates $\hat{\gamma}^2$ and $\hat{\Psi}$ given by (25) and (53), respectively. For the sake of simplicity, our discussion is in terms of $\hat{\gamma}^2$ only. It is appropriate to discuss the choice of these constants within the framework of spectral density estimation. Defining

$$f(\omega) = \frac{1}{2\pi}\nu_0 + \frac{1}{\pi}\sum_{h=1}^{\infty}\nu_h \cos \omega h,$$

we see that $\gamma^2 = 2\pi f(0)$. To estimate $f(\omega)$ we must effectively estimate ν_h for each h. For fixed n_B, ν_h may be estimated for $h = 0, 1, \dots, n_B - 1$. However, because of the fact that for fixed n_B the variance of $\hat{\nu}_h$ increases with h, it is standard to consider estimates of the form

$$\hat{f}(\omega) = \frac{1}{2\pi}\hat{\nu}_0 + \frac{1}{\pi}\sum_{h=1}^{M} c_h\hat{\nu}_h \cos \omega h,$$

where M is much smaller than n_B, and c_h decreases as h increases. For a given value of M the constants c_h are usually given by $c_h = w(h/m)$ for some function w (called the "lag window") defined on $[0, 1]$, satisfying $w(0) = 1$, $w(1) = 0$ and w decreases smoothly. Two commonly used choices are the Blackman–Tukey and the Parzen windows; see Anderson (1971), pages 514–516 for a definition of these. Also, see Anderson (1971), Chapter 9 for a general discussion of estimation of the spectral density. It is clear that as M increases, the bias of $\hat{f}(\omega)$ decreases while its variance increases. For both the Blackman–Tukey and the Parzen windows as well as for most of the commonly used windows, a value of M of the order $n_B^{1/5}$ is usually used, since (under certain conditions on the stationary series) this minimizes the asymptotic mean squared error. See Anderson (1971), Section 9.3.4. Thus, condition (27) is not at all restrictive.

It should be noted that the part of the proof of Theorem 1 that gives the consistency of $\hat{f}(0)$ applies equally well to the estimates $\hat{f}(\omega)$ for any ω, and similarly for the spectral density estimates of the series $\{N_A(B_{i+1}, B_i)\}$ and $\{B_{i+1} - B_i\}$.

It was necessary to give a proof of the consistency of $\hat{f}(0)$ because the currently available consistency results for spectral density estimates are valid under conditions on $\{U_i\}$ that are not implied by Assumptions 1–5 [e.g., existence of all moments in Brillinger (1975), $\{U_i\}$ is a linear process as in Anderson (1971) and in Hannan (1970)].

3. Discussion. The methods described in this paper enable the construction of asymptotic confidence intervals for $K(t_1, t_2)$, for fixed values of t_1 and t_2. The function $K(\cdot, \cdot)$ will usually be of interest over a continuum of values, say $-L \le t_1 < t_2 \le L$, where L is some number much smaller than T. One can plot $\hat{K}(-L, t)$ for $-L \le t \le L$ or, what is sometimes more useful, plot $\hat{K}(t - d/2, t + d/2)$ for $-L + d/2 \le t \le L - d/2$. Here, d is some small number representing the experimenter's guess at the duration or likely duration of the effect of a B point on the A process. The function $K(t - d/2, t + d/2)$ is identically equal to d if N_A and N_B are independent.

We may form the bands

$$\hat{K}(-L, t) \pm z^{(\alpha/2)}\hat{\sigma}(-L, t)/\sqrt{n_B}, \qquad -L \le t \le L,$$

and

$$\hat{K}\left(t - \frac{d}{2}, t + \frac{d}{2}\right) \pm z^{(\alpha/2)}\hat{\sigma}\left(t - \frac{d}{2}, t + \frac{d}{2}\right)/\sqrt{n_B}, \qquad -L + \frac{d}{2} \le t \le L - \frac{d}{2},$$

where $\hat{\sigma}(t_1, t_2)$ is an estimate of $\sigma(t_1, t_2)$ and $z^{(\alpha/2)}$ is the upper $\alpha/2 \cdot 100$ percentile point of a standard normal variable. These bands of course are not simultaneous confidence bands. To form simultaneous confidence bands one would need to carry out two distinct steps:

1. Establish weak convergence of the processes

$$V_{n_B}(t) = \sqrt{n_B}\left(\hat{K}(-L, t) - K(-L, t)\right)$$

and

$$W_{n_B}(t) = \sqrt{n_B}\left(\hat{K}\left(t - \frac{d}{2}, t + \frac{d}{2}\right) - K\left(t - \frac{d}{2}, t + \frac{d}{2}\right)\right)$$

to Gaussian processes $V(t)$ and $W(t)$, respectively.

2. Obtain $v^{(\alpha)}$ and $w^{(\alpha)}$, the upper $\alpha \cdot 100$ percentile points of $\sup_{-L \le t \le L}|V(t)|$ and $\sup_{-L+d/2 \le t \le L-d/2}|W(t)|$, respectively.

The bands

$$\hat{K}(-L, t) \pm v^{(\alpha)}/\sqrt{n_B}, \qquad -L \le t \le L,$$

and

$$\hat{K}\left(t - \frac{d}{2}, t + \frac{d}{2}\right) \pm w^{(\alpha)}/\sqrt{n_B}, \qquad -L + \frac{d}{2} \le t \le L - \frac{d}{2}$$

are then asymptotic simultaneous confidence bands.

A proof of weak convergence appears extremely difficult. Although desirable from a theoretical point of view, weak convergence is not useful statistically unless the distribution of the supremum of the absolute value of the limiting process can be obtained. Unfortunately, this is in general a very difficult problem even if the Gaussian process is stationary [see Cressie and Davis (1981)].

Acknowledgments. I am very grateful to all the reviewers for their constructive criticism, and in particular to a referee for pointing out useful references.

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