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THE BRACKETING CONDITION FOR LIMIT THEOREMS ON STATIONARY LINEAR PROCESSES

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This paper aims at improvement in regularity conditions on spectral densities for the limit theorems of the quasi maximum-likelihood estimator and the quasi likelihood-ratio statistic. The approach parallels the Daniels–Huber–Pollard proofs of central limit theorems under nonstandard conditions for i.i.d. situations. The results of the paper enable one to dispense with excessive regularity conditions on the spectral density.

0. Introduction. In order to establish the asymptotic properties of the quasi maximum-likelihood (QML) estimator and the quasi likelihood-ratio statistics for general stationary time series models, this paper presents a set of regularity conditions which is very much in parallel to the ones under which Daniels (1961), Huber (1967) and Pollard (1985) established the asymptotic normality of the maximum-likelihood estimator for i.i.d. cases. The QML estimator was first introduced and studied by Whittle (1952) and since then generalizations have been tried in various directions in such studies as Walker (1964), Hannan (1973), Hosoya (1974), Dunsmuir and Hannan (1976) and Hosoya and Taniguchi (1982). But those generalizations are mainly about the generating process of observations, and as far as the regularity conditions of the spectral density are concerned, they are more or less based on common assumptions. The aim of that section is an improvement in the latter aspect.

In this respect, Pollard (1985) gave a general setup of a stochastic differentiability condition and especially presented a weaker condition for bracketing the remainder term in the stochastic differentiability condition than Huber did. Lemma 2.4 in Section 2 which is crucial for the following central limit theorems is proved by Pollard's bracketing-function method.

It may be pertinent here to illustrate the usefulness of the results of this paper. Suppose that $I(\omega)$ is the periodogram based on a finite realization of a stationary process which has a spectral density $f(\omega - \theta)$, where θ indicates the location of the peak. The criterion function to be minimized in order to obtain a quasi maximum-likelihood estimate of θ is expressed as

$$D(f_\theta, I) = \int_{-\pi}^{\pi} \left[\log f(\omega - \theta) + I(\omega) f(\omega - \theta)^{-1}\right] d\omega. \tag{0.1}$$

Suppose now that f has a sharp peak such that $f(\omega - \theta) = \exp(-|\omega - \theta|)$; then it is not differentiable at $\theta = \omega$ and the usual regularity condition does not hold.

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On the other hand, the criterion

$$D(f_\theta, f_0) = \int_{-\pi}^{\pi} \left[\log f(\omega - \theta) + f(\omega) f(\omega - \theta)^{-1}\right] d\omega \tag{0.2}$$

is smooth with respect to θ and it holds that, if $|\theta|$ is small,

$$D(f_\theta, f_0) = (2\pi - \pi^2) + \pi\theta^2 + o(\theta^2)$$

and it behaves like a quadratic function in a neighborhood of the origin. This latter property of $D(f_\theta, f_0)$ is the one which is needed in the following limit theorems.

There is another example of application which seems to be practically more interesting. This time, suppose $f(\omega - \theta)$ is sufficiently smooth everywhere and has a peak at θ. It is noted that the minimization of the criterion (0.1) is very much affected by the performance of $I(\omega)$ in a region where f is relatively small; but this is not necessarily desirable if the estimation of the location of the peak is of interest and not the overall fit. For such a case the modified density,

$$f^*(\omega - \theta) = \begin{cases} f(\omega - \theta), & \text{if } f \geq \varepsilon, \\ \varepsilon, & \text{if } f < \varepsilon, \end{cases}$$

might be fitted instead of f [Hosoya (1978) proposed such a modification for the purpose of minimax prediction]. Then f^* is continuous but not differentiable with respect to θ, whereas

$$\int_{-\pi}^{\pi} \left[\log f^*(\omega - \theta) + f(\omega - \theta) f^*(\omega - \theta)^{-1}\right] d\omega$$

can be smooth in general.

Section 1 gives limit theorems of the QML estimator and the quasi likelihood-ratio statistics for fourth-order stationary processes under nonstandard conditions in Huber's sense (Theorems 1.2–1.4). The proofs of the theorems and the related lemmas are all given in Section 2. As for the notations and symbols used in the paper, J denotes the set of integers; degree(s) of freedom is abbreviated as d.f.; $\partial f(x_0)/\partial x$ denotes the derivative of f evaluated at $x = x_0$; R^p is Euclidean p-space. $\delta(x, y) = 1$ if $x = y$ and 0 otherwise. I_k is the identity matrix of order k; A' and A^* denote the transpose and the conjugate transpose of a matrix A, respectively.

1. Limit theorems. Suppose that a vector-valued process $\{z(t): t \in J\}$ is generated as

$$\mathbf{z}(t) = \sum_{j=0}^{\infty} G(j)\mathbf{e}(t-j), \tag{1.1}$$

where the $\mathbf{z}(t)$ and $\mathbf{e}(t)$ have s real components, the matrices $G(j)$ are $s \times s$ such that $\sum_{t=0}^{\infty} |G_{jk}(t)| < \infty$ for all $1 \leq j,\ k \leq s$; the process $\{\mathbf{e}(t), t \in J\}$ is a fourth-order stationary process such that $E(\mathbf{e}(t)) = 0$, $E(\mathbf{e}(t)\mathbf{e}(s)') = \delta(t, s)K$ for a positive-definite matrix K and the joint fourth cumulant of $e_a(t_1), e_b(t_2), e_c(t_3), e_d(t_4)$ is equal to κ_{abcd} if $t_1 = t_2 = t_3 = t_4$ and is equal to 0, otherwise, where e_a, e_b denotes components of **e**. Then the process $\{\mathbf{z}(t)\}$ has a spectral density matrix

$$f(\omega) = (1/2\pi)k(\omega)Kk(\omega)^*, \qquad -\pi \le \omega \le \pi, \tag{1.2}$$

where $k(\omega) = \sum_{j=0}^{\infty} G(j)e^{i\omega j}$.

The following is assumed throughout:

(A) Denote by $\mathscr{B}(t)$ the σ-field generated by $\{\mathbf{e}(l);\ l \le t\}$. Then,

(i) for each β_1, β_2 and m, there exists a positive ε such that

$$\operatorname{Var}\left[E\{e_{\beta_1}(n)e_{\beta_2}(n+m) \mid \mathscr{B}(n-\tau)\} - \delta(m,0)K_{\beta_1\beta_2}\right] = O(\tau^{-2-\varepsilon}),$$

uniformly in n;

(ii)

$$E|E\{e_{\beta_1}(n_1)e_{\beta_2}(n_2)e_{\beta_3}(n_3)e_{\beta_4}(n_4)|\mathscr{B}(n_1-t)\} - E\{e_{\beta_1}(n_1)e_{\beta_2}(n_2)e_{\beta_3}(n_3)e_{\beta_4}(n_4)\}| = O(\tau^{-1-\eta}),$$

uniformly in n_1, where $n_1 \le n_2 \le n_3 \le n_4$ and $\eta > 0$.

For a partial realization $\mathbf{z}(1), \dots, \mathbf{z}(n)$, denote by $\operatorname{Cov}(m)$ and $I(\omega)$, respectively, the serial covariance and the periodogram matrix; namely,

$$\operatorname{Cov}(m) = \frac{1}{n}\sum_{t=1}^{n-m} \mathbf{z}(t)\mathbf{z}(t+m)', \qquad 0 \le m \le n-1,$$

$$\text{and } \operatorname{Cov}(m) = \operatorname{Cov}'(-m) \quad \text{for } -n+1 \le m < 0 \text{ and}$$

$$I_n(\omega) = \frac{1}{2\pi n}\left(\sum_{t=1}^{n} \mathbf{z}(t)e^{it\omega}\right)\left(\sum_{t=1}^{n} \mathbf{z}(t)e^{it\omega}\right)^*.$$

Suppose statistical inference is conducted on a parametric model of $\{\mathbf{z}(t);\ t \in J\}$ which is structured as

$$\mathbf{z}(t) = \sum_{j=0}^{\infty} G(j;\theta)\boldsymbol{\varepsilon}(t-j), \tag{1.3}$$

where the matrices G and the vectors $\boldsymbol{\varepsilon}$ are of the same size as in (1.1), $E(\boldsymbol{\varepsilon}(t)) = 0$, $E(\boldsymbol{\varepsilon}(t)\boldsymbol{\varepsilon}(s)') = \delta(t,s)K(\mu)$; $\theta \in \Theta$ and $\mu \in M$, where Θ and M are open subsets of R^p and R^q, respectively. Set $\psi = (\theta, \mu)$ and $\Psi = \Theta \times M$ and denote by ∞ the point at infinity of the one-point compactification of Ψ in case Ψ is not compact. The notation $G(j, \psi)$ and $K(\psi)$ is also used. The coefficient matrices $G(j, \psi)$ are assumed to satisfy $G(0, \psi) = I_s$ and $\sum_{j=0}^{\infty} \operatorname{tr} G(j, \psi)K(\psi)G(j, \psi)' < \infty$ and thus the model process has a spectral density

$$\begin{aligned} f(\omega; \psi) &= f(\omega; \theta, \mu) \\ &= (1/2\pi)k(\omega;\theta)K(\mu)k(\omega;\theta)^*, \qquad -\pi \le \omega \le \pi, \end{aligned} \tag{1.4}$$

where $k(\omega;\theta) = \sum_{j=0}^{\infty} G(j,\theta)e^{i\omega j}$. Assume throughout that $f(\omega;\psi)$ has an inverse for $|\omega| \le \pi$ and $\psi \in \Psi$.

Assume also that $\log\det K(\mu)$ is differentiable and at each point of ψ $f(\omega;\psi)^{-1}$ is differentiable a.e. ω with respect to ψ. The derivatives are denoted, respectively, as $H_j(\psi) = \partial \log\det K(\mu)/\partial\psi_j$ and $h_j(\omega;\psi) = \partial f^{-1}(\omega;\psi)/\partial\psi_j$ and h_j is assumed to be measurable a.e. ω. The notations $H(\psi)$ and $\mathrm{tr}\{h(\omega;\psi)f(\omega)\}$ represent, respectively, the $(p+q)$-vectors whose jth elements are $H_j(\psi)$ and $\mathrm{tr}\{h_j(\omega;\psi)f(\omega)\}$. The $h_j(\omega;\psi)$ are assumed separable throughout.

Let $S_{nj}(\psi)$ be defined as

$$(1.5)\qquad S_{nj}(\psi) = H_j(\psi) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\{h_j(\omega;\psi)I_n(\omega)\}\,d\omega,\qquad j=1,\dots,p+q,$$

and let $S_n(\psi)$ be the vector $\{S_{nj}(\psi)\}$.

In order to prove Theorem 1.1, assume for each j, $j = 1,\dots,p+q$:

(B.1) At each $\psi \in \Psi$, there are square-integrable Hermitian matrix-valued functions $\bar{h}_j(\omega)$, $\tilde{h}_j(\omega)$ and $r > 0$ such that for $|\psi_1 - \psi| < r$, $\tilde{h}_j(\omega) \le h_j(\omega;\psi_1) \le \bar{h}_j(\omega)$ and

$$(1.6)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\{\bar{h}_j(\omega) - \tilde{h}_j(\omega)\}\,d\omega < \varepsilon,$$

given $\varepsilon > 0$, where the inequality $A \le B$ implies that $B - A$ is nonnegative definite.

(B.2) $\lambda_j(\psi) = H_j(\psi) + (2\pi)^{-1}\int_{-\pi}^{\pi}\mathrm{tr}\{h_j(\omega;\psi)f(\omega)\}\,d\omega$ has a unique zero at $\psi = \psi_0 \in \Psi$.

(B.3) There is a compact neighborhood C of ψ_0 such that for $\psi \in \Psi\setminus C$,

$$(1.7)\qquad c_{1j}(\psi)\tilde{g}_j(\omega) \le h_j(\omega;\theta) \le c_{2j}(\omega)\bar{g}_j(\omega),$$

where c_{1j} and c_{2j} are real-valued and $\tilde{g}_j$ and $\bar{g}_j$ are Hermitian square-integrable matrices.

(B.4) There exists $b(\theta) \ge b_0 > 0$ such that

(i) $\liminf_{\psi\to\infty} |\lambda(\psi)|/b(\psi) \ge 1$,

(ii) $\limsup_{\psi\to\infty}(2\pi)^{-1}\int_{-\pi}^{\pi}\sum_j \mathrm{tr}[\{c_{2j}(\psi)\bar{g}_j(\omega) - c_{1j}(\psi)\tilde{g}_j(\omega)\}f(\omega)]\,d\omega/b(\psi)^2 < 1$,

(iii) $\limsup_{\psi\to\infty}|c_{1j}(\psi)|/b(\theta) < \infty$ and $\limsup_{\psi\to\infty}|c_{2j}(\psi)|/b(\theta) < \infty$.

(B.5) The spectral density f satisfies for some $\alpha > 0$,

$$(1.8)\qquad \sup_{|\lambda|\to\varepsilon}\int_{-\pi}^{\pi}\mathrm{tr}[\{f(\omega) - f(\omega-\lambda)\}\{f(\omega) - f(\omega-\lambda)\}^*]\,d\omega = O(\varepsilon^\alpha)$$

as $\varepsilon \to 0$, where f is extended so $f(\omega) = f(\omega + 2\pi)$.

THEOREM 1.1. *Suppose $\tilde{\psi}_n$ is a sequence of measurable functions of* $(\mathbf{z}(1),\dots,\mathbf{z}(n))$ *taking values in Ψ such that $S_n(\tilde{\psi}_n) \to 0$ in probability as $n \to \infty$. Then under Assumptions* (B.1)–(B.5), *$\tilde{\psi}_n$ tends to ψ_0 in probability.*

A value $\hat{\psi}_n$ such that $S_n(\hat{\psi}_n) = 0$ is said to be a quasi maximum-likelihood (QML) estimate of ψ and the above theorem asserts the consistency of it. The central limit theorems and the related results are established below under the following assumptions [(C.4) is for Lemma 2.4]:

(C.1) For some $\alpha > 1$, (1.8) holds.

$$\text{(C.2)} \quad \lim_{r \to 0} \int_{-\pi}^{\pi} \sup_{|\psi| \leq r} \operatorname{tr}\left[\{h_j(\omega; \psi) - h_j(\omega; \psi_0)\} \times \{h_j(\omega; \psi) - h(\omega; \psi_0)\}^*\right] d\omega = 0, \quad j = 1, \dots, p+q.$$

(C.3) Given $\varepsilon > 0$, there exist an integer $m(\varepsilon)$ and a partition $U^1(r), \dots, U^{m(\varepsilon)}(r)$ of the ball in Ψ with center ψ_0 and radius r and square-integrable Hermitian matrix-valued functions $\bar{h}_j^i(\omega)$, $\tilde{h}_j^i(\omega)$ such that, for all sufficiently small r and for all j, $\tilde{h}_j^i(\omega) \leqq h(\omega; \psi) \leqq \bar{h}_j^i(\omega)$ if $\psi \in U^i(r)$ and

$$\text{(1.9)} \quad \int_{-\pi}^{\pi} \operatorname{tr}\{\bar{h}_j^i(\omega) - \tilde{h}_j^i(\omega)\} \leqq \varepsilon r.$$

(C.4) $|\lambda(\psi)| \geqq a_1|\psi - \psi_0|$ for some $a_1 > 0$ in a neighborhood of ψ_0.

THEOREM 1.2. *Suppose both* $\sqrt{n}\, S_n(\tilde{\psi}_n) \to 0$ *and* $\tilde{\psi}_n - \psi_0 \to 0$ *in probability as* $n \to \infty$, *and* (C.1) *and* (C.3) *are true. If* λ *is differentiable at* $\psi = \psi_0$ *and the matrix of the derivatives* $\Lambda_{ij} = \partial\lambda_i/\partial\psi_j$ *is denoted as* Λ, $\sqrt{n}(\tilde{\psi} - \psi_0)$ *has the asymptotic normal distribution with mean* 0 *and covariance matrix* $\Lambda^{-1}U(\Lambda')^{-1}$, *where* U *is the matrix whose* (j, l)*th element is represented as*

$$\text{(1.10)} \quad U_{jl} = \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{tr}\left[f(\omega)h_j(\omega; \psi_0)f(\omega)h_l(\omega; \omega_0)\right] d\omega + \frac{1}{(2\pi)^2} \sum_{a, b, c, d, = 1}^{p+q} \kappa_{abcd} \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} k^*(\omega)h_j(\omega; \psi_0)k(\omega)\, d\omega\right]_{ab} \times \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} k^*(\omega)h_l(\omega; \psi_0)k(\omega)\, d\omega\right]_{cd},$$

where $[\ ]_{ab}$ *denotes the* (a, b)*th element.*

Assume henceforth that the process $\{\mathbf{z}(t)\}$ is the process as defined in the first paragraph of this section but also that its spectral density given in (1.2) satisfies $f(\omega) = f(\omega; \psi_0)$, $\psi_0 \in \Psi$. Moreover assume the following:

(D.1) The $h_j(\omega; \psi)$ are jointly measurable with respect to $(\omega; \psi)$, where Ψ is endowed with the Borel subsets.

(D.2) There exists a neighborhood N of ψ_0 such that

(i) $\int_{-\pi}^{\pi} \operatorname{tr}[(\partial/\partial\psi_j)f(\omega; \psi)(\partial/\partial\psi_k)f(\omega; \psi)^*]\, d\omega$ is bounded in N for $j, k = 1, \dots, p+q$,

(ii) $\int_{-\pi}^{\pi} \operatorname{tr}[f(w; \psi)f(\omega; \psi)^*]\, d\omega < \infty$ for $\psi \in N$ and

(iii) $\int_{-\pi}^{\pi} \operatorname{tr}[f(\omega; \psi)^* h_j(\omega; \psi)h_k(\omega; \psi)^*]\, d\omega < \infty$ for $\psi \in N$.

(D.3) $V_{jk}(\psi) \overset{\text{def}}{=} (1/2)\pi\int_{-\pi}^{\pi} \operatorname{tr}[f(\omega;\psi)h_j(\omega;\psi)f(\omega;\psi)h_k(\omega;\psi)]\,d\omega$ is continuous at $\psi = \psi_0$, and the matrix $V(\psi_0) = \{V_{jk}(\psi_0),\ 1 \le j,\ k \le p+q\}$ is invertible.

(D.4) $H(\psi)$ is continuous in a neighborhood of ψ_0.

THEOREM 1.3. *Suppose that* (C.1)–(C.3) *and* (D.1)–(D.4) *are true and suppose also that* $\sqrt{n}\,S_n(\tilde{\psi}_n) \to 0$ *and* $\tilde{\psi}_n - \psi_0 \to 0$ *in probability as* $n \to \infty$,. *Then* $\sqrt{n}(\tilde{\theta}_n - \theta_0)$ *and* $\sqrt{n}(\tilde{\mu}_n - \mu_0)$ *are asymptotically independently normally distributed with mean* 0 *and the covariance matrices which are given, respectively, by* $V_{(1)}^{-1}$ *and* $V_{(2)}^{-1}U_{(2)}V_{(2)}^{-1}$, *where* $\tilde{\psi}_n = (\tilde{\theta}_n, \tilde{\mu}_n)$ *and* $V_{(1)}, V_{(2)}, U_{(2)}$ *are submatrices of* V *and* U *such that* $V_{(1)} = \{V_{ij};\ 1 \le i, j \le p\}$, $V_{(2)} = \{V_{ij};\ p+1 \le i, j \le p+q\}$ *and* $U_{(2)} = \{U_{jk};\ p+1 \le j, k \le p+q\}$.

The following theorem is important in order to derive the asymptotic distribution of the likelihood ratio statistics. Set

$$(1.11)\quad \bar{L}_n(\psi) = -n\left[\log\det K(\psi) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr}\{f^{-1}(\omega;\psi)I_n(\omega)\}\,d\omega\right].$$

THEOREM 1.4. *Under the assumptions of Theorem* 1.3,

$$(1.12)\quad \begin{aligned} &\bar{L}_n(\tilde{\psi}_n) - \bar{L}_n(\psi_0)\\ &\quad= \tfrac{1}{2}\sum_i\sum_j V_{ij}(\psi_0)\sqrt{n}(\tilde{\psi}_n - \psi_0)_i\sqrt{n}(\tilde{\psi}_n - \psi_0)_j + o_p(1)\\ &\quad= \tfrac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} V_{ij}(\psi_0)\sqrt{n}(\tilde{\theta}_n - \theta_0)_i\sqrt{n}(\tilde{\theta}_n - \theta_0)_j\\ &\qquad+ \tfrac{1}{2}\sum_{k=1}^{q}\sum_{l=1}^{q} V_{q+k,q+l}(\psi_0)\sqrt{n}(\tilde{\mu}_n - \mu_0)_k\sqrt{n}(\tilde{\mu}_n - \mu_0)_l + o_p(1). \end{aligned}$$

Suppose a nested model is given as follows. Let $q_0, q_1, \ldots, q_p$ be positive integers such that $q_0 < q_1 < \cdots < q_p$. Let Θ_p be a nonempty open region of R^{q_p} and let $\Theta_l = \{\alpha \in \Theta_p;\ \alpha_{q_l+1} = 0, \ldots, \alpha_{q_p} = 0\}$, where α_{q_i} denotes the q_ith coordinate of the q_p-vector α. Let $\Theta_l^* = \{\alpha \in \Theta_l;\ \alpha \notin \Theta_{l-1}\}$ for $l = 1, \ldots, p$ and let $\Theta_0^* = \Theta_0$. Denote by $\alpha(l)$ a q_l-vector, $l = 0, \ldots, p$, and sometimes $\alpha \in \Theta_p$ such that $\alpha_i = \alpha_i(l)$, $i = 1, \ldots, q_l$, and $\alpha_i = 0$ for $i > q_l$ is identified with $\alpha(l)$. Then consider again the vector-valued stationary process $\{\mathbf{z}(t): t \in J\}$ given in the first paragraph; suppose that the basic parametric model is structured as (1.3) but this time Θ has the hierarchical structure $\Theta_0 \subset \Theta_1 \subset \cdots \subset \Theta_p$, where the Θ_j's are defined above. Let M be an open set in R^r and define the alternative hypothesis H_j^* as $H_j^* = \{(\theta, \mu): \theta \in \theta_j^* \text{ and } \mu \in M\}$ and the null hypothesis as $H_0^* = \{(\theta, \mu): \theta \in \Theta_0 \text{ and } \mu \in M\}$.

Set $\psi = (\theta, \mu)$ as before. Assume that for each l, $l = 0, 1, \ldots, p$, there exists a sequence of statistics $\hat{\psi}_n(l) = (\hat{\theta}_n(l), \hat{\mu}_n(l))$ in $\Theta_l^* \times M$ such that

(E.1) $\sqrt{n}\,S_n(\hat{\psi}_n(l)) \to 0$ in probability as $n \to \infty$ and

(E.2) $\hat{\psi}_n(l) \to \psi_0 = (\theta_0, \mu_0) \in \Theta_0 \times M$ in probability as $n \to \infty$.

Define the quasi log-likelihood ratio $\bar{L}_{n,ij}$ as $\bar{L}_{n,ij} = \bar{L}_n(\hat{\theta}_n(i), \hat{\mu}_n(i)) - \bar{L}_n(\hat{\theta}_n(j), \hat{\mu}_n(j))$, where $\bar{L}_n$ is defined in (1.11).

THEOREM 1.5. *Assume the conditions* (C.1)–(C.4) *and* (D.1)–(D.3) *are true and also suppose that* (E.1) *and* (E.2) *hold. Then* $\bar{L}_{n,01}, \dots, \bar{L}_{n,0p}$ *are asymptotically jointly distributed as* $-\frac{1}{2}\sum_{j=1}^{i}\chi_j^2$, $i = 1, \dots, p$, *where the* χ_j^2*'s are independent* χ^2 *random variables with* $q_j - q_{j-1}$ *d.f.*

2. Proofs of theorems. This section provides the proofs of theorems in Section 1 and also the necessary lemmas and their proofs. Let $\{\mathbf{z}(t)\}$ be the fourth-order process given in the first paragraph there. Let L^2 be the Banach space of square-integrable complex-valued function with respect to the Lebesgue measure on $(-\pi, \pi]$ and let $\|\ \|$ be the norm. Set

$$\sigma_n(\lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\omega)K_n(\lambda - \omega)\, d\omega,$$

where K_n is the Fejér kernel.

LEMMA 2.1. *Suppose* $g \in L^2$ *and that there is an* $\alpha > 0$ *such that*

$$\sup_{|\lambda|<\varepsilon}\int_{-\pi}^{\pi} |g(\omega) - g(\omega - \lambda)|^2\, d\omega = O(\varepsilon^\alpha) \tag{2.1}$$

as $\varepsilon \to 0$. *Then*

$$\|\sigma_n - g\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma_n(\omega) - g(\omega)|^2\, d\omega = O(n^{-\alpha}).$$

PROOF. For any $h \in L^2$ such that $\|h\| < 1$, the relationship

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} (\sigma_n(\omega) - g(\omega))h(\omega)\, d\omega\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\omega)\|g - g_{-\omega}\|\, d\omega$$

holds where $f_{-\omega}$ denotes the shift $g_{-\omega}(\lambda) = g(\lambda - \omega)$. On the other hand, it follows from the relationship

$$K_n(\omega) \le \min\{(n+1),\ a_1/[(n+1)\omega^2]\}$$

for a constant a_1 that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\omega)\|g - g_{-\omega}\|\, d\omega \\
&\quad \le \frac{n+1}{2\pi}\int_{-1/n}^{1/n}\|g - g_{-\omega}\|\, d\omega + \frac{a_2}{2\pi}\int_{1/n\le|\omega|\le\pi}\frac{\|g - g_{-\omega}\|}{(n+1)\omega^2}\, d\omega,
\end{aligned} \tag{2.2}$$

where $a_2 > 0$ is a constant. Now for sufficiently large n, (2.1) implies that

$$\frac{n+1}{2\pi}\int_{-1/n}^{1/n}\|g - g_{-\omega}\|\, d\omega \le \left(\frac{1}{n}\right)^{\alpha/2}\frac{n+1}{\pi n}, \tag{2.3}$$

whereas there exists $a_3 > 0$ such that

$$
\begin{aligned}
(2.4)\quad \frac{1}{n+1}\int_{1/n\le|\omega|\le\pi}\|g - g_{-\omega}\|/\omega^2\,d\omega &\le \frac{a_3}{n+1}\int_{1/n\le|\omega|\le\pi}|\omega|^{\alpha/2}/\omega^2\,d\omega \\
&= 2a_3\left\{\pi^{(\alpha/2-1)} - (1/n)^{(\alpha/2-1)}\right\} \\
&\quad \div\{(n+1)(\alpha/2-1)\}.
\end{aligned}
$$

The lemma then is a consequence of (2.2)–(2.4). □

LEMMA 2.2. *If the spectral density f satisfies* (2.1), *then*

$$E\int_{-\pi}^{\pi}|EI_{n,kl}(\omega) - f_{kl}(\omega)|^2\,d\omega = O(n^{-\alpha}).$$

LEMMA 2.3. *Under Assumptions* (B.1)–(B.5), *there exist a positive constant* ε *and a compact neighborhood* C *of* ψ_0 *such that*

$$\lim_{n\to\infty} P_r\left\{\inf_{\psi\in\Psi\setminus C}|S_n(\psi)| > \varepsilon\right\} = 1.$$

PROOF. Thanks to (B.4)(i), given $\varepsilon_1 > 0$ there is a compact set C such that $|\lambda|/b(\psi) > 1 - \varepsilon_1$ on $\psi \in \Psi\setminus C$. Then

$$
\begin{aligned}
&\sup_{\Psi\setminus C}\{S_{nj}(\psi) - \lambda_j(\psi)\}/b(\psi) \\
&\quad\leqq \left\{\sup_{\Psi\setminus C}|c_{2j}(\psi)|/b(\psi)\right\}(2\pi)^{-1}\left|\int_{-\pi}^{\pi}\operatorname{tr}\{\bar{g}_j(\omega)(I_n - f)\}\,d\omega\right| \\
&\qquad + (2\pi)^{-1}\int_{-\pi}^{\pi}\operatorname{tr}\left\{\left(c_{2j}(\psi)\bar{g}_j(\omega) - c_{1j}(\psi)\tilde{g}_j(\omega)\right)f(\omega)\right\}d\omega/b(\theta),
\end{aligned}
$$

where the first term in the right-hand side $\to 0$ in probability, since the integral is equal to

$$\int_{-\pi}^{\pi}\operatorname{tr}\{\bar{g}_j(\omega)(I_n - E(I_n))\}\,d\omega + \int_{-\pi}^{\pi}\operatorname{tr} g_j(\omega)\{E(I_n) - f(\omega)\}\,d\omega$$

and the first term $\to 0$ in probability and the second term $\to 0$ in view of the Schwarz inequality and Lemma 2.1. A similar relationship holds for $\inf\{S_{nj}(\psi) - \lambda_j\}/b(\psi)$ and so in view of Assumption (B.4)(ii) there is an $\varepsilon_2 > 0$ such that given $\eta > 0$,

$$P_r\{|S_n(\psi) - \lambda(\psi)|/b(\psi) < 1 - \varepsilon_2\} > 1 - \eta$$

for sufficiently large n. The result then follows from the relationship

$$|S_n(\psi)| \geqq \{|\lambda(\psi)|/b(\psi) - |S_n(\psi) - \lambda(\psi)|/b(\psi)\}b_0$$

by the choice of ε_1 and ε so that $\varepsilon_1 < \varepsilon_2$ and $\varepsilon < (\varepsilon_2 - \varepsilon_1)b_0$. □

PROOF OF THEOREM 1.1. Given $\varepsilon_1 > 0$, let $B(r(\psi))$ be the open ball $\{\psi: |\psi_1 - \psi| < r(\psi)\}$ and $\bar{h}_j(\omega)$ and $\tilde{h}_j(\omega)$ be the bracketing functions which satisfy (1.6) for ε_1 and $r = r(\psi)$. Since there is a constant c and the identity matrix I such that $cI - E(I_n)$ is positive definite for all n,

$$\sup_{B(r(\psi))} |S_{nj}(\psi_1) - S_{nj}(\psi)| \leqq (2\pi)^{-1} c \int_{-\pi}^{\pi} \operatorname{tr}\left(\bar{h}_j(\omega) - \tilde{h}_j(\omega)\right) d\omega$$

$$+ (2\pi)^{-1} \int_{-\pi}^{\pi} \operatorname{tr}\left\{\bar{h}_j(\omega) - \tilde{h}_j(\omega)\right\}\{I_n - E(I_n)\}\, d\omega$$

$$\leqq c\varepsilon_1 + o_p(1)$$

so that

$$\sup_{B(r(\psi))} |S_n(\psi_1) - S_n(\psi)| \leqq c(p+q)\varepsilon_1 + o_p(1).$$

Since $\psi(\lambda)$ is continuous thanks to Assumption (B.1), given an open neighborhood N of ψ_0, there is $\varepsilon_2 > 0$ such that $\inf_{C \setminus N} |\lambda(\psi)| > \varepsilon_2$. Suppose that $B_j = B(r(\psi_j))$, $j = 1, \dots, K$, be a open finite subcover of $C \setminus N$. Then

$$\inf_{C \setminus N} |S_n(\psi)| \geqq \inf_j |\lambda(\psi)| - \sup_j \sup_{B_j} |S_n(\psi) - S_n(\psi_j)| + \sup_j |S_n(\psi_j) - \lambda(\psi)_j|$$

$$\geqq \varepsilon_2 - c(p+q)\varepsilon_1 + o_p(1),$$

since $\sup_j |S_n(\psi_j) - \lambda(\psi_j)| \to 0$ in probability. Now choose ε_1 so that $\varepsilon_2 - c(p+q)\varepsilon_1 > 0$ and set $\varepsilon = \varepsilon_2 c(p+q)\varepsilon_1$. Then the proof is complete. □

The next lemma essentially parallels Pollard's Lemma 4.

LEMMA 2.4. *If* (C.1)–(C.4) *are true,*

$$\sup_{|\psi - \psi_0| \leqq d_0} |S_n(\psi) - S_n(\psi_o) - \lambda(\psi)| / \{n^{-1/2} + |\lambda(\psi)|\} \to 0$$

in probability as $n \to \infty$ for small enough d_0, where ψ_0 is the value defined in (B.2).

PROOF. Denote for simplicity $h_j(\omega; \psi) - h_j(\omega; \psi_0)$ as $h - h_0$ and set $T_n(g) = (2\pi)^{-1} \int_{-\pi}^{\pi} \operatorname{tr}\{(g - h_0)(I_n - E(I_n)\}\, d\omega$ and denote $\|g_1 - g_2\|^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} \operatorname{tr}\{(g_1 - g_2)(g_1 - g_2)^*\}\, d\omega$. Also set

$$Q(r) = (2\pi)^{-1} \int_{-\pi}^{\pi} \sup_{|\psi| \leqq r} \operatorname{tr}\{(h - h_0)(h - h_0)^*\}\, d\omega.$$

Without loss of generality assume $d_0 = 1$ and $Q(1) < \infty$. Since

$$\sup_{|\psi| \leqq 1} \left| (2\pi)^{-1} \int_{-\pi}^{\pi} \operatorname{tr}\{(h - h_0)(I_n - f)\}\, d\omega \right|$$

$$\leqq \sup_{|\psi| \leqq 1} |T_n(h)| + \sup_{|\psi| \leqq 1} \left| \int_{-\pi}^{\pi} \operatorname{tr}\{(h - h_0)(EI_n - f)\}\, d\omega \right|$$

and since the second term above is not greater than $Q(1)\|EI_n - f\|$ which is of order $O(n^{-\alpha/2})(\alpha > 1)$ in view of Lemma 2.2, it suffices to show that as $n \to \infty$,

$$\sup_{|\psi| \leq 1} |T_n(h)|/\left(n^{-1/2} + |\lambda(\psi)|\right) \to 0.$$

Choose k_0 such that $n/2 < 4^{k_0+1} < n$ and let $B(k)$ be the ball with center ψ_0 and radius 2^{-k}, $k = 0, 1, \ldots, k_0$, and let $A(k)$ be the difference $B(k) \setminus B(k-1)$. Given $\varepsilon > 0$, let $U^1, \ldots, U^m$ be a partition of $B(k)$ for which (1.9) holds for ε' which is determined below. Without loss of generality, the bracketing functions $\bar{h}^i_j$, $\tilde{h}^i_j$ can be assumed to satisfy $\|\bar{h}^i - h_0\| \leq Q(2^{-k})$ and $\|\tilde{h}^i - h_0\| \leq Q(2^{-k})$. As in the proof of Theorem 1.1, since $E(I_n)$ is bounded uniformly in n and ω, there is a constant c_1 such that $c_1 I - E(I_n)$ is positive definite where I is the identity matrix. Then since h and I_n are Hermitian and I_n is nonnegative definite,

$$\begin{aligned} T_n(h) &\leq T_n(\bar{h}^i) + (2\pi)^{-1} \int_{-\pi}^{\pi} \operatorname{tr}\{(\bar{h}^i - \tilde{h}^i) E(I_n)\}\, d\omega \\ &\leq T_n(\bar{h}^i) + c_1 \varepsilon' 2^{-k} \end{aligned} \tag{2.5}$$

for $\psi \in U^i$. Set $\varepsilon' = a_1\varepsilon/(4c_1)$ for a_1 in Assumption (C.4). It follows from (2.5) that

$$\begin{aligned} &P_r\left[\sup_{A(k)} T_n(h)/\{n^{-1/2} + |\lambda(\psi)|\} > \varepsilon\right] \\ &\quad \leq m(\varepsilon') \max_i P_r\{\sqrt{n}\, T_n(\bar{h}^i) > \varepsilon a_1 \sqrt{n}\, 2^{-(k+1)}\}. \end{aligned} \tag{2.6}$$

By an argument similar to that in Walker [(1964), pages 373–374], a constant c_2 can be fixed so that

$$\operatorname{Var}\{T_n(\bar{h}^i)\} < c_2 \|\bar{h}^i - h_0\|^2 < c_2 Q(2^{-k}).$$

Thus in view of the Chebyshev inequality, the right-hand side in (2.6) is not greater than

$$m(\varepsilon') c_2 Q(2^{-k}) / \left(\varepsilon a_1 \sqrt{n}\, 2^{-(k+1)}\right)^2 = m(\varepsilon') c_2 Q(2^{-k}) 4^{k+1}/(n\varepsilon^2).$$

A similar bracketing method is applied to bound $P_r\{\inf_{A(k)} T_n(h) > -\varepsilon\}$ and consequently

$$P_r\left[\sup_{A(k)} |T_n(h)|/\{n^{-1/2} + |\lambda(\psi)|\} > \varepsilon\right] \leq 8m(\varepsilon') c_2 Q(2^{-k}) 4^k/(n\varepsilon^2).$$

Furthermore, it is shown in a similar way that

$$P_r\left[\sup_{B(k_0)} |T_n(h)|/\{n^{-1/2} + |\lambda(\psi)|\} > \varepsilon\right] \leq c_2 Q(2^{-k_0})/\varepsilon^2.$$

Set k' so that for $k \geqq k'$, $8m(\varepsilon')c_2Q(2^{-k})/\varepsilon^2 < \varepsilon$; then

$$P_r\left[\sup_{B_0}|T_n(h)|/\{n^{-1/2}+|\lambda(\psi)|\} > \varepsilon\right]$$

$$\leqq \left(\sum_{k=0}^{k'-1} + \sum_{k'}^{k_0}\right)P_r\left[\sup_{A(k)}|T_n(h)|/\left(n^{-1/2}+|\lambda(\psi)|\right) > \varepsilon\right]$$

$$+P_r\left[\sup_{B(k_0)}|T_n(h)|/\left(n^{-1/2}+|\lambda(\psi)|\right) > \varepsilon\right]$$

$$\leqq 8m(\varepsilon')c_2Q(1)(4^{k'}-1)/(3n\varepsilon^2) + \varepsilon(4^{k_0+1}-1)/(3n) + c_2Q(2^{-k_0})/\varepsilon^2.$$

Since k' is independent of n, the first and the third terms above $\to 0$ as $n \to \infty$ and the second term is less than ε, whence the lemma follows. □

LEMMA 2.5. *$\sqrt{n}\{S_n(\psi_0) + \lambda(\tilde{\psi}_n)\} \to 0$ in probability as $n \to \infty$ if $\sqrt{n}\,S_n(\tilde{\psi}) \to 0$ in probability and $P_r\{|\tilde{\psi}_n - \psi_0| \le d_0) \to 1$ as $n \to \infty$*

PROOF. The lemma can be deduced from Lemma 2.4 in a way quite similar to that in Huber [(1967), page 230]. □

PROOF OF THEOREM 1.2. It is shown that $\sqrt{n}\,S_n(\psi_0)$ tends to a multivariate normal distribution with mean 0 and with covariance matrix U; then the theorem is a straightforward consequence of Lemma 2.5. First of all, note that $E\{\sqrt{n}\,S_n(\psi_0)\} \to 0$ as $n \to \infty$ since

$$|E\sqrt{n}\,S_n(\psi_0)|^2 \le \frac{n}{2\pi}\int_{-\pi}^{\pi}\left\{\operatorname{tr}\sum_j h_j(\omega;\psi_0)h_j(\omega;\psi_0)^*\right\}d\omega$$

$$\times \frac{1}{2\pi}\int_{-\pi}^{\pi}\operatorname{tr}\left(\tilde{f}_n(\omega)-f(\omega)\right)\left(\tilde{f}_n(\omega)-f(\omega)\right)^* d\omega$$

and Lemma 2.3 is applied to the second factor in the right-hand side, where $\tilde{f}_n(\omega) = 1/2\pi\int_{-\pi}^{\pi}K_n(\lambda)f(\omega-\lambda)\,d\lambda$. Set

$$\alpha^{(a)}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(i\omega t)h_a(\omega;\psi_0)\,d\omega, \qquad a = 1,\dots,p+q$$

and denote by $\alpha_{j,k}^{(a)}(t)$ the elements of the matrix. Given arbitrarily fixed constants λ_a, $a = 1,\dots,p+q$, consider the linear combination

$$Y_n = \sum_a \lambda_a[S_n(\psi_0) - E(S_n(\psi_0))]_a$$

$$= \sum_{s=-(n-1)}^{n-1}\operatorname{tr}\left[\left\{\sum_a \lambda_a\alpha^{(a)}(s)\right\}\sqrt{n}\,\{\operatorname{Cov}(s) - E(\operatorname{Cov}(s))\}\right]$$

and set $Y_n = Y_{n,m} + w_{n,m}$, where

$$Y_{n,m} = \sum_{|s|\le m} \operatorname{tr}\left[\left\{\sum_a \lambda_a \alpha^{(a)}(s)\right\}\sqrt{n}\left\{\operatorname{Cov}(s) - E(\operatorname{Cov}(s))\right\}\right].$$

It can be shown that

$$\limsup_{m\to\infty}\limsup_{n\to\infty} E(w_{n,m})^2 = 0, \tag{2.7}$$

because

$$\begin{aligned}
E\{w_{n,m}\}^2 &= \sum_{h,i,j,k=1}^{p+q} n \sum_{m<|s|,|t|\le n-1} \sum_{a,b=1}^{p+q} \lambda_a\lambda_b \alpha_{hi}^{(a)}(s)\alpha_{jk}^{(b)}(t) \\
&\quad \times E\left[\{\operatorname{Cov}_{hi}(s) - E(\operatorname{Cov}_{hi}(s))\}\{\operatorname{Cov}_{jk}(t) - E(\operatorname{Cov}_{jk}(t))\}\right] \\
&\le \sum_{h,i,j,k}\sum_{s,t}\sum_{a,b} |\lambda_a||\lambda_b||\alpha_{hi}^{(a)}(s)||\alpha_{jk}^{(b)}(t)| \\
&\quad \times \sum_{l=-\infty}^{\infty}\left[|\gamma_{hj}(l)||\gamma_{ik}(l+t-s)| + |\gamma_{hk}(l+t)\gamma_{ij}(l-s)|\right. \\
&\qquad\qquad \left. + |\kappa_{hijk}^{z}(0,s,l,l+t)|\right],
\end{aligned}$$

where $\gamma_{ij}(t) = E\{z_i(0)z_j(t)\}$ and $\kappa_{hijk}^{z}(0,s,l,l+t)$ is the fourth cumulant of $z_h(0)$, $z_i(s)$, $z_j(l)$, $z_k(l+t)$, whereas it holds that

$$\begin{aligned}
&\sum_{s,t}|\alpha_{hi}^{(a)}(s)||\alpha_{jk}^{(b)}(t)|\sum_l\left\{|\gamma_{hj}(l)||\gamma_{ik}(l+t-s)| + |\gamma_{hk}(l+s)||\gamma_{ij}(l-s)|\right\} \\
&\quad \le 2\left\{\sum_{l=-\infty}^{\infty}|\gamma_{hj}(l)|\right\}^2\left\{\sum_{|s|=m}^{\infty}\alpha_{hi}^{(a)}(s)|^2\sum_{j=-\infty}^{\infty}|\alpha_{jk}(t)|^2\right\}^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{s,t}|\alpha_{hi}^{(a)}(s)||\alpha_{jk}^{(b)}(t)||\kappa_{hijk}^{z}(0,s,l,l+t)| \\
&\quad \le \max_{a,b,c,d}|K_{abcd}|\left\{\max_{|s|>m}|\alpha_{hi}^{(a)}(s)|\right\}\left\{\max_{|t|\ge m}|\alpha_{jk}^{(b)}(t)|\right\} \\
&\quad \times\left\{\sum_{i,j=1}^{s}\sum_{l=0}^{\infty}\sum_l|G_{ij}(l)|\right\}^4;
\end{aligned}$$

whence (2.7) follows in view of $\sum_{s=-\infty}^{\infty}|\alpha_{ij}^{(a)}(s)|^2 < \infty$.

For fixed m, the application of Theorem 2.2 of Hosoya and Taniguchi (1982) can show that $V_{n,m}$ has the limiting normal distribution with mean 0 and variance

$$\sigma_m^2 = \sum_{h,i,j,k} \sum_{0 \le |s|, |t| \le m} \sum_{a,b} \lambda_a \lambda_b \alpha_{hi}^{(a)}(s) \alpha_{jk}^{(b)}(t) \times \sum_{l=-\infty}^{\infty} \Big\{ \gamma_{hj}(l)\gamma_{ik}(l+t-s) + \gamma_{hk}(l+t)\gamma_{ij}(l-s) + \sum_{c,d,e,f} \sum_{r=-\infty}^{\infty} K_{cdef} G_{hc}(r) G_{id}(r+s) G_{je}(r+l) G_{kf}(r+l+t) \Big\}.$$

Since the series

$$\sum_{s,t=-\infty}^{\infty} \alpha_{hi}^{(a)} \alpha_{jk}^{(b)}(t) \Big\{ \sum_{l=-\infty}^{\infty} \gamma_{hj}(l) \gamma_{ik}(l+t-s) \Big\}$$

is absolutely convergent, the repeated use of the Parseval equality leads to

$$\lim_{m\to\infty} \sum_{h,i,j,k} \sum_{1 \le |s|, |t| \le m} \alpha_{hi}^{(a)}(s) \alpha_{jk}^{(b)}(t) \sum_l \gamma_{hj}(l) \gamma_{ik}(l+t-s) = 2\pi \int_{-\pi}^{\pi} \operatorname{tr}[f(\omega) h_a(\omega; \psi_0) f(\omega) h_b(\omega; \psi_0)]\, d\omega. \tag{2.8}$$

Similarly

$$\sum_{h,i,j,k} \sum_{1 \le |s|, |t| \le m} \alpha_{hi}^{(a)}(s) \alpha_{jk}^{(b)}(t) \sum_l \gamma_{hk}(l+t) \gamma_{ij}(l-s)$$

tends to the same quantity in the right-hand side of (2.8). On the other hand, by the application of the Parseval equality again,

$$\lim_{m\to\infty} \sum_{|s|, |t| \le m} \sum_{p=-\infty}^{\infty} \alpha_{hi}^{(a)}(s) \alpha_{jk}^{(b)}(t) \times \sum_{l=-\infty}^{\infty} G_{hc}(p) G_{id}(p+s) G_{je}(p+l) G_{hf}(p+l+t) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} k^*(\omega) h_a(\omega; \psi_0) k(\omega)\, d\omega \right\}_{cd} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} k^*(\omega) h_b(\omega; \psi_0) k(\omega)\, d\omega \right\}_{ef}.$$

Consequently

$$\lim_{m\to\infty} \sigma_m^2 = \sum_{a,b=1}^{p+q} \lambda_a \lambda_b U_{ab}$$

and since the constants λ_a are arbitrary, the theorem follows. □

The following three lemmas are for Theorems 1.3 and 1.4. Set

$$\lambda_j^*(\psi_1, \psi_2) = H_j(\psi_1) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{tr}\{h_j(\omega; \psi_1) f(\omega; \psi_2)\}\, d\omega.$$

LEMMA 2.6. *If* (D.1)–(D.4) *hold, there exists a neighborhood* N_1 *of* ψ_0 *such that*

$$\lambda_j^*(\psi, \psi) = 0 \quad \text{for } \psi \in N_1 \text{ and } j = 1, \dots, p + q.$$

PROOF. The following equalities hold:

$$\begin{aligned}
&\log K(\psi_1) - \log K(\psi_2) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[f^{-1}(\omega; \psi_1) - f^{-1}(\omega; \psi_2) \right] f(\omega; \psi_2)\, d\omega \\
&\quad = \sum_j \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big[H_j(\psi_t) - H_j(\psi_2) \\
&\qquad + \int_0^1 \operatorname{tr}\{ h_j(\omega; \psi_t) f(\omega; \psi_2)\}\, dt \Big]\, d\omega \cdot (\psi_1 - \psi_2)_j \\
&\quad = \sum_j \int_0^1 \lambda_j^*(\psi_t, \psi_2)\, dt \cdot (\psi_1 - \psi_2)_j,
\end{aligned} \tag{2.9}$$

where $\psi_t = t\psi_1 + (1 - t)\psi_2$ and the order of intergration is interchangeable in the second equality because

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |\operatorname{tr}\{ h_j(\omega; \psi_t) f(\omega; \psi_2)\}|\, d\omega &\le \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr}\{ h_j(\omega; \psi_t) h^*(\omega; \psi_t)\}\, d\omega \right]^{1/2} \\
&\quad \times \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr}\{ f(\omega; \psi_2) f^*(\omega; \psi_2)\}\, d\omega \right]^{1/2}
\end{aligned}$$

and each factor is finite, thanks to the assumptions. Since, according to Hosoya and Taniguchi [(1982), page 149], the quantity in the left-hand side of (2.9) cannot be positive and since $\lambda_j^*(\psi_t, \psi_2)$ is continuous with respect to t, $\lambda_j^*(\psi_2, \psi_2)$ must be 0. □

LEMMA 2.7. *If Assumptions* (C.2)(ii) *and* (D.1)–(D.4) *hold, the* $\lambda_j(\psi)$ *are differentiable at* $\psi = \psi_0$ *and* $\partial \lambda_j(\psi_0)/\partial \psi_k = -V_{jk}(\psi_0)$, *where*

$$V_{jk}(\psi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr}\{ f(\omega; \psi) h_j(\omega; \psi) f(\omega; \psi) h_k(\omega; \psi)\}\, d\omega.$$

PROOF. Set $\psi_t = (1 - t)\psi_0 + t\psi$ and set $f_j(\omega; \psi) = \partial f(\omega; \psi)/\partial \psi_j$. In view of the previous lemma, in a neighborhood of ψ_0,

$$H_j(\psi) + \frac{1}{2\pi}\int_{-\pi}^{\pi} h_j(\omega; \psi) f(\omega; \psi)\, d\omega = 0;$$

hence

$$\lambda_j(\psi) - \lambda_j(\psi_0)$$
$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\{h_j(\omega;\psi)(f(\omega;\psi) - f(\omega;\psi_0))\}\,d\omega$$
$$= -\frac{1}{2\pi}\sum_k \int_{-\pi}^{\pi} \mathrm{tr}\left[h_j(\omega;\psi)\int_0^1 f(\omega;\psi_t)h_k(\omega;\psi_t)f(\omega{:}\psi_k)\,dt\right]d\omega(\psi-\psi_0)_k.$$

On the other hand,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 \mathrm{tr}\{h_j(\omega;\psi)f(\omega;\psi_t)h_k(\omega;\psi_t)f(\omega;\psi_t)\}\,dt\,d\omega$$
$$= \int_0^1 V_{jk}(\psi_t)\,dt + R(\psi),$$

where

$$|R(\psi)| \le \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 \mathrm{tr}\{(h_j(\omega;\psi_t) - h_j(\omega;\psi))f_k(\omega;\psi_t)\}\,dt\,d\omega\right|$$
$$\le \left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 \mathrm{tr}\{(h_j(\omega;\psi_t) - h_j(\omega;\psi)) \times (h_j(\omega;\psi_t) - h_j(\omega;\psi))^*\}\,dt\,d\omega\right]^{1/2}$$
$$\times\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 \mathrm{tr}\{f_k(\omega;\psi_t)f_k(\omega;\psi_t)^*\}\,dt\,d\omega\right]^{1/2}.$$

Since the first factor in the right-hand side above tends to 0 as $\psi \to \psi_0$ thanks to (C.2) and the second term is bounded, $|R(\psi)| \to 0$ as $\psi \to \psi_0$. Consequently, since V_{jk} is continuous at $\psi = \psi_0$, the result follows. □

LEMMA 2.8. *If* $1 \le i \le p$, *then*

$$U_{ij} = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{tr}\left[f(\omega;\psi)h_i(\omega;\psi)f(\omega;\psi)h_j(\omega;\psi)\right]_{\psi=\psi_0} d\omega \tag{2.10}$$

and if $1 \le i \le p$ *and* $p+1 \le k \le p+q$, $U_{ij} = 0$.

PROOF. The first assumption holds because

$$k^*(\omega;\psi)h_i(\omega;\psi)k(\omega;\psi) = K(\mu)^{-1}k(\omega;\psi)^{-1}(\partial k(\omega;\psi)/\partial\psi_i)$$
$$+(\partial k^*(\omega;\psi)/\partial\psi_i)k^*(\omega;\psi)^{-1}K(\mu)^{-1},$$

and the Fourier coefficients of $k^{-1}(\omega;\psi)(\partial k(\omega;\psi)/\partial\psi_i)$ are 0 for the nonpositive indices while those of $(\partial k^*(\omega;\psi)/\partial\psi_i)k^*(\omega;\psi)^{-1}$ are 0 for the nonnegative indices. On the other hand, if $1 \le i \le p$ and $p+1 \le j \le p+q$, the integrand of

U_{ij} is equal to

$$\operatorname{tr}[f_i(\omega;\psi)h_j(\omega;\psi)] = \operatorname{tr}\left[\left\{\frac{\partial}{\partial\psi_i}k(\omega;\psi)\right\}K(\mu)\left\{\frac{\partial}{\partial\psi_j}K^{-1}(\mu)\right\}k^{-1}(\omega;\psi)\right]$$
$$+\operatorname{tr}\left[K(\mu)\left\{\frac{\partial}{\partial\psi_i}k^*(\omega;\psi)\right\}k^*(\omega;\psi)\left\{\frac{\partial}{\partial\psi_j}K^{-1}(\mu)\right\}\right],$$

where the first term in the right-hand side has nonzero Fourier coefficients only for the positive indices and the second term only for the negative indices. Thus the result follows. □

PROOF OF THEOREM 1.3. The theorem is obtained by incorporation of Lemmas 2.6–2.8 in Theorem 1.2 in a straightforward way. □

PROOF OF THEOREM 1.4. It holds that

$$\partial\bar{L}_n/\partial\psi_j = -H_j(\psi) - \frac{1}{2\pi}\int_{-\pi}^{\pi} h_j(\omega;\psi)I_n(\omega)\,d\omega \tag{2.11}$$

in a neighborhood of ψ_0, because

$$\bar{L}_n(\psi_1) - \bar{L}_n(\psi_2) = -\{\log\det K(\psi_1) - \log\det K(\psi_2)\}$$
$$-\frac{1}{2\pi}\sum_j\int_{-\pi}^{\pi}\int_0^1 \operatorname{tr}\{h_j(\omega;\psi_t^*)I_n(\omega)\}\,dt\,d\omega\cdot(\psi_1-\psi_2)_j,$$

where $\psi_t^* = t\psi_1 + (1-t)\psi_2$ and the integral

$$\int_{-\pi}^{\pi}\operatorname{tr}\{h_j(\omega;\psi)I_n(\omega)\}\,d\omega$$

is continuous at $\psi = \psi_2$ in view of Assumption (C.2) and the Schwarz inequality. It follows from (2.11) that

$$\bar{L}_n(\tilde{\psi}) - \bar{L}_n(\psi_0)$$
$$= -\sum_j\int_0^1\sqrt{n}\,\{S_{nj}(\psi_t) - S_{nj}(\tilde{\psi})\}\,dt\cdot\sqrt{n}\,(\tilde{\psi}-\psi_0)_j + o_p(1), \tag{2.12}$$

where $S_{nj}(\psi)$ is defined in (1.5) and $\psi_t = t\psi_0 + (1-t)\tilde{\psi}$. On the other hand,

$$\sup_{0\le t\le 1}\sqrt{n}\left|S_{nj}(\psi_t) - S_{nj}(\tilde{\psi}) - \{\lambda_j(\psi_t) - \lambda_j(\tilde{\psi})\}\right| \to 0 \tag{2.13}$$

in probability since $\sqrt{n}\,S_{nj}(\tilde{\psi})$ and $\sqrt{n}\,\{S_{nj}(\psi_0) + \lambda_j(\tilde{\psi})\} \to 0$ in probability and since $\sup_{0\leqq t\leqq 1}\sqrt{n}\,|S_{nj}(\psi_t) - S_{nj}(\psi_0) + \lambda_j(\psi_t)|$ is also shown to tend to 0 in probability by an argument similar to Lemma 2.4. Consequently the relationships (2.12) and (2.13) imply that

$$\bar{L}_n(\tilde{\psi}) - \bar{L}_n(\psi_0) = -\sum_j\int_0^1\sqrt{n}\,\{\lambda_j(\psi_t) - \lambda_j(\tilde{\psi})\}\,dt\cdot\sqrt{n}\,(\tilde{\psi}-\psi_0)_j + o_p(1).$$

But since

$$\sqrt{n}\left(\lambda_j(\psi_t) - \lambda_j(\tilde{\psi})\right) = (t-1)\sum_k V_{jk}\sqrt{n}\left(\tilde{\psi} - \psi_0\right)_j + o\left(\sqrt{n}\,|\tilde{\psi} - \psi_0|\right),$$

it follows that

$$\bar{L}_n(\tilde{\psi}) - \bar{L}_n(\psi_0) = \tfrac{1}{2}\sum_j \sum_k V_{jk}\sqrt{n}\left(\tilde{\psi} - \psi_0\right)_j \sqrt{n}\left(\tilde{\psi} - \psi_0\right)_k + o_p(1). \qquad \square$$

PROOF OF THEOREM 1.5. It follows from Theorems 1.3 and 1.4 that $2(\bar{L}_n(\hat{\psi}_n(l)) - \bar{L}_n(\psi_0))$, $l = 1, \dots, p$, have the same limiting distribution as

$$\sum_{i=1}^{q_l} \sum_{j=1}^{q_l} \sqrt{n}\left(\hat{\theta}_n(l) - \theta_0\right)_i \sqrt{n}\left(\hat{\theta}_n(l) - \theta_0\right)_j V_{ij}(\psi_0)$$
$$+ \sum_{i=1}^{r} \sum_{j=1}^{r} \sqrt{n}\left(\hat{\mu}_n(l) - \mu_0\right)_i \sqrt{n}\left(\hat{\mu}_n(l) - \mu_0\right)_j V_{(2)ij}(\psi_0).$$

On the other hand, it follows from Lemma 2.5 and the proof of Theorem 1.2 that $\{\sqrt{n}(\hat{\theta}_n(l) - \theta_0), \sqrt{n}(\hat{\mu}_n(l) - \mu_0)\}$, $l = 1, \dots, p$, are jointly distributed as $\{V(l)^{-1}\sqrt{n}\,S_n(l), V_{(2)}^{-1}\sqrt{n}\,T_n\}$, where $V(l) = \{V_{ij}, i, j = 1, \dots, q_l\}$ and $S_n(l)$ and T_n are column vectors $\{S_{nj}(\psi_0), j = 1, \dots, q_l\}$ and $\{S_{nj}(\psi_0), j = q_p + 1, \dots, q_p + r\}$, respectively. Therefore $\bar{L}_{n,01}, \dots, \bar{L}_{n,0p}$ have the same limiting distribution as

$$-\tfrac{1}{2}\left\{\sum_{k=1}^{q_j} \sum_{l=1}^{q_j} V^{kl}(j)\sqrt{n}\,S_{nk}(\psi_0)\sqrt{n}\,S_{nl}(\psi_0)\right.$$
$$\left. - \sum_{k=1}^{q_0} \sum_{l=1}^{q_0} V^{kl}(0)\sqrt{n}\,S_{nk}(\psi_0)\sqrt{n}\,S_{nl}(\psi_0)\right\}, \qquad j = 1, \dots, p,$$

where $V^{kl}(j)$ signifies the (k, l) element of $V(j)^{-1}$. Since $V_{(1)}(\psi_0)$ is symmetric and positive definite, there exists a lower-triangular nonsingular real matrix C such that $C'C = V_{(1)}(\psi_0)^{-1}$. Denote by C_{ij} the (i, j) element of C and denote by $C^{(i)}$ the triangular matrix $\{C_{k,j};\ k, j = i, \dots, q_i\}$. Set $w(n) = C^{-1}\sqrt{n}\,S_n(p)$, denote by $w_i(n)$ the ith element and denote by $w^{(i)}(n)$ the column q_i-vector $\{w_j(n),\ j = 1, \dots, q_i\}$. Then it follows from the triangularity of $C^{(i)}$ that $C^{(i)-1}V(j)C^{(i)'-1} = i_{q_i}$. Consequently, the $\bar{L}_{n,0i}$'s are expressed as

$$\bar{L}_{n,0i} = -\tfrac{1}{2}\left\{w^{(i)}(n)'C^{(i)'-1}V(i)^{-1}C^{(i)-1}w^{(i)}(n)\right.$$
$$\left. - w^{(0)}(n)'C^{(0)'-1}V(0)^{-1}C^{(0)-1}w^{(0)}(n)\right\}$$
$$= -\tfrac{1}{2}\sum_{j=q_0+1}^{q_i} w_j(n)^2, \qquad i = 1, \dots, p.$$

Since $\sqrt{n}\,S_n(p)$ has the asymptotic normal distribution with mean 0 and covariance matrix $V_{(1)}^{-1}$ in view of Theorem 1.2, $w_j(n)$, $j = 1, \dots, q_p$, are asymptotically independently standard-normally distributed; thus the theorem follows. $\square$

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