The Annals of Statistics
1989, Vol. 17, No. 1, 236–251

ON THE ADMISSIBILITY AND CONSISTENCY OF TESTS FOR HOMOGENEITY OF VARIANCES

BY ARTHUR COHEN[1] AND JOHN I. MARDEN[2]

Rutgers University and University of Illinois at Urbana-Champaign

Consider the one-way fixed and balanced analysis of variance model under the assumptions of independence and normality. The problem is to test for homogenity of variances. A necessary and sufficient condition for admissibility of a test among the class of scale invariant tests is given. Hartley's test and Cochran's test are not in the class and therefore are inadmissible. Various scale invariant tests are examined for parameter consistency. Parameter consistency in this case means the following: Consider a sequence of values of the maximal invariant parameter in the alternative space. If the Kullback–Leibler distance from this sequence to the null point tends to infinity then the power of the test tends to 1. Several well known tests are shown to be parameter consistent (PC) for all significance levels. Some well known tests however may not be PC or may be PC only for certain significance levels. Extensions of PC results to nonnormal cases are indicated.

1. Introduction and summary. The problem of testing homogeneity of variances in a one-way fixed analysis of variance model has been studied for the past 50 years. If we let s_i^2 denote the sample variance for the ith population, $i = 1, 2, \dots, p$, then the s_i^2's are independently distributed and each estimates its corresponding variance σ_i^2. Under the assumption of normality of each of the p populations and the further assumption that each s_i^2 is based on $(n+1)$, the same number of observations, Laue (1965) studied a two parameter family of tests $T(\lambda, \eta)$ which reject the homogeneity hypothesis when $R(\lambda, \eta) \ge d$, where

$$R(\lambda, \eta) = M(\lambda)/M(\eta), \qquad -\infty \le \eta < \lambda \le \infty, \tag{1.1}$$

$M(t) = [(1/p)\sum_{i=1}^p s_i^{2t}]^{1/t}$, $t \ne 0$, $M(0) = (\prod_{i=1}^p s_i^2)^{1/p}$, $M(\infty) = \max s_i^2$, $M(-\infty) = \min s_i^2$. In addition to $T(\lambda, \eta)$, $\lambda \ne \eta$, $T(\lambda, \lambda)$ is determined by considering $[pn/(\lambda - \eta)]\log R(\lambda, \eta)$, setting $\eta = \lambda - \delta$, letting $\delta \to 0$ and using L'Hospital's rule. Letting $\lambda \to 0$ and using L'Hospital's rule again determines $T(0,0)$ [see (3.10)]. We assume throughout that $p \ge 3$. Many well known tests are contained in the $T(\lambda, \eta)$ family. For example, $T(1,0)$ is equivalent to the likelihood ratio test; $T(2,1)$ is equivalent to a test suggested by Stevens (1936); $T(0,0)$ has been suggested by Bechhofer (1960) and Bartlett and Kendall (1946); $T(\infty, 1)$ is equivalent to Cochran's test (1941) and $T(\infty, -\infty)$ is equivalent to Hartley's test (1950). Laue proved that tests in the family share some of the properties of the likelihood ratio test. That is, they are consistent, similar, they are based on statistics which are asymptotically distributed as chi-square, and as

---

Received June 1984; revised March 1988.
[1]Research supported by NSF Grant DMS-84-18416.
[2]Research supported by NSF Grant MCS-82-01771.
AMS 1980 *subject classifications*. Primary 62F99; secondary 62F03, 62C07.
Key words and phrases. Homogeneity of variances, admissibility, likelihood ratio test, Hartley's test, Cochran's test, locally most powerful unbiased test, parameter consistency, robust tests.

Box (1953) showed for the likelihood ratio test, they are sensitive to the normality assumption.

Cohen and Strawderman (1971) showed that if $\lambda \geq 0 \geq \eta$, then the corresponding test in (1.1) is unbiased. Kiefer and Schwartz (1965) proved that $T(1,0)$ yields an admissible Bayes test. It follows from results in Cohen, Sackrowitz and Strawderman (1985) that $T(2,1)$ is the unique locally most powerful test among tests which are permutation invariant and locally unbiased. A recent result of Cohen and Sackrowitz (1987) implies that $T(2,1)$ is locally most powerful unbiased among permutation invariant tests. Such a property implies its admissibility since the permutation group is finite and power functions are continuous.

It is easily seen that all tests in the family $T(\lambda, \eta)$ are scale invariant. In this paper we study the question of admissibility of tests among the class of scale invariant tests. Clearly any inadmissible test within this class will be inadmissible among the class of all tests. A necessary and sufficient condition is given for admissibility within the class. The necessary and sufficient condition entails a characterization of a minimal complete class, which is obtained by using a theorem of Brown and Marden (1989). It is interesting to note that two of the more popular tests, Cochran's test and Hartley's test [see Seber (1977), page 147], are inadmissible. The result is of additional interest since it can be shown that Cochran's test and Hartley's test lie in the (nonminimal) complete class one gets using a theorem of Matthes and Truax (1967).

We remark that the complete class result is developed for the situation where sample sizes for each population need not be the same. The interesting applications here however are to tests which are appropriate when the sample sizes from each population are the same.

Anderson and Perlman (1988) introduced the notion of parameter consistency of a test. We define parameter consistency for scale invariant tests as follows: Consider a sequence of values of the maximal invariant parameter in the alternative space. Suppose that the sequence of Kullback–Leibler (KL) distances between the null point and the sequence of parameter points tends to ∞. Then if the power of the test tends to 1 for this sequence of parameter points it is said to be parameter consistent (PC). (Kullback–Leibler distance is defined in Section 3.)

In this paper we study parameter consistency of some of the parametric tests within the class $T(\lambda, \eta)$ determined by (1.1). The normal model with equal sample sizes from each population is assumed, although for the likelihood ratio test one need not assume equal sample sizes. The likelihood ratio test, $T(0,0)$, Hartley's test and $T(\lambda, \eta)$ with $-\infty \leq \eta \leq 0 \leq \lambda \leq \infty$ are PC. Some tests, such as Cochran's test, are PC for some significance levels but not PC for other significance levels.

A remark is made concerning the PC property of tests when the normality assumption is relaxed. An additional remark concerns the PC property of nonparametric tests for homogeneity of variances.

In Section 2 we give the minimal complete class result and prove the inadmissibility of Cochran's test and Hartley's test. We indicate that some other tests in $T(\lambda, \eta)$ must lie in the minimal complete class but an in depth study of the relation between $T(\lambda, \eta)$ and the complete class theorem is not made. Even if a test lies in the minimal complete class, that does not imply its admissibility except among scale invariant tests. The useful application of the complete class theorem is for inadmissibility results. In Section 3 the results concerned with parameter consistency are given.

2. Complete class. Suppose that s_i^2 denotes the sample variance based on $(\nu_i + 1)$ observations for the ith population, $i = 1, 2, \ldots, k$. Let $v_i = \nu_i s_i^2$ and assume v_i/σ_i^2 has a chi-square distribution with ν_i degrees of freedom. Write $v' = (v_1, v_2, \ldots, v_p)$ and $\sigma^{2'} = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_p^2)$. The problem considered is that of testing H_0: $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_p^2$ versus H_1: $\sigma_i^2 \neq \sigma_j^2$ for some (i, j). The problem is invariant under the scale transformations $v \to cv$; $\sigma^2 \to c\sigma^2$, for $c > 0$. One representation of the maximal invariant statistic is $u' = (u_1, u_2, \ldots, u_{p-1})$, where $u_i = v_i/v_p$. A representation of the maximal invariant parameter is $\tau' = (\tau_1, \tau_2, \ldots, \tau_{p-1})$, where $\tau_i = [(1/\sigma_i^2)/\Sigma_{i=1}^p(1/\sigma_i^2)]$. The density of u may be written as

$$g_\tau(u) = C(\tau) \prod_{i=1}^{p-1} u_i^{(\nu_i/2)-1}\left(1 + \sum_{i=1}^{p-1} \tau_i(u_i - 1)\right)^{-\beta}, \tag{2.1}$$

$$0 < u_i < \infty, 0 < \tau_i < 1,$$

where $\beta = [(\Sigma_{i=1}^p \nu_i)/2]$. A relevant quantity in the development of the minimal complete class of scale invariant tests is the ratio of the densities under H_0 and H_1. Since H_0 in terms of τ is $\tau_i = 1/p$, use (2.1) to find that this ratio is

$$R_\tau(u) = \bar{C}(\tau)(1/p)^\beta\left[\left(1 + \sum_{i=1}^{p-1} u_i\right)\Big/\left(1 + \sum_{i=1}^{p-1} \tau_i(u_i - 1)\right)\right]^\beta. \tag{2.2}$$

It is convenient to make the 1–1 transformations $u \to x' \equiv (x_1, x_2, \ldots, x_{p-1})$, $\tau \to \theta' \equiv (\theta_1, \theta_2, \ldots, \theta_{p-1})$, where $x_i = (u_i - 1)/(1 + \Sigma_{j=1}^{p-1} u_j)$ and $\theta_i = p\tau_i - 1$. Then the bracketed quantity in (2.2) becomes $[p/(1 + \Sigma_{i=1}^{p-1} x_i\theta_i)]$ so that (2.2) may be written as

$$R_\theta(x) = C^*(\theta)\left(1 + \sum_{i=1}^{p-1} x_i\theta_i\right)^{-\beta}, \tag{2.3}$$

for $x \in \mathscr{X}$, where $\mathscr{X} = \{x\colon -1 < x_i < 1,\ -p < \Sigma_{i=1}^{p-1} x_i < 1\}$. The range of the maximal invariant parameter θ is

$$\Theta = \left\{\theta\colon -1 < \theta_i < (p - 1),\ -1 < \sum_{i=1}^{p-1} \theta_i < (p - 1)\right\}.$$

In terms of θ, the testing problem may be expressed as testing

$$H_0\colon \theta = 0 \quad \text{versus} \quad H_1\colon \theta \neq 0.$$

Define

$$R_\theta^*(x) = \left[1\Big/\left(1 + \sum_{i=1}^{p-1} x_i\theta_i\right)\right]^\beta,$$

$$h(x,\theta) = \left(R_\theta^*(x) - 1 + \beta\sum_{i=1}^{p-1} x_i\theta_i\right)\Big/\|\theta\|^2, \tag{2.4}$$

$$d(x;\lambda,M,\theta,J) = \lambda'x + x'Mx + \int_{\overline{\Theta}-\{0\}} h(x,\theta)J(d\theta),$$

where $\lambda \in R^{p-1}$, M is in S_{p-1}, the set of $(p-1)\times(p-1)$ symmetric nonnegative definite matrices and J is in $\mathscr{F}$, the set of finite measures on $\overline{\Theta} - \{0\}$, where $\overline{\Theta}$ is the closure of Θ.

THEOREM 2.1. *The test $\phi(x)$ is an admissible test among scale invariant tests if and only if there exist*

$$(\lambda, M, J, c) \in R^{p-1}\times S_{p-1}\times\mathscr{F}\times R - \{(0,0,0,0)\} \tag{2.5}$$

such that

$$\phi(x) = \begin{cases} 1, & \text{if } d(x;\lambda,M,J) > c,\\ 0, & \text{otherwise (almost everywhere)}.\end{cases} \tag{2.6}$$

For ease of presentation we defer the proof to the Appendix.

COROLLARY 2.2. *A necessary condition for a test to be admissible among scale invariant tests is that it has a convex acceptance region in x.*

PROOF. The quantity $d(x;\lambda,M,J)$ is clearly a convex function of x. This follows since $\lambda'x$, $x'Mx$ and $(1/(1+\Sigma x_i\theta_i))^\beta$ are convex, so $h(x,\theta)$ is convex as well. Hence $\{x: d(x;\lambda,M,J)\le c\}$ is convex. □

Our main goal in this section is to use the complete class theorem to prove the inadmissibility of Cochran's test and Hartley's test. To facilitate our proof we establish an additional corollary to Theorem 2.1. The corollary enables us to state a necessary and sufficient condition in terms of variables which are an arbitrary linear transformation of x. Later on Cochran's test and Hartley's test will be expressed in terms of variables which are a linear transformation of x. Hence now define y via $x = Ay + b$, where A is an arbitrary $(p-1)\times(p-1)$ nonsingular matrix and b is an arbitrary $(p-1)\times 1$ vector. Also define

$$\bar{h}(y,\theta) = \left\{\left[1/(1+\theta'(Ay+b))^\beta\right] - 1 - \beta\theta'(Ay+b)\right\}\Big/\|\theta\|^2.$$

COROLLARY 2.3. *A necessary and sufficient condition for a test $\varphi(y)$ to be admissible among scale-invariant tests is that there exist $\bar{\lambda}$, $\overline{M}$, J, and $|\bar{c}| < \infty$*

such that

$$\varphi(y) = \begin{cases} 1, & \text{if } \bar{d}(y; \bar{\lambda}, \bar{M}, J) > \bar{c}, \\ 0, & \text{otherwise (almost everywhere)}, \end{cases} \tag{2.7}$$

where

$$\bar{d}(y; \bar{\lambda}, \bar{M}, J) = \bar{\lambda}'y + y'\bar{M}y + \int_{\bar{\Theta} - \{0\}} \bar{h}(y, \theta) J(d\theta). \tag{2.8}$$

PROOF. Use Theorem 2.1 and the transformation $x = Ay + b$. The quantities $\bar{\lambda}, \bar{M}, \bar{c}$ are expressed in terms of λ, M, c as follows: $\bar{\lambda} = A'\lambda + 2A'Mb$, $\bar{M} = A'MA$, $\bar{c} = c - \lambda'b - b'Mb$. □

We note that the range of values of y for which $\bar{h}$, (2.7) and (2.8) are defined depends on A and b. The range will be specified when Corollary 2.3 is applied. In order to apply Corollary 2.3 we need the following lemma.

LEMMA 2.4. *The quantity $\bar{d}(y; \bar{\lambda}, \bar{M}, J)$ given in* (2.8) *must satisfy at least one of the following conditions: $\bar{d}$ is* (a) *strictly convex in y_1;* (b) *strictly increasing in y_1;* (c) *strictly decreasing in y_1; and* (d) *independent of y_1.*

PROOF. Examine the three terms of $\bar{d}$. First $\bar{\lambda}'y$ is either:

$$\text{strictly increasing in } y_1 \text{ if } \bar{\lambda}_1 > 0, \tag{2.9}$$

$$\text{independent of } y_1 \text{ if } \bar{\lambda}_1 = 0, \tag{2.10}$$

$$\text{strictly decreasing in } y_1 \text{ if } \bar{\lambda}_1 < 0. \tag{2.11}$$

The second term $y'\bar{M}y$ is either:

$$\text{strictly convex in } y_1 \text{ if } \bar{M}_{11} > 0, \tag{2.12}$$

$$\text{independent of } y_1 \text{ if } \bar{M}_{11} = 0, \tag{2.13}$$

since $\bar{M}_{11} = 0$ implies $\bar{M}_{1j} = 0$ for $j = 2, \dots, p - 1$.

Finally from (2.7) we see that $\bar{h}(y, \theta)$ is independent of y_1 if $(\theta'A)_1 = 0$ or strictly convex in y_1 if $(\theta'A)_1 \neq 0$. This in turn implies that $\int_{\bar{\Theta} - \{0\}} \bar{h}(y, \theta) J(d\theta)$ is either:

$$\text{strictly convex in } y_1 \text{ if } J\{\theta: (\theta'A)_1 \neq 0\} > 0, \tag{2.14}$$

$$\text{independent of } y_1 \text{ if } J\{\theta: (\theta'A)_1 \neq 0\} = 0. \tag{2.15}$$

Combine the twelve possible combinations for the three terms. For example (2.9), (2.13) and (2.14) imply (a); (2.10), (2.13) and (2.14) imply (a); (2.10), (2.12) and (2.14) imply (a). Similarly each of the other nine combinations imply (a), (b), (c) or (d). □

We now apply the above results to specific tests. First consider Cochran's test which accepts H_0 if

$$\left[\max v_i \Big/ \sum_{i=1}^{p} v_i\right] \le B, \tag{2.16}$$

where $(1/p) < B < 1$ for any test whose level $\alpha \in (0,1)$. Clearly if $0 < B < (1/p)$, the acceptance region is empty. The test in (2.16) is equivalent to accepting H_0 if

$$\max(z_1, z_2, \dots, 1 - z_1 - z_2 - \cdots - z_{p-1}) \le B, \tag{2.17}$$

where $z_i = v_i/\sum_{j=1}^{p} v_j$, $i = 1,2,\dots,p$. We let $z' = (z_1,\dots,z_{p-1})$ and note that $z_p = 1 - \sum_{i=1}^{p-1} z_i$. The sample space for z is

$$\mathscr{Z} \equiv \left\{z: 0 < z_i \text{ for } i = 1,2,\dots,p-1, \sum_{i=1}^{p-1} z_i < 1\right\}.$$

THEOREM 2.5. *Cochran's test is inadmissible for* $\alpha \in (0,1)$.

PROOF. First recall that for $i = 1,2,\dots,p-1$, $x_i = (u_i - 1)/(1 + \sum_{j=1}^{p-1} u_j)$, $u_i = v_i/v_p$, and so $x = Az - b$, where $A = I_{p-1} + J^*$, J^* is a $(p-1) \times (p-1)$ matrix all of whose elements are 1, $b = 1$, $1' = (1,1,\dots,1)$. Thus Corollary 2.3 and Lemma 2.4 are applicable where the role of y will be played by z. The acceptance region of Cochran's test in $\mathscr{Z}$ is $A \equiv \{z \in \mathscr{Z} | z_i \le B, i = 1,2,\dots,p\}$. Assume that the test in (2.17) is admissible, so that by Corollary 2.3 there exist $\bar{d}$ and $\bar{c}$, called $\bar{d}_c$ and $\bar{c}_c$ here, such that (2.17) is equivalent to test (2.7). It can be shown that on the boundary ∂A of A in $\mathscr{Z}$, $\bar{d}_c = \bar{c}_c$. We will use Lemma 2.4 to arrive at a contradiction. Consider the set in $\mathscr{Z}$ which has fixed $z_2 = B$, $z_3 = \cdots = z_{p-1} = \delta$ for arbitrary fixed δ satisfying

$$\max(0, (1 - 3B)/(p-3)) < \delta < \min((1 - B)/(p-3), B) \tag{2.18}$$

and z_1 satisfying

$$\max(0, 1 - 2B - (p-3)\delta) < z_1 < \min(B, 1 - B - (p-3)\delta). \tag{2.19}$$

(For $p = 3$, there is no need for δ.) This set is contained in ∂A since $z_i < B$ for $i \ne 2$ and $z_2 = B$. Since z_1 ranges over a nonempty interval (recall $1/p < B < 1$) and $\bar{d}_c$ is constant as z_1 ranges over that interval, conditions (a), (b) and (c) of Lemma 2.4 fail. The lemma thus implies that (d) holds, i.e., that $\bar{d}_c$ is independent of z_1. However, the test cannot be independent of z_1, since it is symmetric in the z_i's, hence would have to be independent of all the z_i's. Thus we have a contradiction, proving that the test (2.17) is inadmissible. □

Next, consider Hartley's test, which accepts H_0 if

$$[\max v_i/\min v_i] \le B, \tag{2.20}$$

where $B > 1$. This test is equivalent to accepting H_0 if

$$[\max z_i/\min z_i] \le B.$$

THEOREM 2.6. *Hartley's test is inadmissible for* $\alpha \in (0,1)$.

PROOF. The acceptance region for Hartley's test within the sample space of z is

$$\begin{aligned}\{z \in \mathscr{Z}\colon (z_1/z_2) \le B, (z_2/z_1) \le B, \ldots, (z_1/z_p) \le B,\\ (z_p/z_1) \le B, (z_2/z_3) \le B, (z_3/z_2) \le B, \ldots,\\ (z_{p-1}/z_p) \le B, (z_p/z_{p-1}) \le B\}.\end{aligned} \tag{2.21}$$

Thus the acceptance region is the intersection of $\binom{p}{2}$ convex sets. Each convex set is determined by a pair of hyperplanes. (At this point the reader may find it helpful to see Figure 1 which shows the acceptance region in the $\mathscr{Z}$ space when $p = 3$.)

Consider the linear transformation $w_1 = z_1$, $w_2 = z_2 - Bz_1$, $w_j = z_j$, $j = 3, \ldots, p - 1$. Since z is linear transformation of x and w is a linear transformation of z it follows that w is a linear transformation of x and so we may

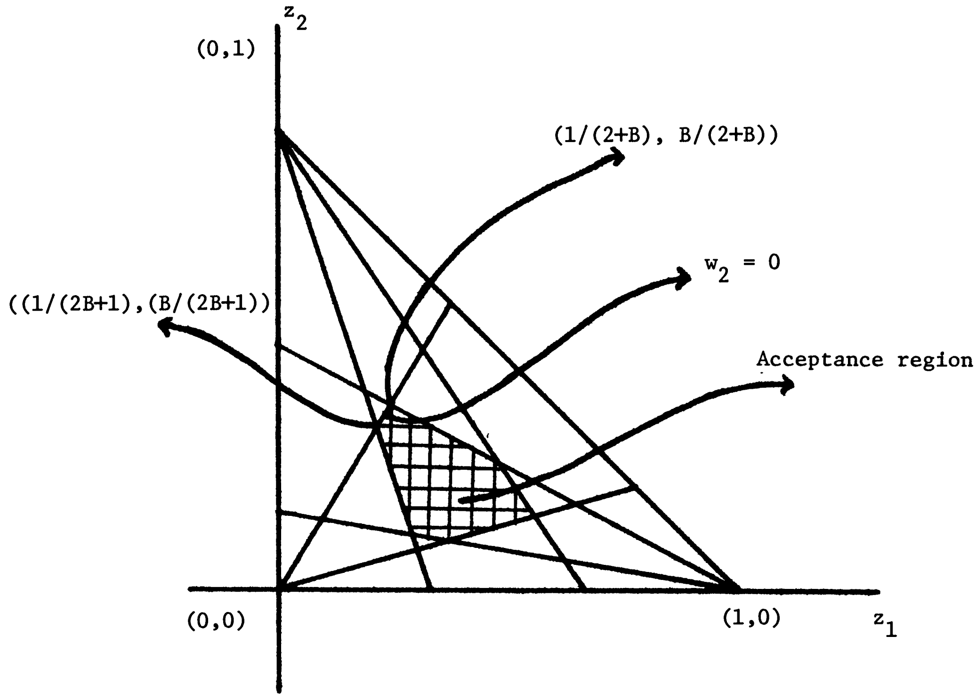

FIG. 1. *Acceptance region in* $\mathscr{Z}$ *for Hartley's test* $p = 3$.

apply Corollary 2.3 and Lemma 2.4 where the role of y, $\bar{d}$ and $\bar{c}$ will be played by w, $\bar{d}_H$ and $\bar{c}_H$, respectively. The sample space of w is

$$W = \{w | w_i > 0,\ i \neq 2,\ w_2 > -Bw_1 \text{ and } w_1(1 + B) + w_2 + \cdots + w_{p-1} < 1\}.$$

Suppose test (2.21) is admissible, so that by Corollary 2.3 $\bar{d}_H = \bar{c}_H$ on the boundary of the acceptance region of the test in W. Consider the set determined by fixing $w_2 = 0$, for $p > 3$ fixing $w_3 = \cdots = w_{p-1} = \frac{1}{3}$, and let

$$\frac{3}{p(1 + 2B)} < w_1 < \frac{3}{p(2 + B)}. \tag{2.22}$$

The open interval (2.22) is nonempty since $B > 1$ and is contained on the boundary of the acceptance region since on that set $z_2 > z_i > z_1$ for $i = 3, \dots, p$ and $z_2/z_1 = B$. As in the proof of Theorem 2.5, we can invoke Lemma 2.4 to show that $\bar{d}_H$ must be independent of w_1, which can be shown to be impossible. Thus we have a contradiction, proving test (2.21) inadmissible. □

REMARK 2.7. The main result of the paper is the inadmissibility of Cochran's test and Hartley's test. These tests were originally proposed for the case where the number of observations in each population is the same. This is the primary reason we assumed $\nu_i = n$ in Section 1. We note that the results of Section 2 did not require such an assumption. The tests $T(\lambda, \eta)$, defined by (1.1), were also intended to treat the case $\nu_i = n$. Some of them could be modified and make sense for the case of arbitrary ν_i. For example, the likelihood ratio test statistic for arbitrary ν_i is

$$\left\{\left(\sum_{i=1}^{p} v_i\right)^{\sum_{i=1}^{p}\nu_i} \middle/ \left[\prod_{i=1}^{p} v_i^{\nu_i}\right]\right\}. \tag{2.23}$$

The case where $\nu_i = n$ is certainly an important one since such a design enables the problem to be permutation invariant and the tests $T(\lambda, \eta)$ are permutation invariant.

REMARK 2.8. One natural question to ask is which tests $T(\lambda, \eta)$ are admissible. As mentioned in the introduction, $T(1,0)$ and $T(2,1)$ are. The likelihood ratio test is admissible even when ν_i are not equal. [See Kiefer and Schwartz (1965).] Our theorem however cannot be used to determine which tests are admissible since the theorem is limited to scale invariant tests. Even if a test is admissible among scale invariant tests it need not be admissible since the scale group is not compact. Hence the important question of admissibility of tests $T(\lambda, \eta)$ is open except for $T(1,0)$, $T(2,1)$, $T(\infty, 1)$ and $T(\infty, -\infty)$. The test $T(-1, -\infty)$ can be proved inadmissible as well. The question of admissibility among scale invariant tests is also open for many of $T(\lambda, \eta)$.

3. Parameter consistency of scale invariant tests. The model under consideration is the same as in Section 2. To begin with, the parameter space is $\{\sigma^2\colon 0 < \sigma_i^2 < \infty,\ i = 1, 2, \dots, p\}$. Since we will be interested in scale invariant tests the only statistic we need consider is the maximal invariant statistic z. The maximal invariant parameter space is denoted by

$$\Omega = \left\{ \rho\colon \rho' = (\rho_1, \rho_2, \dots, \rho_{p-1}),\ \rho_i = \left(\sigma_i^2 \Big/ \sum_{j=1}^{p} \sigma_j^2 \right), \right. \tag{3.1}$$
$$\left. i = 1, 2, \dots, p-1;\ 0 < \rho_i < 1 \right\}.$$

In terms of the parameter ρ, the null hypothesis is H_0: $\rho = (1/p)1 \equiv \rho_0$ and the alternative is H_1: $\rho \neq \rho_0$. Now let $f(z, \rho)$ denote the probability density function of z. Consider $\rho_1 \in \Omega$ and $\rho_2 \in \Omega$, $\rho_1 \neq \rho_2$.

DEFINITION 3.1. The Kullback–Leibler (KL) distance between ρ_1 and ρ_2 is

$$I(\rho_1, \rho_2) = E_{\rho_1} \log[f(z, \rho_1)/f(z, \rho_2)]. \tag{3.2}$$

Now let $\beta_\varphi(\rho)$ denote the power of the test function φ.

DEFINITION 3.2. A test function φ is said to be PC if given any sequence of alternatives ρ_n such that

$$\lim_{n \to \infty} I(\rho_0, \rho_n) = \infty, \tag{3.3}$$

$$\lim_{n \to \infty} \beta_\varphi(\rho_n) = 1. \tag{3.4}$$

LEMMA 3.1. *The KL distance between* ρ_0 *and* ρ *is*

$$I(\rho_0, \rho) = \sum_{i=1}^{p} (\nu_i/2) \log \rho_i + q E_{\rho_0} \left\{ \log \left[\sum_{i=1}^{p} (z_i/\rho_i) \right] \right\} \tag{3.5}$$

where $q = \sum_{i=1}^{p} \nu_i/2$.

PROOF. Use the fact that v_i/σ_i^2 has a chi-square distribution with ν_i degrees of freedom to derive the density of z, namely,

$$f(z, \rho) \alpha \left\{ \prod_{i=1}^{p} z_i^{(\nu_i/2)-1} \Big/ \prod_{i=1}^{p} \rho_i^{\nu_i/2} \right\} \left[\sum_{i=1}^{p} (z_i/\rho_i) \right]^q. \tag{3.6}$$

Now use (3.6) in (3.2), replacing ρ_1 with ρ_0 and ρ_2 with ρ, to derive (3.5). □

LEMMA 3.2. *Let* ρ_n *be a sequence of alternatives and* $m_n = \min_{1 \le i \le p} \rho_{in}$. *Then* $I(\rho_0, \rho_n) \to \infty$ *if and only if* $m_n \to 0$.

PROOF. Since each $\rho_{in} < 1$, and $\sum_{i=1}^{p} z_i/\rho_{in} \le \sum_{i=1}^{p} z_i/m_n = 1/m_n$, from (3.5),

$$I(\rho_0, \rho_n) \le -q \log m_n. \tag{3.7}$$

Suppose m_n does not approach 0. Then there exists a subsequence of $\{\rho_n\}$ on which m_n is bounded away from 0, and hence on which, $-\log m_n$ is bounded.

Thus $I(\rho_0, \rho_n)$ does not approach ∞. This completes the "only if" part of the proof.

Now suppose $m_n \to 0$. Since z_i has a marginal Beta$(\nu_i/2, (q-\nu_i)/2)$ distribution under ρ_0, $K \equiv \sup_{1\le i\le p}|E_{\rho_0}(\log z_i)| < \infty$, hence

$$E_{\rho_0}\left(\log \sum_{i=1}^{p} \frac{z_i}{\rho_{in}}\right) \ge -K - \log m_n. \tag{3.8}$$

Thus from (3.5) and (3.8)

$$I(\rho_0, \rho_n) \ge \sum \frac{\nu_i}{2}\log \rho_{in} - q\log m_n - qK$$
$$= \sum_{i=1}^{p} \frac{\nu_i}{2}\log\frac{\rho_{in}}{m_n} - qK.$$

Since $m_n \le \rho_{in}$ for each i and n and $\Sigma_{i=1}^{p}\rho_{in} = 1$, the final summation approaches ∞ as $m_n \to 0$. Thus the "if" part of the proof is finished, which completes the proof of the lemma. □

We proceed to study the PC property of scale invariant tests. By virtue of Lemma 3.2, an invariant test is PC if its power tends to 1 whenever the sequence $\{m_n\}$ of $\{\rho_n\}$ converges to 0.

We remark here that often we assume ν_i's are equal. The assumption is not necessarily used in proving a result. It is made because the test statistic under consideration makes sense or was recommended for the equal sample size situation. Sometimes, as in the likelihood ratio test, a modification of the test statistic can be made so that it makes sense for the unequal sample size case.

Let us examine PC for the tests $T(\lambda, \eta)$ determined by (1.1).

THEOREM 3.3. *Suppose* $\nu_i = n$, $i = 1, \dots, p$. *If* $\infty \ge \lambda \ge 0 \ge \eta \ge -\infty$, *then the test* $T(\lambda, \eta)$ *is PC for all sizes* α, $0 < \alpha \le 1$. *If* $\infty \ge \lambda > \eta > 0$ *or* $0 > \lambda > \eta \ge -\infty$, *then the test is PC if and only if* $\alpha \ge \alpha^*$, *where* α^* *is the size of the test with rejection region*

$$R(\lambda, \eta) \ge [p/(p-1)]^{1/\eta - 1/\lambda}. \tag{3.9}$$

PROOF. Since Ω is bounded, to prove a test is PC it is enough to show that (3.4) holds for any *convergent* sequence $\{\rho_n\}$ satisfying (3.3).

First assume $\infty \ge \lambda > 0 > \eta \ge -\infty$ and $\{\rho_n\}$ satisfies (3.3) with $\rho_n \to \rho^*$. Let $\eta^* = -\eta > 0$ so that $R(\lambda, \eta) = (\Sigma v_i^\lambda/p)^{1/\lambda}[(\Sigma 1/v_i^{\eta^*})/p]^{1/\eta^*}$. It is easily verified that the range of R is $[1, \infty)$. Thus the test rejects when $R(\lambda, \eta) > k$ for $k \ge 1$. Let $w_1, \dots, w_p$ be independent χ^2 variables with n degrees of freedom so that $R(\lambda, \eta)$ is distributed as $[\Sigma(\rho_{in}w_i)^\lambda/p]^{1/\lambda}[\Sigma(1/\rho_{in}w_i)^{\eta^*}/p]^{1/\eta^*}$, when $\rho = \rho_n$. Since $\rho_n \to \rho^*$ and $\Sigma_{i=1}^{p}\rho_i^* = 1$,

$$\left\{\sum_{i=1}^{p}(\rho_{in}w_i)^\lambda/p\right\}^{1/\lambda} \to_{\mathscr{D}} \left\{\sum_{i=1}^{p}(\rho_i^* w_i)^\lambda/p\right\}^{1/\lambda},$$

which has a continuous distribution on $(0, \infty)$. Also, since $m_n \to 0$,

$$\left\{\sum_{i=1}^{p} (1/\rho_{in} w_i)^{\eta^*}/p\right\}^{1/\eta^*}$$

converges in probability to ∞. Thus $R(\lambda, \eta)$ converges in probability to ∞ showing that (3.4) holds.

When $\lambda > \eta = 0$, $R(\lambda, \eta)$ is distributed as

$$\frac{\left(\Sigma(\rho_{in} w_i)^{\lambda}/p\right)^{1/\lambda}}{\left(\prod_{i=1}^{p} w_i\right)^{1/p}\left(\prod_{i=1}^{p}\rho_{in}\right)^{1/p}}$$

and when $\lambda = 0 > \eta$, $R(\lambda, \eta)$ is distributed as

$$\left(\prod w_i\right)^{1/p}\left\{\frac{1}{p}\Sigma\left(1/\left[\rho_{in}^{(1-1/p)} w_i\right]^{\eta^*}\right\}^{1/\eta^*}$$

with $\eta^* = -\eta$. For either case, $R(\lambda, \eta) \to \infty$ as $m_n \to 0$, hence the test is PC.

The test $T(0,0)$ rejects if

$$B(v) = \sum_{i=1}^{p}\left(\log v_i - \left(\sum_{i=1}^{p}\log v_i\right)/p\right)^2 > C \tag{3.10}$$

for $0 \le C < \infty$. Thus $B(v)$ is distributed as

$$\sum_{i=1}^{p}\left(\left(\log w_i - \sum_{i=1}^{p}\log w_i/p\right) - \left(\log \rho_{in} - \sum \log \rho_{in}/p\right)\right)^2.$$

As $m_n \to 0$, $\Sigma(\log \rho_{in} - \Sigma \log \rho_{in}/p)^2 \to \infty$ since $\Sigma \rho_{in} = 1$, hence $B(v) \to \infty$, proving that the test is PC.

Next suppose that $\infty \ge \lambda > \eta > 0$. Then $R(\lambda, \eta)$ is distributed as

$$\left\{\left[\sum_{i=1}^{p}(\rho_{in} w_i)^{\lambda}/p\right]^{1/\lambda}\right\}\Big/\left\{\left[\sum_{i=1}^{p}(\rho_{in} w_i)^{\eta}/p\right]^{1/\eta}\right\}. \tag{3.11}$$

As $\rho_n \to \rho^*$,

$$R(\lambda, \eta) \to \left\{\left[\Sigma(\rho_i^* w_i)^{\lambda}/p\right]^{1/\lambda}\right\}\Big/\left\{\left[\Sigma(\rho_i^* w_i)^{\eta}/p\right]^{1/\eta}\right\} \equiv U. \tag{3.12}$$

Let r equal the number of nonzero components of ρ^*. Then the random variable

$$\left\{\left[(1/r)\Sigma(\rho_i^* w_i)^{\lambda}\right]^{1/\lambda}\Big/\left[(1/r)\Sigma(\rho_i^* w_i)^{\eta}\right]^{1/\eta}\right\} \tag{3.13}$$

has range $[1, r^{(1/\eta)-(1/\lambda)}]$. Hence the random variable U has range $[(p/r)^{(1/\eta)-(1/\lambda)}, p^{(1/\eta)-(1/\lambda)}]$. Thus $P(U > k) = 1$ if and only if $k \le (p/r)^{(1/\eta)-(1/\lambda)}$. Therefore the test with rejection region $\{R(\lambda, \eta) > k\}$ is PC if and only if $P(U > k) = 1$ for all $r = 1, 2, \ldots, (p-1)$, which implies that $k \le [p/(p-1)]^{(1/\eta)-(1/\lambda)}$. This completes the part of the theorem when $\infty \ge \lambda > \eta > 0$. For $0 > \lambda > \eta \ge -\infty$ the proof is similar. □

Theorem 3.3 shows that when all $\nu_i = n$, the LRT ($T(1,0)$), Hartley's test ($T(\infty, -\infty)$) and the test $T(0,0)$ are PC for all levels $\alpha > 0$. It can be similarly shown that the LRT and Hartley's test are PC when the ν_i's are unequal as well. We also have that Cochran's test is *not* PC if the cutoff point B in (2.16) exceeds $1/(p-1)$, and the $T(2,1)$ test is not PC if its cutoff point exceeds $\sqrt{p/(p-1)}$.

To get some idea about the significance levels for which the tests $T(\lambda, \eta)$ are PC we approximate α^*. For large n,

$$(np)/(\lambda - \eta)\log R(\lambda, \eta) \tag{3.14}$$

is approximately distributed as χ^2_{p-1} so that

$$\alpha^* \cong P\left\{\chi^2_{p-1} \geq (np/\eta\lambda)\log(p/(p-1))\right\}. \tag{3.15}$$

As $n \to \infty$, $\alpha^* \to 0$, so for large enough n the test will be PC for the usual significance levels. Let us consider some specific examples. Suppose $p = 3$, $n = 10$, $(\lambda, \eta) = (2, 1)$, so that we are considering the locally best permutation invariant unbiased test. The term in brackets on the right-hand side of the inequality in (3.17) is 6.08 so $\alpha^* \cong 0.05$. On the other hand if $p = 5$, $n = 5$, $(\lambda, \eta) = (2, 1)$, $\alpha^* \cong 0.7$.

REMARK 3.4. The assumption that v_i/σ_i^2 are independent chi-square variables could be relaxed in Theorem 3.3. The following more general assumptions would suffice: The variables $v_1, v_2, \dots, v_p$ are independent with density of v_i denoted by $f(v_i, \sigma_i^2) = (1/\sigma_i^2)f(v_i/\sigma_i^2)$. That is, σ_i^2 is a scale parameter. The null and alternative hypotheses are as before. The density f must satisfy the following property: Let $g_{\rho_0}(z)$ be the density of the maximal invariant z under H_0. Let $g_{\rho_n}(z)$ be a sequence of densities of z under alternatives ρ_n, where $\rho_n \to \rho^*$. Then the KL distance $I(\rho_0, \rho_n)$, defined in terms of g_{ρ_0} and g_{ρ_n} tends to ∞ if and only if at least one element of ρ_n tends to 0.

REMARK 3.5. There are many nonparametric tests for homogeneity of variances in the one-way fixed analysis of variance model. For such a model one observes x_{ij}, $i = 1, 2, \dots, p$; $j = 1, 2, \dots, n+1$, where x_{ij} has density $f(x_{ij}; \mu_i, \sigma_i) = (1/\sigma_i)f((x_{ij} - \mu_i)/\sigma_i)$, $\operatorname{cov}(x_{ij}, x_{i'j'}) = 0$ unless $i = i'$, $j = j'$, $E((x_{ij} - \mu_i)/\sigma_i)^4 < \infty$. In nonparametric models one does not wish to make further assumptions regarding f. Motivated by Lemma 3.2 one could define parameter consistency of a scale invariant test in such cases as follows: Suppose ρ_n is a sequence of alternatives for which $m_n = \min \rho_{in} \to 0$. A scale invariant test φ is parameter consistent if $\beta_\varphi(\rho_n) \to 1$ for such a sequence.

For this definition it can be shown that the three nonparametric tests studied by Layard (1973) are PC. Brown and Forsythe (1974) study additional nonparametric tests for homogeneity of variances. It can be shown that the test based on W_0 [see Brown and Forsythe (1974), page 364] is not PC, although a modification, replacing their z_{ij} by $\log z_{ij}$, is PC.

APPENDIX

PROOF OF THEOREM 2.1. We will use results from Brown and Marden (BM) (1989). In that paper they define the class of tests Φ in BM Theorem 2.4 which contains all tests of the form

$$(\text{A.1}) \qquad \phi(x) = \begin{cases} 1, & \text{if } d(x; \lambda, M, J) > c, \\ 0, & \text{if } d(x; \lambda, M, J) < c \text{ (almost everywhere)}, \end{cases}$$

for (λ, M, J, c) satisfying (2.5) and

$$(\text{A.2}) \qquad |d(x; \lambda, M, J)| < \infty \quad \forall x.$$

[See Example 4.4 in BM to verify that (2.5) gives the correct range for (λ, M, J, c).] We will use BM Theorem 2.4 to show that Φ is an essentially complete class for our problem and BM Lemma 3.2 to show that Φ is in fact minimal complete. We will then argue that Φ is the class given in Theorem 2.1.

To apply BM Theorem 2.4, we need to verify BM Assumptions 2.1–2.3. The function corresponding to the one in BM (2.1) is $R_\theta^*(x)$ of (2.4) above. BM Assumptions 2.1 and 2.2 are straightforward, except for BM (2.2) in Assumption 2.2. We will prove a stronger result, i.e., that for some $\alpha > 0$ and for all x, there exists $k < \infty$ such that

$$(\text{A.3}) \qquad \sup_{\theta \in \Theta(\alpha)} \left| \frac{R_\theta^*(x) - 1 + \beta \Sigma_{i=1}^{p-1} x_i \theta_i - \beta(\beta + 1)(\Sigma x_i \theta_i)^2/2}{\|\theta\|^2} \right| \leq k,$$

where $\Theta(\alpha) = \{\theta | 0 < \|\theta\| \leq \alpha\}$. To prove (A.3), note that since $(1 + z)^{-\beta}$ has a continuous second derivative at $z = 0$, for small enough $\varepsilon > 0$, there exists $k < \infty$ such that

$$(\text{A.4}) \qquad \sup_{|z| < \varepsilon} \left| \frac{(1 + z)^{-\beta} - 1 + \beta z - \beta(\beta + 1)z^2/2}{z^2} \right| = k.$$

Since $\mathscr{X}$ is bounded, if we take $\alpha > 0$ small enough such that

$$\sup_{\|\theta\| \leq \alpha} |\textstyle\sum \theta_i x_i| \leq \varepsilon,$$

(A.4) will imply (A.3). BM Assumption 2.3 is immediate since Θ is bounded, hence $\mathscr{C}$ in the assumption consists of just $\mathscr{X}$ and the empty set. Hence Φ is essentially complete.

We now verify BM Assumption 3.1 needed for BM Lemma 4.2. The assumption has four parts. Parts (i) and (iii) are immediate sinced $\mathscr{C} = \{\mathscr{X}, \phi\}$. Part (iv) requires that for ϕ in Φ,

$$(\text{A.5}) \qquad P_0(\{x | d(x; \lambda, M, J) = c\}) = 0,$$

where P_0 is the distribution of x under the null hypothesis. To check (A.5), refer to $d(x; \lambda, M, J)$ of (2.4). First note that if $(M, J) \neq (0, 0)$, then d is strictly convex in at least one x_i, since $x'Mx$ and $R_\theta^*(x) - 1 + \beta x'\theta$ are strictly convex in at least one x_i. If $(M, \pi) = (0, 0)$ and $\lambda \neq 0$, then d is either strictly increasing in some x_i (if $\lambda_i > 0$) or strictly decreasing in some x_i (if $\lambda_i < 0$). Thus if $(\lambda, M, J) \neq (0,0,0)$, $P_0\{x\colon d(x) = c\} = 0$, since the distribution of x is absolutely continuous with respect to Lebesgue measure. If $(\lambda, M, J) = (0,0,0)$, then $c \neq 0$ by (2.5). Hence the set in (A.5) is empty, proving (A.5).

Finally we need to verify part (ii). Toward this end, take α from (A.3) sufficiently small so that $\overline{\Theta}(\alpha)$, the closure of $\Theta(\alpha)$, is contained in Θ. Also, α should be chosen so that J(boundary $(\Theta(\alpha))) = 0$. Now rewrite the last expression in (2.4) as

$$d(x) - c = \lambda^{*\prime}x + x'Mx + \int_{\Theta(\alpha)} h(x,\theta)J_0(d\theta) + \int_{\overline{\Theta} - \overline{\Theta}(\alpha)} R_\theta^*(x)J_1(d\theta) - c^*,$$

where c^* is a constant, $\lambda^* = \lambda + \int_{\overline{\Theta} - \overline{\Theta}(\alpha)}(\beta\theta/\|\theta\|^2)J(d\theta)$, $J_0 = JI_{\overline{\Theta}(\alpha)-\{0\}}$ and $J_1 = (J/\|\theta\|^2)I_{\overline{\Theta}-\overline{\Theta}(\alpha)}$. From Lemma 2.5 of BM we can find a sequence $\{J_{0i}\}$ of measures on $(\Theta(\alpha))$ such that for each x,

$$a_i(x) \equiv \int_{\Theta(\alpha)} [f_\theta(x)/f_0(x)]\, J_{0i}(d\theta) - J_{0i}\{0\}$$

converges to

$$a(x) = \lambda^{*\prime}x + x'Mx + \int_{\Theta(\alpha)} h(x,\theta)J_0(d\theta) - c^*. \tag{A.6}$$

Note that BM (2.51) implies that $H_{0i}\{\Theta(\alpha)\} \to \operatorname{tr} M$, hence $\{H_{0i}(\Theta(\alpha))\}$ is a bounded sequence. Thus by (A.3) and the boundedness of $\mathscr{X}$ there exists a $\overline{K} < \infty$ such that

$$|a_i(x)| \le \overline{K} \quad \text{for every } i, x \tag{A.7}$$

and

$$|a(x)| \le \overline{K} \quad \text{for every } x. \tag{A.8}$$

Let J_{1i} be the measure on $[i/(i+1)]\overline{\Theta}$, where this latter set consists of the points $[i/(i+1)]\theta$ for all $\theta \in \overline{\Theta}$, defined by $J_{1i}(\Gamma) = J_1([i/(i+1)]^{-1}\Gamma)$ for $\Gamma \subseteq [i/(i+1)]\overline{\Theta}$. Then

$$\begin{aligned} A_i(x) \equiv \int_\Theta R_\theta^*(x)J_{1i}(d\theta) &= \int_{\Theta - \overline{\Theta}(\alpha)} R^*_{[i/(i+1)]\theta}(x)J_1(d\theta) \\ &= \int_{\overline{\Theta} - \overline{\Theta}(\alpha)} R_\theta^*([i/(i+1)]x)J_1(d\theta), \end{aligned} \tag{A.9}$$

since $R_\theta^*(x) = (1 + \theta'x)^{-\beta}$. Furthermore, since R_θ^* is bounded and continuous in θ for each fixed x, the first line of (A.9) shows that $A_i(x) \to A(x)$, where $A(x) \equiv \int_{\overline{\Theta} - \overline{\Theta}(\alpha)} R_\theta^*(x)J_1(d\theta)$.

Now define the sequence $\{\phi_i\}$, to be used as in BM Assumption 3.1(ii), by

$$\phi_i(x) = \begin{cases} 1, & \text{if } a_i(x) + A_i(x) > 0, \\ 0, & \text{if } a_i(x) + A_i(x) \le 0, \end{cases}$$

and note that

$$\phi(x) = \begin{cases} 1, & \text{if } a(x) + A(x) > 0, \\ 0, & \text{if } a(x) + A(x) \le 0, \end{cases}$$

almost everywhere by (A.5). To complete the verification of (ii), we need to show that BM (3.5) is true. That is we need to show that

$$\lim_{i \to \infty} \int (\phi_i(x) - \phi(x))(a_i(x) + A_i(x)) f_0(x)\, dx = 0. \tag{A.10}$$

First note

$$A_i(0) = A(0) = J_1(\overline{\Theta} - \overline{\Theta}(\alpha)) < \infty, \tag{A.11}$$

since J_1 is finite. Next note that

$$\text{if } t \ge J_1(\overline{\Theta} - \overline{\Theta}(\alpha)), \text{ then } A(x) \le t \text{ implies } A_i(x) \le t. \tag{A.12}$$

To see this, recognize that $D = \{x | A(x) \le t\}$ is convex since $R_\theta^*(x)$ is convex. Also, since $t \ge J_1(\overline{\Theta} - \overline{\Theta}(\alpha))$, (A.6) implies $0 \in D$. Thus if $x \in D$, $[i/(i+1)]x \in D$, i.e., $A([i/(i+1)]x) \le t$, but $A_i(x) = A([i/(i+1)]x)$.

Third, note that

$$\begin{aligned} \{&\text{if } N > \overline{K}, \text{ then } A(x) > N \text{ implies } \varphi(x) = 1 \\ &\qquad\text{and } A_i(x) > N \text{ implies } \varphi_i(x) = 1\}, \end{aligned} \tag{A.13}$$

where $\overline{K}$ is given in (A.7).

Now take $N > \max[\overline{K}, J_1(\overline{\Theta} - \overline{\Theta}(\alpha)]$. Then

$$\begin{aligned} &\int (\phi_i(x) - \phi(x)) A_i(x) f_0(x)\, dx \\ &= \int_{A_i(x) \le N} (\phi_i(x) - \phi(x)) A_i(x) f_0(x)\, dx \\ &\quad + \int_{A_i(x) > N} (\phi_i(x) - \phi(x)) A_i(x) f_0(x)\, dx. \end{aligned} \tag{A.14}$$

By (A.12) $A_i(x) > N$ implies $A(x) > N$ and (A.13), $A_i(x) > N$ implies $\phi_i(x) = \phi(x) = 1$. Hence the left-hand side of (A.10) can be written as

$$\lim_{i \to \infty} \int (\phi_i(x) - \phi(x)) \left[a_i(x) + A_i(x) I_{\{A_i(x) \le N\}}(x) \right] f_0(x)\, dx. \tag{A.15}$$

From (A.7) and (A.11–A.14), the integrand in (A.15) is bounded and the dominated convergence theorem implies (A.10). Thus Lemma 4.2 can be applied, which with Theorem 2.4 proves that Φ is minimal complete. See the first paragraph of BM Section 3.

Finally, we argue that Φ is the class in Theorem 2.1. (A.8) and the boundedness of J_1 show that for any x, $|a(x) + A(x)|$ is bounded. Hence $|d(x; \lambda, M, J)|$ is bounded. Thus (A.2) is a superfluous condition. Also, (A.5) shows that ϕ in (A.1) equals ϕ in (2.5) almost everywhere. Hence the result. □

REFERENCES

ANDERSON, T. W. and PERLMAN, M. (1988). Consistency of invariant tests for the multivariate analysis of variance. In *Proc. of the Second International Tampere Conference in Statistics* (T. Pukkila and S. Puntanen, eds.). Dept. Mathematical Sciences, Univ. Tampere, Finland.

BARTLETT, M. S. and KENDALL, D. G. (1946). The statistical analysis of variance—heterogeneity and the logarithmic transformation. *J. Roy. Statist. Soc. Suppl.* **8** 128–138.

BECHHOFER, R. E. (1960). A multiplicative model for analyzing variances which are affected by several factors. *J. Amer. Statist. Assoc.* **55** 245–264.

BOX, G. E. P. (1953). Nonnormality and tests on variances. *Biometrika* **40** 318–335.

BROWN, L. D. and MARDEN, J. I. (1989). Complete class results for hypothesis testing problems with simple null hypotheses. *Ann. Statist.* **17** 209–235.

BROWN, M. B. and FORSYTHE, A. B. (1974). Robust tests for equality of variances. *J. Amer. Statist. Assoc.* **69** 364–367.

COCHRAN, W. G. (1941). The distribution of the largest of a set of estimated variances as a fraction of their total. *Ann. Eugenics* **11** 47–52.

COHEN, A. and SACKROWITZ, H. B. (1987). Unbiasedness of tests for homogeneity. *Ann. Statist.* **15** 805–816.

COHEN, A. and STRAWDERMAN, W. E. (1971). Unbiasedness of tests for homogeneity of variances. *Ann. Math. Statist.* **42** 355–360.

COHEN, A., SACKROWITZ, H. B. and STRAWDERMAN, W. E. (1985). Multivariate locally most powerful unbiased tests. In *Multivariate Analysis VI* (P. R. Krishnaiah, ed.) 121–144. North-Holland, Amsterdam.

HARTLEY, H. O. (1950). The maximum F ratio as a shortcut test for homogeneity of variances. *Biometrika* **37** 308–312.

KIEFER, J. and SCHWARTZ, R. (1965). Admissible Bayes character of T^2, R^2, and other fully invariant tests for classical multivariate normal problems. *Ann. Math. Statist.* **36** 747–770.

LAUE, R. V. (1965). Heterogeneity of variances in a one-way and two-way classification model. Ph.D. dissertation, Rutgers Univ.

LAYARD, M. W. J. (1973). Robust large-sample tests for homogeneity of variances. *J. Amer. Statist. Assoc.* **68** 195–198.

MATTHES, T. K. and TRUAX, D. R. (1967). Tests of composite hypotheses for the multivariate exponential family. *Ann. Math. Statist.* **38** 681–697.

SEBER, G. A. F. (1977). *Linear Regression Analysis*. Wiley, New York.

STEVENS, W. L. (1936). Heterogeneity of a set of variances. *J. Genetics* **33** 398–399.

DEPARTMENT OF STATISTICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903

DEPARTMENT OF STATISTICS
UNIVERSITY OF ILLINOIS
725 SOUTH WRIGHT STREET
CHAMPAIGN, ILLINOIS 61801