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SECOND ORDER AND L^p-COMPARISONS BETWEEN THE BOOTSTRAP AND EMPIRICAL EDGEWORTH EXPANSION METHODOLOGIES

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The bootstrap estimate of distribution functions of studentized statistics is shown to be more accurate than even the two-term empirical Edgeworth expansion, thus strengthening the claim of superiority of the bootstrap over the normal approximation method. The two methods are compared not only with respect to bounded bowl-shaped loss functions but also with respect to squared error loss and, more generally, in L^p-norms.

1. Introduction. Efron's bootstrap [Efron (1979)] often provides "better than normal" estimates of distribution functions of studentized statistics T_n. This has been proved by Singh (1981) [also see Bickel and Freedman (1980)] by deriving a two-term Edgeworth expansion of the Student's statistic under the empirical. One may use this last expansion, called here a two-term *E.E-expansion* (or, *empirical Edgeworth expansion*), as an alternative to the bootstrap. Since the errors of the two estimates are both $o(n^{-1/2})$ a.s., it is important to compare the $O(n^{-1})$ error terms of the two. Crucial to this and higher order comparisons is a result of Babu and Singh (1984), as extended in Bhattacharya (1987), according to which the bootstrap estimate differs from an $(s-1)$-term E.E-expansion $(s \geq 3)$ only by $o(n^{-(s-2)/2})$ a.s., if Cramér's condition holds and sufficiently many moments are finite. It is shown in Section 2 with the help of this result that the bootstrap outperforms the two-term E.E-expansion for classical statistics, if comparisons are made with respect to bounded bowl-shaped loss functions. In view of the importance of the squared error loss, in Section 3 we carry out the comparison in the L^p-norm $(1 \leq p < \infty)$ for the case of the Student's t, although the method is easily seen to generalize. The uniform integrability considerations for this last comparison are nontrivial, and several estimates of independent interest are obtained in the Appendix.

Certain other advantages enjoyed by the bootstrap over the E.E-expansion are mentioned in Hall (1986).

Finally, Qumsiyeh (1986) has recently shown that the three-term and higher order E.E-expansion are not in general inferior to the bootstrap.

2. Second order comparison with bounded loss. Let $\{Z_j\}$ be a sequence of k-dimensional i.i.d. random vectors. We will assume that the distribution of Z_1 satisfies *Cramér's condition*:

$$\limsup_{\|\xi\| \to \infty} |E \exp\{i\xi \cdot Z_1\}| < 1. \tag{2.1}$$

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Here $\|x\|$ and $x \cdot y$ denote Euclidean norm and inner product. Consider a real valued studentized statistic

$$T_n := \sqrt{n}H(\bar{Z}), \tag{2.2}$$

where H is an s-times continuously differentiable function on an open set containing EZ_1 and $H(EZ_1) = 0$. Then the distribution function of T_n has an $(s - 1)$-term *E-expansion* (or, *Edgeworth expansion*) with a density [Bhattacharya and Ghosh (1978)]

$$\psi_{s-2,n}(x) = \left(1 + \sum_{r=1}^{s-2} n^{-r/2} p_r(x)\right)\phi(x), \tag{2.3}$$

provided $E\|Z_1\|^s < \infty$. Here ϕ is the standard normal density and p_r is a polynomial of degree $3r$ whose coefficients are functions of moments of Z_1. The corresponding $(s - 1)$-term E.E-expansion has density

$$\tilde{\psi}_{s-2,n}(x) = \left(1 + \sum_{r=1}^{s-2} n^{-r/2} \tilde{p}_r(x)\right)\phi(x), \tag{2.4}$$

where $\tilde{p}_r$ is obtained by substituting sample moments for population moments in p_r.

Let $P^*(T_n^* \le x)$ denote the (bootstrap) distribution function of the resampled statistic T_n^* under the empirical. Then according to Babu and Singh (1984) and Bhattacharya (1987),

$$P^*(T_n^* \le x) = \int_{-\infty}^{x} \tilde{\psi}_{s-2,n}(y)\,dy + o(n^{-(s-2)/2}) \quad \text{a.s.} \tag{2.5}$$

Therefore, with $s = 4$ above, the error of estimation of the bootstrap estimate of $P(T_n \le x)$ is

$$\begin{aligned} &P^*(T_n^* \le x) - P(T_n \le x) \\ &\quad = n^{-1}\left[n^{1/2}\int_{-\infty}^{x} (\tilde{p}_1(y) - p_1(y))\phi(y)\,dy\right] \\ &\qquad + n^{-1}\left(\int_{-\infty}^{x} (\tilde{p}_2(y) - p_2(y))\phi(y)\,dy\right) + o(n^{-1}) \quad \text{a.s.} \\ &\quad = n^{-1}\left[n^{1/2}\int_{-\infty}^{x} (\tilde{p}_1(y) - p_1(y))\phi(y)\,dy\right] + o(n^{-1}) \quad \text{a.s.} \end{aligned} \tag{2.6}$$

On the other hand, the error of estimation of the two-term E.E-expansion is

$$\begin{aligned} \int_{-\infty}^{x} \tilde{\psi}_{1,n}(y)\,dy - P(T_n \le x) &= n^{-1}\left[n^{1/2}\int_{-\infty}^{x} (\tilde{p}_1(y) - p_1(y))\phi(y)\,dy\right] \\ &\quad - n^{-1}\int_{-\infty}^{x} p_2(y)\phi(y)\,dy + o(n^{-1}) \quad \text{a.s.} \end{aligned} \tag{2.7}$$

Now one has [see Bhattacharya and Ghosh (1978)]

$$\int_{-\infty}^{x} p_1(y)\phi(y)\,dy = (c_1 + c_2x^2)\phi(x), \tag{2.8}$$

where, on computing moments by the delta method, one can easily show that c_1, c_2 are polynomial functions of the derivatives of H of orders three and less at EZ_1 and of moments of Z_1 of orders *four* and less. Therefore, by a first order Taylor expansion around the true moments and using the classical central limit theorem [see Cramér (1946), page 367], one gets

$$n^{1/2}\int_{-\infty}^{x} (\tilde{p}_1(y) - p_1(y))\phi(y)\,dy \xrightarrow[\text{weakly}]{n\to\infty} N(0, \sigma_b^2(x)), \tag{2.9}$$

if $E\|Z_1\|^8 < \infty$ and H is four times continuously differentiable. Here $\sigma_b^2(x) = a(x)\phi^2(x)$, $a(x)$ being a polynomial of degree four. Also write

$$\int_{-\infty}^{x} p_2(y)\phi(y)\,dy = q_2(x)\phi(x), \tag{2.10}$$

where q_2 is a *fifth* degree polynomial in x. From (2.5)–(2.11) we arrive at the following theorem. For this assume studentization, i.e.,

$$H(EZ_1) = 0, \qquad (\operatorname{grad} H)(EZ_1) \cdot V(\operatorname{grad} H)(EZ_1) = 1, \tag{2.11}$$

where V is the *dispersion matrix* of Z_1.

THEOREM 2.1. *Assume* $E\|Z_1\|^8 < \infty$ *and Cramér's condition* (2.1). *If, in addition,* H *is a real valued function on* $\mathbb{R}^k$ *which is four times continuously differentiable in a neighborhood of* EZ_1 *and* (2.11) *holds then, as* $n \to \infty$,

$$\begin{gathered} n(P^*(T_n^* \le x) - P(T_n \le x)) \xrightarrow[\text{weakly}]{} N(0, \sigma_b^2(x)), \\ n\left(\int_{-\infty}^{x} \tilde{\psi}_{1,n}(y)\,dy - P(T_n \le x)\right) \xrightarrow[\text{weakly}]{} N(-q_2(x)\phi(x), \sigma_b^2(x)). \end{gathered} \tag{2.12}$$

Theorem 2.1 has the following immediate corollary [see Pfanzagl (1980), page 20 or Anderson (1955), page 172, Corollary 2]. Define a function L on $\mathbb{R}^p$ to be *bowl-shaped* if $\{x \in \mathbb{R}^p\colon L(x) \le c\}$ is convex for all c, *symmetric* if $L(x) = L(-x)$ for all x and a *loss function* if $L(0) = 0$, $L(x) > 0$, for all $x \ne 0$.

COROLLARY 2.2. *Under the hypotheses of Theorem* 2.1 *one has, for every symmetric, bowl-shaped, bounded loss function* L, *the inequality*

$$\begin{aligned} &\lim_{n\to\infty} EL(n[P^*(T_n^* \le x) - P(T_n \le x)]) \\ &\quad < \lim_{n\to\infty} EL\left(n\left[\int_{-\infty}^{x} \tilde{\psi}_{1,n}(y)\,dy - P(T_n \le x)\right]\right) \end{aligned} \tag{2.13}$$

for every x *such that* $q_2(x) \ne 0$. *If* $q_2(x) = 0$, *then equality holds in* (2.13).

By regarding the left sides of (2.12) as stochastic processes, indexed by $x \in \mathbb{R}^1$, one may assert their weak convergence (in sup norm) to rather trivial types of Gaussian processes, under the hypothesis of Corollary 2.2.

The above results immediately extend to vector valued statistics T_n.

3. L^p-comparisons. The calculations are carried out and results stated only for the Student's statistic in this section, for the sake of simplicity. The method can be extended to all statistics $\sqrt{n}\, H(\bar{Z})$ mentioned in Section 2.

Let $X_1, X_2, \dots, X_n, \dots,$ be independent random variables with common distribution F, mean μ and variance $\sigma^2 > 0$. Conditionally given $X_1, \dots, X_n$, let $X_1^*, \dots, X_n^*$ be n independent random variables with common distribution F_n, where F_n is the *empirical distribution* of $X_1, \dots, X_n$.

We will write P^* and E^* to denote conditional probabilities and expectations (given $X_1, \dots, X_n$) associated with $X_1^*, \dots, X_n^*$, while P, E will denote the corresponding unconditional quantities. A *superscript* * indicates a statistic obtained by replacing unstarred observations by starred ones in statistics such as $\bar{X}$, $s_n^2 = \Sigma(X_i - \bar{X})^2/n$. A *wiggle* $\tilde{}$ on top indicates replacement of F by F_n in a functional $T(F)$. If a statistic also involves population moments, in the starred version these are replaced by corresponding sample moments.

Define the statistics

$$T_n = \sqrt{n}\,(\bar{X} - \mu)/s_n, \qquad T_n^* = \sqrt{n}\,(\bar{X}^* - \bar{X})/s_n^*. \tag{3.1}$$

If one writes

$$\begin{aligned} Z_i &= \left(X_i, X_i^2\right), \qquad Z_i^* = \left(X_i^*, X_i^{*2}\right), \qquad \bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i, \\ \bar{Z}^* &= \frac{1}{n}\sum_{i=1}^{n} Z_i^*, \qquad \underset{\sim}{\mu} = EZ_i = \left(\mu, \sigma^2 + \mu^2\right), \end{aligned} \tag{3.2}$$

then $T_n = n^{1/2}H(\bar{Z})$, $T_n^* = n^{1/2}\tilde{H}(\bar{Z}^*)$ where for $z = (z^{(1)}, z^{(2)}) \in \mathbb{R}^2$,

$$H(z^{(1)}, z^{(2)}) = \left(z^{(1)} - \mu\right)/\left(z^{(2)} - \left(z^{(1)}\right)^2\right)^{1/2}. \tag{3.3}$$

Denote by V the dispersion matrix of (X_1, X_1^2) and by $\tilde{V}$ the dispersion matrix of (X_1^*, X_1^{*2}) under F_n.

Also write

$$\eta_s = E\left(\left(Z_1 - \underset{\sim}{\mu}\right)'V^{-1}\left(Z_1 - \underset{\sim}{\mu}\right)\right)^{s/2}. \tag{3.4}$$

Let $\gamma_{s-2,n}$ denote the density of the $(s-1)$-term E-expansion of $n^{1/2}(\bar{Z} - \underset{\sim}{\mu})$ and $\tilde{\gamma}_{s-2,n}$ that of $n^{1/2}(\bar{Z}^* - \bar{Z})$ under P^*. Also [see, e.g., Qumsiyeh (1986)],

$$\begin{aligned} \int_{-\infty}^{x} \psi_{1,n}(y)\,dy &= \Phi(x) + n^{-1/2}\left(\frac{\mu_3}{6\sigma^3}\right)(2x^2 + 1)\phi(x), \\ \int_{-\infty}^{x} \psi_{2,n}(y)\,dy &= \int_{-\infty}^{x} \psi_{1,n}(y)\,dy + n^{-1}q_2(x)\phi(x), \end{aligned} \tag{3.5}$$

where q_2 is a polynomial of degree five whose coefficients are constant multiples of 1, μ_3^2/μ_2^3, μ_4/μ_2^2. Here $\mu_r = E(X_1 - \mu)^r$ and $\Phi(x)$ is the standard normal distribution function.

By a result of Babu and Singh (1984) one has

$$\sup_{A \in \mathscr{A}} \left| P^*\left(\sqrt{n}\,(\bar{Z}^* - \bar{Z}) \in A\right) - \int_A \tilde{\gamma}_{s-2,n}(z)\,dz \right| = o\left(n^{-(s/2)/2}\right), \quad \text{a.s.}, \tag{3.6}$$

provided Cramér's condition (2.1) holds and $E\|Z_1\|^s < \infty$. Here $\mathscr{A}$ is any class of Borel subsets of $\mathbb{R}^2$ satisfying

$$\sup_{A \in \mathscr{A}} \int_{(\partial A)^\varepsilon} \phi_I(z)\,dz = O(\varepsilon^a) \quad \text{for some } a > 0,\ \varepsilon \downarrow 0. \tag{3.7}$$

We have denoted by ∂A the *boundary* of the set A and by $(\partial A)^\varepsilon$ the set of all points at a distance less than ε form ∂A. Also, ϕ_I is the standard normal density on $\mathbb{R}^2$. The result (3.6) has been extended to smooth functions of $\bar{Z}$ in Bhattacharya (1987), and as a special case one has

$$\sup_{B \in \mathscr{B}} \left| P^*(T_n^* \in B) - \int_B \tilde{\psi}_{s-2,n}(x)\,dx \right| = o\left(n^{-(s-2)/2}\right) \quad \text{a.s.}, \qquad \text{as } n \to \infty. \tag{3.8}$$

Here $\mathscr{B}$ is any class of Borel subsets of $\mathbb{R}^1$ satisfying

$$\sup_{B \in \mathscr{B}} \int_{(\partial B)^\varepsilon} \phi(x)\,dx = O(\varepsilon^a) \quad \text{for some } a > 0,\ \varepsilon \downarrow 0. \tag{3.9}$$

Our first result provides L^p-analogues of (3.6) and (3.8). We write, for any random variable Y,

$$\|Y\|_p = \left(E|Y|^p\right)^{1/p}, \qquad 1 \le p < \infty. \tag{3.10}$$

PROPOSITION 3.1. *Assume* (2.1) *and* $E|X_1|^{s^2} < \infty$. *Then, for* $1 \le p < \infty$,

$$\sup_{A \in \mathscr{A}} \left\| P^*\left(\sqrt{n}\,(\bar{Z}^* - \bar{Z}) \in A\right) - \int_A \tilde{\gamma}_{s-2,n}(z)\,dz \right\|_p = o\left(n^{-(s-2)/2}\right), \tag{3.11}$$

for every class $\mathscr{A}$ *of Borel subsets of* $\mathbb{R}^2$ *satisfying* (3.7). *Also, for every class* $\mathscr{B}$ *of Borel subsets of* $\mathbb{R}^1$ *satisfying* (3.9) *one has*

$$\sup_{B \in \mathscr{B}} \left\| P^*(T_n^* \in B) - \int_B \tilde{\psi}_{s-2,n}(x)\,dx \right\|_p = o\left(n^{-(s-2)/2}\right). \tag{3.12}$$

The proof of this proposition is given in the Appendix.

Proposition 3.1 enables us to prove the main result of this section.

THEOREM 3.2. *Let* $1 \le p < \infty$. *Assume* (2.1) *and* $E|X_1|^{12p+14} < \infty$. *Then one has the strict inequality*

$$\begin{aligned} &\lim_{n \to \infty} n \| P^*(T_n^* \le x) - P(T_n \le x) \|_p \\ &\quad < \lim_{n \to \infty} n \left\| \int_{-\infty}^x \tilde{\psi}_{1,n}(y)\,dy - P(T_n \le x) \right\|_p \end{aligned} \tag{3.13}$$

for all points x except the roots of the polynomial equation $q_2(x) = 0$. At the 0's of $q_2(x)$ equality holds in (3.13).

REMARK 3.2.1. If $s_n^2 = 0$, the expressions within $\|\cdot\|_p$ in (3.13) may be given arbitrary values between 0 and 1. A similar consideration applies to (3.11) if $\tilde{V}$ is singular.

We need the following lemma whose proof is given in the Appendix.

LEMMA 3.3. (a) *If $E|X_1|^{2rm(p+1)+2} < \infty$ for some integers r, m with $r \geq 2$ and $m \geq 1$, and for some $p \in [1, \infty)$, then*

$$E\left[\left|\frac{\tilde{\mu}_r^m}{s_n^{rm}} - \frac{\mu_r^m}{\sigma^{rm}}\right|^p \cdot 1_{\{s_n^2 > 0\}}\right] \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

(b) *If $E|X_1|^{12p+14} < \infty$ for some $p \in [1, \infty)$, then for all $c \in \mathbb{R}^1$, $d > 0$,*

$$\lim_{n \to \infty} E\left|d\sqrt{n}\left(\frac{\tilde{\mu}_3}{s_n^3} - \frac{\mu_3}{\sigma^3}\right) \cdot 1_{\{s_n^2 > 0\}} - c\right|^p = d^p \int \left|y - \frac{c}{d}\right|^p d\Phi_{\delta^2}(y) \tag{3.15}$$

for some $\delta^2 \geq 0$. Here Φ_{δ^2} denotes the normal distribution with mean 0 and variance δ^2.

PROOF OF THEOREM 3.2. By Theorem 2 in Bhattacharya and Ghosh (1978), one may replace $P(T_n \leq x)$ by $\int_{-\infty}^x \psi_{2,n}(y)\,dy$. In view of (3.12),

$$\begin{aligned} &\|P^*(T_n^* \leq x) - P(T_n \leq x)\|_p \\ &\quad = \left\|\int_{-\infty}^x \tilde{\psi}_{2,n}(y)\,dy - \int_{-\infty}^x \psi_{2,n}(y)\,dy\right\|_p + o(n^{-1}). \end{aligned} \tag{3.16}$$

By Lemma 3.3(a), (b) it follows that [see (3.5)]

$$\begin{aligned} &n\left\|\int_{-\infty}^x \tilde{\psi}_{2,n}(y)\,dy - \int_{-\infty}^x \psi_{2,n}(y)\,dy\right\|_p \\ &\quad = \frac{1}{6}\left(\int_{-\infty}^x |y|^p\,d\Phi_{\delta^2}(y)\right)^{1/p}(2x^2+1)\phi(x) + o(1). \end{aligned} \tag{3.17}$$

Similarly,

$$\begin{aligned} &n\left\|\int_{-\infty}^x \tilde{\psi}_{1,n}(y)\,dy - P(T_n \leq x)\right\|_p \\ &\quad = n\left\|\int_{-\infty}^x \tilde{\psi}_{1,n}(y)\,dy - \int_{-\infty}^x \psi_{2,n}(y)\,dy\right\|_p + o(1) \\ &\quad = d\left(\int_{-\infty}^{\infty}\left|y - \frac{c}{d}\right|^p d\Phi_{\delta^2}(y)\right)^{1/p}\phi(x) + o(1), \end{aligned} \tag{3.18}$$

where $d = (2x^2+1)/6$, $c = 6q_2(x)$. □

APPENDIX

To prove Proposition 3.1 we begin with some lemmas. Below the *constants* c_r $(r = 1, 2, \dots)$ are positive nonrandom quantities which do not depend on n or arguments x, y, z, ξ, etc.

First, the following lemma follows easily from standard moderate deviation estimates [Bhattacharya and Ranga Rao (1986), Corollary 17.12] for sample moments $\Sigma X_i^r/n$.

LEMMA A.1. *If V is nonsingular and $E|X_1|^{s^2} < \infty$ for some $s \geq 6$, then there exists a constant c_1 such that*

$$P(\tilde{\eta}_s > c_1) = o(n^{-(s-2)/2}). \tag{A.1}$$

Also,

$$P\left(\|\tilde{V}^{-1} - V^{-1}\| > \tfrac{1}{2}\|V^{-1}\|\right) = o(n^{-(s-2)/2}). \tag{A.2}$$

LEMMA A.2. *Let $r \geq 2$, $k \geq 1$, be integers.*

(a) *Assume that $EX_1^{2(rk+1)} < \infty$. Then writing $\beta_r = E|X_1 - \mu|^r$,*

$$\limsup_{n\to\infty} E\left(\left(\frac{\tilde{\beta}_r}{s_n^r}\right)^k \cdot 1_{\{s_n^2 > 0\}}\right) < \infty. \tag{A.3}$$

(b) *If $\tilde{V}$ is nonsingular, then*

$$n^{-(r-2)/2}\tilde{\eta}_r \leq 2^{r/2}. \tag{A.4}$$

PROOF. (a) One has

$$\begin{aligned} \frac{\tilde{\beta}_r}{s_n^r} &= \frac{(1/n)\Sigma_{i=1}^n |X_i - \overline{X}|^r}{\left((1/n)\Sigma_{i=1}^n (X_i - \overline{X})^2\right)^{r/2}} \\ &\leq \frac{(1/n)\left(\Sigma_{i=1}^n (X_i - \overline{X})^2\right)^{r/2}}{n^{-r/2}\left(\Sigma_{i=1}^n (X_i - \overline{X})^2\right)^{r/2}} = n^{(r/2)-1}. \end{aligned} \tag{A.5}$$

Next,

$$E\left(\left(\frac{\tilde{\beta}_r}{s_n^r}\right)^k \cdot 1_{\{s_n^2 > 0\}}\right) = E\left\{\left(\frac{\tilde{\beta}_r}{s_n^r}\right)^k \cdot 1_{A_1}\right\} + E\left\{\left(\frac{\tilde{\beta}_r}{s_n^r}\right)^k \cdot 1_{A_2}\right\} = J_1 + J_2, \tag{A.6}$$

say, where

$$A_1 = \left\{0 < \sum_{i=1}^n (X_i - \overline{X})^2 < \sigma^2\right\}, \qquad A_2 = \left\{\sum_{i=1}^n (X_i - \overline{X})^2 \geq \sigma^2\right\}. \tag{A.7}$$

Using the assumption $EX_1^{2(rk+1)} < \infty$ it follows by moderate deviations that

$$P(A_1) = o(n^{-rk}). \tag{A.8}$$

By (A.5) and (A.8), $\limsup J_1 = 0$. Also,

$$J_2 \le \left(E\tilde{\beta}_r^{2k}\right)^{12}\left(E\left(s_n^{-2rk}\cdot 1_{A_2}\right)\right)^{1/2} \le c_2 E\left(s_n^{-2rk}\cdot 1_{A_2}\right)^{1/2}. \tag{A.9}$$

Furthermore, writing $G_n(t) = P(\sigma^2/s_n^2 \le t)$,

$$E\left(s_n^{-2rk}\cdot 1_{A_2}\right) = \sigma^{-2rk}\int_0^n t^{rk}\,dG_n(t) \le \sigma^{-2rk}rk\int_0^n t^{rk-1}(1-G_n(t))\,dt. \tag{A.10}$$

For $2 \le t \le n$ one has, by a moderate deviation estimate [see (A.8)], $1 - G_n(t) = o(n^{-rk})$. Hence $\limsup J_2 < \infty$.

(b) One has

$$\begin{aligned}\tilde{\eta}_r &= \frac{1}{n}\sum_{i=1}^n \left((Z_i - \bar{Z})'\tilde{V}^{-1}(Z_i - \bar{Z})\right)^{r/2}\\ &\le \frac{1}{n}\left(\sum_{i=1}^n (Z_i - \bar{Z})'\tilde{V}^{-1}(Z_i - \bar{Z})\right)^{r/2}\\ &= n^{r/2-1}\tilde{\eta}_2^{r/2} = n^{r/2-1}2^{r/2},\end{aligned} \tag{A.11}$$

since $\tilde{\eta}_2 = \text{trace of } \tilde{V}^{-1}\tilde{V} = 2$. □

The next lemma follows from Lemma A.2(b) and the nature of the algebraic dependence of the coefficients of the polynomials in $\gamma_{s-2,n}$ on cumulants [see Bhattacharya and Ranga Rao (1986), Lemma 6.3 and page 52].

LEMMA A.3. *On the set $\{\omega: V_n(\omega)$ nonsingular$\}$ one has*

$$\int_{\mathbb{R}^2} |\tilde{\gamma}_{s-2,n}(z)|\,dz \le c_3. \tag{A.12}$$

PROOF OF PROPOSITION 3.1. Let $\tilde{G}_n$ denote the common distribution of $Z_i^* - \bar{Z}$ under P^* and $\tilde{Q}_n$ the distribution of $\sqrt{n}(\bar{Z}^* - \bar{Z})$ under P^*. Writing $\hat{m}$ for the Fourier transform of a finite signed measure m, one has $\hat{\tilde{Q}}_n(\xi) = \hat{\tilde{G}}_n^n(\xi/\sqrt{n})$. Let $\tilde{\Gamma}_{s-2,n}$ denote the signed measure with density $\tilde{\gamma}_{s-2,n}$. By Corollary 11.5 in Bhattacharya and Ranga Rao (1986), one has, for every Borel set A and every $\varepsilon > 0$, the inequality

$$\begin{aligned}&|\tilde{Q}_n(A) - \tilde{\Gamma}_{s-2,n}(A)|\\ &\quad\le (2\alpha' - 1)^{-1}\left[\|(\tilde{Q}_n - \tilde{\Gamma}_{s-2,n})*K_\varepsilon\|_v + \int_{(\partial A)^{2\varepsilon}} |\tilde{\gamma}_{s-2,n}(z)|\,dz\right],\end{aligned} \tag{A.13}$$

where $\|m\|_v$ denotes *variation norm* of a signed measure m, K_1 is a probability measure on $\mathbb{R}^2$ having a finite third absolute moment, $\hat{K}_1$ vanishes outside the unit ball, $\alpha' > 1/2$, and K_ε is the probability measure $K_\varepsilon(A) = K_1(\varepsilon^{-1}A)$. By Lemma 11.6 in Bhattacharya and Ranga Rao (1986), one has, writing $|\nu|$ for the sum of coordinates of the multiindex ν,

$$\text{(A.14)}\quad \begin{aligned}&\|(\tilde{Q}_n - \tilde{\Gamma}_{s-2,n})^* K_\varepsilon\|_v \\ &\le c_4 \max_{|\nu|=0,3} \int |D^\nu\{(\hat{\tilde{Q}}_n(\xi) - \hat{\tilde{\Gamma}}_{s-2,n}(\xi)) \cdot \hat{K}_\varepsilon(\xi)\}|\, d\xi.\end{aligned}$$

Now let R_1 denote the set

$$\text{(A.15)}\qquad R_1 = \{\|\tilde{V}^{-1} - V^{-1}\| \le \tfrac{1}{2}\|V^{-1}\|, \tilde{\eta}_s \le c_1\}.$$

By Theorem 9.10 in Bhattacharya and Ranga Rao (1986) (with $\eta_{s+1} < \infty$) one has on R_1, for all $|\nu'| \le s$,

$$\text{(A.16)}\quad |D^{\nu'}(\hat{\tilde{Q}}_n(\xi) - \hat{\tilde{\Gamma}}_{s-2,n}(\xi))| \le \frac{c_5}{n^{(s-1)/2}}(1 + \|\xi\|^{3(s-1)+|\nu'|})e^{-\|\xi\|^2/8}$$

for $\|\xi\| \le c_6 n^{1/2}$. Let θ be defined by

$$\text{(A.17)}\qquad \sup_{\|\xi\| \ge c_6} |Ee^{i\xi_1 X_1 + \xi_2 X_1^2}| = 1 - \theta.$$

By a result of Babu and Singh (1984) there exists a constant $\delta > 0$ such that

$$\text{(A.18)}\qquad P\left(\sup_{c_6 \le \|\xi\| \le e^{n\delta}} |\hat{G}_n(\xi)| \le 1 - \frac{\theta}{2}\right) = 1 - o(n^{-(s-1)/2}).$$

Let R_2 denote the set within parentheses on the left side in (A.18), and let

$$\text{(A.19)}\qquad R = R_1 \cap R_2.$$

Take $\varepsilon = e^{-n\delta}$ in (A.14) and use (A.16) and (A.18) to get, on R,

$$\text{(A.20)}\qquad \|(\tilde{Q}_n - \tilde{\Gamma}_{s-2,n})^* K_\varepsilon\|_v \le c_7 n^{-(s-1)/2}.$$

Also, on R_1,

$$\text{(A.21)}\qquad \sup_{A \in \mathscr{A}} \int_{(\partial A)^{2\varepsilon}} |\tilde{\gamma}_{s-2,n}(z)|\, dz \le c_8 \varepsilon^a, \qquad \varepsilon \downarrow 0.$$

Using (A.20) and (A.21) in (A.13) one gets

$$\text{(A.22)}\qquad \sup_{A \in \mathscr{A}} |\tilde{Q}_n(A) - \tilde{\Gamma}_{s-2,n}(A)| \le c_9 n^{-(s-2)/2} \delta_n \quad \text{on } R,$$

where δ_n are nonrandom, $\delta_n \to 0$. By Lemma A.3,

$$\text{(A.23)}\quad \sup_{A \in \mathscr{A}} |\tilde{\mathbf{Q}}_n(A) - \tilde{\Gamma}_{s-2,n}(A)| \le 1 + c_3 \quad \text{on } R^c \cap \{\tilde{V}_n \text{ nonsingular}\}.$$

Since $P(R^c) = o(n^{-(s-2)/2})$, the proof of (3.11) is complete. In order to prove (3.12) one may use the method Bhattacharya and Ghosh (1978), applied to (A.22). □

Proof of Lemma 3.3. (a) By Lemma A.2(a), the left side of (3.14), with p replaced by $p + 1$, is bounded. Since $\tilde{\mu}_r^m / s_n^{rm} - \mu_r^m / \sigma^{rm} \to 0$ a.s. as $n \to \infty$, (3.14) follows.

(b) Note that $\sqrt{n}(\tilde{\mu}_{3,n}/s_n^3 - \mu_3/\sigma^3) \cdot 1_{\{s_n^2 > 0\}}$ converges to a normal law $N(0, \delta^2)$ if $EX_1^6 < \infty$ [see Cramér (1946), page 367]. Also, the $(p+1)$th absolute moment of

$$\sqrt{n}\left(\frac{\tilde{\mu}_{3,n}}{s_n^3} - \frac{\mu_3}{\sigma^3}\right) \cdot 1_{\{s_n^2 > 0\}} = \sqrt{n}\,\frac{\tilde{\mu}_{3,n}}{s_n^3}\left(1 - \frac{s_n^3}{\sigma^3}\right) \cdot 1_{\{s_n^2 > 0\}} + \sqrt{n}\left\{(\tilde{\mu}_{3,n} - \mu_3)/\sigma^3\right\} \cdot 1_{\{s_n^2 > 0\}} \tag{A.24}$$

is easily shown to be a bounded sequence using the Schwarz inequality and Lemma A.2(a).

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