The Annals of Statistics
1984, Vol. 12, No. 3, 1117–1123

BOUNDS FOR THE BAYES RISK FOR TESTING SEQUENTIALLY THE SIGN OF THE DRIFT PARAMETER OF A WIENER PROCESS

By Ashim Mallik and Yi-Ching Yao

Massachusetts Institute of Technology

Let $x(t)$ be a Wiener process with drift μ and variance 1 per unit time. The following problem is treated; test $H:\mu \le 0$ vs. $A:\mu > 0$ with the loss function $|\mu|$ if the wrong decision is made and 0 otherwise, and with c = cost of observation per unit time, where μ has a prior distribution which is normal with mean 0 and variance σ_0^2. An idea of Bickel and Yahav is followed to obtain a lower bound for the Bayes risk which is strict as $\sigma_0 \to \infty$ for all c. An upper bound is also derived.

1. Introduction. Let $X(t)$ be a Wiener process with drift μ and variance 1 per unit time. Chernoff [2] considered the following problem: test

$$H:\mu \le 0 \text{ vs. } A:\mu > 0$$

with the loss function $|\mu|$ if the wrong decision is made and 0 otherwise, c = cost of observation per unit time, and μ has a prior distribution which is normal with mean 0 and variance σ_0^2. Chernoff [3] showed that the Bayes risk

$$B(\sigma_0^2) = c^{2/3}[K\sigma_0^{-1} - 6c^{1/3}\sigma_0^{-2}\ln\sigma_0(1 + o(1))] \tag{1.1}$$

as $\sigma_0 \to \infty$, where K is an unknown constant.

By considering the above testing problem with the additional information of the magnitude of μ, Bickel and Yahav [1] obtained a lower bound for the Bayes risk for the case of μ having the improper prior distribution (i.e. $\sigma_0 = \infty$) and conjectured that the lower bound can be attained as $c \downarrow 0$. In Section 2, we consider the case of σ_0 finite. By using similar techniques as in Bickel and Yahav [1], we obtain a lower bound for the Bayes risk and show that this lower bound is not asymptotically achievable as $\sigma_0 \to \infty$ for all $c > 0$. In Section 3, we consider the case of μ having the improper prior distribution and show that Bickel and Yahav's lower bound is not asymptotically achievable as $c \downarrow 0$. In Section 4, we derive an upper bound for the Bayes risk.

2. Lower bound for finite σ_0. From Chernoff [3], the posterior cost of wrong decision is given by

$$Y_t = (t + \sigma_0^{-2})^{-1/2}\{\phi(\alpha) - |\alpha|\Phi(-|\alpha|)\}, \tag{2.1}$$

where $\alpha = (t + \sigma_0^{-2})^{-1/2}X(t)$ and ϕ and Φ are the standard normal density and

Received March 1983; revised January 1984.

AMS 1980 *subject classifications.* Primary 62L10; secondary 62C10.

Key words and phrases. Sequential tests, S.P.R.T., Bayes, stopping times, lower bound, asymptotic expansion.

cumulative distribution functions, respectively. Let the posterior risk at time t be

$$R(c, t) = Y_t + ct. \tag{2.2}$$

We are interested in a stopping rule τ_0 for which

$$E[R(c, \tau_0)] = \inf_{\tau \in T} E[R(c, \tau)]$$

where T is the class of all stopping times.

Using the idea of Bickel and Yahav [1], let us consider the following problem of testing

$$H: \mu = \mu_0 \text{ vs. } A: \mu = -\mu_0$$

with $|\mu_0|$ for the cost of wrong decision and prior distribution $P(\mu = \mu_0) = P(\mu = -\mu_0) = 1/2$. Then the posterior cost of wrong decision is

$$\tilde{Y}_t = |\mu_0| P(X(t)\mu < 0 \mid X(t)).$$

Let

$$\tilde{R}(c, t) = \tilde{Y}_t + ct.$$

To solve the above Bayes problem, we have to find a stopping rule $\tau^*(|\mu_0|)$ such that

$$E(\tilde{R}(c, \tau^*)) = \inf_{\tau \in T} E(\tilde{R}(C, \tau)).$$

From the property of the S.P.R.T., Bickel and Yahav [1] derived the following lemma.

LEMMA 2.1. *The stopping rule τ^*: stop at the first time τ that $|X(t)| = a$, where a is determined by the minimization of*

$$|\mu_0|(1 + \exp(2a|\mu_0|))^{-1} + ca|\mu_0|^{-1}(1 - 2(1 + \exp(2a|\mu_0|)))^{-1},$$

is the optimal stopping rule for the above problem.

LEMMA 2.2.

$$(2\pi\sigma_0^2)^{-1/2} \int_{-\infty}^{\infty} E_\mu[\tilde{R}(c, \tau^*)]\exp(-\mu^2/2\sigma_0^2)\, d\mu \le E[R(c, \tau_0)].$$

PROOF. The stopping rule τ_0 is a Bayes rule for a symmetric problem and hence is symmetric in μ. Therefore

$$E_\mu[R(c, \tau_0)] \ge E_\mu[\tilde{R}(c, \tau^*)] \quad \text{for all } \mu,$$

and the lemma follows.

THEOREM 2.3.

$$(2\pi\sigma_0^2)^{-1/2} \int_{-\infty}^{\infty} E_\mu[\tilde{R}(c, \tau^*)]\exp(-\mu^2/2\sigma_0^2)\, d\mu$$

$$= c^{2/3}[K'\sigma_0^{-1} - 3/2\, c^{1/3}\sigma^{-2}\ln \sigma_0(1 + o(1))]$$

as $\sigma_0 \longrightarrow \infty$, *where*

$$K' = (2\pi)^{-1/2}2^{1/3}3^{-1}\int_1^\infty (z - z^{-1} + 2\ln z)^{-4/3} \cdot (1 + z^{-2} + 2z^{-1})\cdot(1 + \ln z - z^{-1})\,dz \approx 1.885.$$

PROOF. Let

$$z = e^{2a\mu} \tag{2.3}$$

where a is the solution of the minimization problem in Lemma 2.1. Then z should satisfy the relation

$$2\mu^3 = c(z - z^{-1} + 2\ln z). \tag{2.4}$$

By using (2.3), (2.4) and Lemma 2.1, it follows that

$$\int_{-\infty}^{\infty} E_\mu[\tilde{R}(c, \tau^*)]\exp(-\mu^2/2\sigma_0^2)\,d\mu = 2^{1/3}3^{-1}c^{2/3}\int_1^\infty (z - z^{-1} + 2\ln z)^{-4/3}(1 + \ln z - z^{-1}) \cdot (1 + 2z^{-1} + z^{-2})\exp[-c^{2/3}(z - z^{-1} + 2\ln z)^{2/3}\sigma_0^{-2}2^{-5/3}]\,dz.$$

Let

$$\gamma = 2^{-5/3}c^{2/3}\sigma_0^{-2}$$

$$I(z) = (z - z^{-1} + 2\ln z)^{-4/3}(1 + \ln z - z^{-1})(1 + 2z^{-1} + z^{-2}).$$

We have

$$\int_{-\infty}^{\infty} E_\mu[\tilde{R}(c, \tau^*)]\exp(-\mu^2/2\sigma_0^2)\,d\mu = 2^{1/3}3^{-1}c^{2/3} \cdot \int_1^\infty I(z)\exp(-\gamma(z - z^{-1} + 2\ln z)^{2/3})\,dz. \tag{2.5}$$

To complete the proof of Theorem 2.3, two further lemmas are needed.

LEMMA 2.4.

$$\int_1^{1/\gamma} I(z)\exp(-\gamma(z - z^{-1} + 2\ln z)^{2/3})\,dz = \int_1^\infty I(z)\,dz + 3\gamma^{1/3}\ln\gamma - 12\gamma^{1/3} + O(\gamma^{2/3}\ln\gamma).$$

PROOF. Calculate as follows:

$$\int_1^{1/\gamma} I(z)\exp(-\gamma(z - z^{-1} + 2\ln z)^{2/3})\,dz$$

$$= \int_1^{\gamma^{-1}} I(z)[1 - \gamma(z - z^{-1} + 2\ln z)^{2/3}(1 + o(1))]\,dz$$

$$= \int_1^{\gamma^{-1}} I(z)\,dz - \gamma(1 + o(1)) \int_1^{\gamma^{-1}} I(z)(z - z^{-1} + 2\ln z)^{2/3}\,dz$$

$$= \int_1^{\infty} I(z)\,dz - \int_{\gamma^{-1}}^{\infty} I(z)\,dz - \gamma(1 + o(1))O(\gamma^{-1/3}\ln\gamma)$$

$$= \int_1^{\infty} I(z)\,dz + 3\gamma^{1/3}\ln\gamma - 12\gamma^{1/3} + O(\gamma^{2/3}\ln\gamma).$$

LEMMA 2.5.

$$\int_{1/\gamma}^{\infty} I(z)\exp(-\gamma(z - z^{-1} + 2\ln z)^{2/3})\,dz$$

$$= 12\gamma^{1/3} - 3\gamma^{1/3}\ln\gamma + 9\cdot 2^{-1}\pi^{1/2}\gamma^{1/2}\ln\gamma(1 + o(1)).$$

PROOF. Let $w = \gamma(z - z^{-1} + 2\ln z)^{2/3}$. Then

$I(z)\exp(-\gamma(z - z^{-1} + 2\ln z)^{2/3})\,dz = 3\cdot 2^{-1}\gamma^{1/2}w^{-3/2}(1 + \ln z - z^{-1})e^{-w}\,dw.$
Let

$$u = z - z^{-1} + 2\ln z = (w/\gamma)^{3/2}.$$

For $z \geq \gamma^{-1}$,

$$1 + \ln z - z^{-1} = 1 + \ln u + O(u^{-1}\ln u)$$
$$= 1 + 3\cdot 2^{-1}\ln(w/\gamma) + O((w/\gamma)^{-3/2}\ln(w/\gamma)).$$

Then

$$\int_{\gamma^{-1}}^{\infty} I(z)\exp(-\gamma(z - z^{-1} + 2\ln z)^{2/3})\,dz$$

$$= 3\cdot 2^{-2}\int_{\gamma(\gamma^{-1}-\gamma-2\ln\gamma)^{2/3}}^{\infty} \gamma^{1/2}w^{-3/2}(3\ln w - 3\ln\gamma + 2)e^{-w}\,dw$$

$$+ O(\gamma^{4/3}\ln\gamma)$$

$$= 12\gamma^{1/3} - 3\gamma^{1/3}\ln\gamma + 9\cdot 2^{-1}\pi^{1/2}\gamma^{1/2}\ln\gamma(1 + o(1)).$$

From (2.5), Lemma 2.4, and Lemma 2.5, we get Theorem 2.3.

From (1.1), Lemma 2.2 and Theorem 2.3, $K \geq K'$. Suppose that $K = K'$. From (1.1), Lemma 2.2 and Theorem 2.3, for σ_0 sufficiently large,

$$-6c\sigma_0^{-2}\ln\,\sigma_0 > -\,{}^3\!/_2\, c\sigma_0^{-2}\ln\,\sigma_0,$$

a contradiction. Therefore $K > K'$, i.e. the lower bound is not asymptotically achievable as $\sigma_0 \to \infty$.

REMARK. Chernoff [4] estimated $K \approx 2.38$ by a questionable least squares fit to the asymptotic expansion. We found $K' \approx 1.885$ by numerical integration. It seems rather interesting that, with the additional information of the magnitude of μ, the Bayes risk is reduced by only about 21%. That is to say, this information is not as substantial as the authors expected.

3. Lower bound for $\sigma_0 = \infty$. Consider the case of μ having the improper prior distribution given by Lebesgue measure. For any stopping rule τ,

$$\int_{-\infty}^{\infty} R(\mu,\,\tau)\,d\mu = \lim_{\sigma_0\to\infty}(2\pi\sigma_0^2)^{1/2}[(2\pi\sigma_0^2)^{-1/2}\int_{-\infty}^{\infty} R(\mu,\,\tau)\exp(-\mu^2/2\sigma_0^2)\,d\mu]$$

$$\geq \lim_{\sigma_0\to\infty}(2\pi\sigma_0^2)^{1/2}B(\sigma_0^2) = (2\pi)^{1/2}Kc^{2/3},$$

so the Bayes risk with respect to Lebesgue measure satisfies

$$\inf_\tau \int_{-\infty}^{\infty} R(\mu,\,\tau)\,d\mu \geq (2\pi)^{1/2}Kc^{2/3} > (2\pi)^{1/2}K'c^{2/3}$$

for all $C > 0$.

Here, $(2\pi)^{1/2}K'c^{2/3}$ is the lower bound derived in [1]. Therefore, we have shown that Bickel and Yahav's lower bound cannot be attained.

4. An upper bound. We first consider $\sigma_0 < \infty$. Modifying the argument in Bickel and Yahav [1], we shall derive an asymptotic upper bound for the Bayes risk as $\sigma_0 \to \infty$.

Define the stopping time τ_1 to be the first time for which

$$A^{-1/2}(t + \sigma_0^{-2})^{-3/2}\exp(-X(t)^2/2(t + \sigma_0^{-2})) = c, \tag{4.1}$$

where A is a positive constant. When $A = 8\pi$, this stopping rule is suggested as an approximation to the Bayes rule for small t in [3]. In order that this rule be meaningful, we require $Ac^2\sigma_0^{-6} < 1$.

LEMMA 4.1. *As* $\sigma_0 \to \infty$,

$$cE[\tau_1] \leq (1 + o(1))2\cdot 3^{1/2}\cdot A^{-1/6}c^{2/3}\sigma_0^{-1}.$$

PROOF. From the definition of τ_1, it follows that

$$X(\tau_1)^2 = -(\tau_1 + \sigma_0^{-2})[\ln\,Ac^2(\tau_1 + \sigma_0^{-2})^3].$$

Since $E_\mu[X(\tau_1)^2] \geq \mu^2E_\mu^2[\tau_1]$ and

$$E[(\tau_1 + \sigma_0^{-2})\ln(\tau_1 + \sigma_0^{-2})] \geq E[\tau_1 + \sigma_0^{-2}]\ln\,E[\tau_1 + \sigma_0^{-2}],$$

we get

$$\mu^2 E_\mu^2[\tau_1]/(E\mu[\tau_1] + \sigma_0^{-2}) \le -\ln Ac^2 - 3\ln(E_\mu[\tau_1] + \sigma_0^{-2}).$$

Define $g(y) \equiv \mu^2 y^2/(y + \sigma_0^{-2}) + \ln Ac^2 + 3\ln(y + \sigma_0^{-2})$. Since $g(y)$ is increasing for positive y, we have $E_\mu[\tau_1] \le z$, where $g(z) = 0$. Hence

$$\begin{aligned} cE[\tau_1] &= c\int_{-\infty}^{\infty} E_\mu[\tau_1](2\pi\sigma_0^2)^{-1/2}\exp(-\mu^2/2\sigma_0^2)\,d\mu \\ &= 2c(2\pi\sigma_0^{-2})^{-1/2}\int_0^\infty E_\mu[\tau_1]\exp(-\mu^2/2\sigma_0^2)\,d\mu \\ &\le 2c(2\pi\sigma_0^{-2})^{-1/2}\int_0^\infty z\exp(-\mu^2/2\sigma_0^2)\,d\mu. \end{aligned}$$

Since $g(z) = 0$,

$$\mu^2 = -[\ln Ac^2 + 3\ln(z + \sigma_0^{-2})](z + \sigma_0^{-2})/z^2,$$

so

$$cE[\tau_1] \le 2c(2\pi\sigma_0^{-2})^{-1/2}\int_0^{(Ac^2)^{-1/3} - \sigma_0^{-2}} z\exp(-\mu^2/2\sigma_0^2)\left(-\frac{d\mu}{dz}\right)dz. \tag{4.2}$$

Here,

$$-d\mu/dz = (z + \sigma_0^{-2})(2\mu z^2)^{-1}[3(z + \sigma_0^{-2})^{-1} + \mu^2(z^2 + 2z\sigma_0^{-2})(z + \sigma_0^{-2})^{-2}].$$

After a careful inspection, it becomes clear that, as $\sigma_0 \to \infty$, the integral in (4.2) can be approximated by replacing σ_0^{-2} with 0. That is to say,

$$\begin{aligned} cE[\tau_1] &\le (1 + o(1))2c(2\pi\sigma_0^2)^{-1/2} \\ &\quad \cdot \int_0^{(Ac^2)^{-1/3}} 2^{-1}(-z^{-1}\ln Ac^2z^3)^{-1/2}z^{-1}(3 - \ln Ac^2z^3)\,dz. \end{aligned}$$

Let $-\ln Ac^2z^3 = u^2$. Then

$$\begin{aligned} cE[\tau_1] &\le (1 + o(1))2c(2\pi\sigma_0^2)^{-1/2}\int_0^\infty (Ac^2)^{-1/6}(1 + u^2/3)\exp(-u^2/6)\,du \\ &= (1 + o(1))2 \cdot 3^{1/2} \cdot A^{-1/6}c^{2/3}\sigma_0^{-1}. \end{aligned}$$

LEMMA 4.2. *As* $\sigma_0 \to \infty$,

$$E[Y_{\tau_1}] \le (1 + o(1))(2\pi)^{-1/2}A^{1/2}cE[\tau_1].$$

PROOF. From (2.1) and (4.1), we have

$$Y_{\tau_1} \le (\tau_1 + \sigma_0^{-2})^{-1/2}\phi((\tau_1 + \sigma_0^{-2})^{-1/2}X(\tau_1)) = (2\pi)^{-1/2}A^{1/2}c(\tau_1 + \sigma_0^{-2}),$$

which proves the lemma.

THEOREM 4.3. *As* $\sigma_0 \to \infty$,

$$B(\sigma_0^2) \le (1 + o(1))2^{1/6}3^{3/2}\pi^{-1/6}c^{2/3}\sigma_0^{-1}.$$

PROOF. Using Lemmas 4.1 and 4.2, we have

$$B(\sigma_0^2) \le E[R(c, \tau_1)] = cE[\tau_1] + E[Y_{\tau_1}]$$
$$\le (1 + o(1))(1 + (2\pi)^{-1/2}A^{1/2})\ 2 \cdot 3^{1/2} \cdot A^{-1/6}c^{2/3}\sigma_0^{-1}.$$

Setting $A = \pi/2$, we get the Theorem.

From (1.1) and Theorem 4.3,

$$K \le 2^{1/6} \cdot 3^{3/2} \cdot \pi^{-1/6} \approx 4.819.$$

For the case of μ having a prior distribution given by Lebesgue measure, we may consider the boundary $(\pi/2)^{-1/2}t^{-3/2}\exp(-X(t)^2/2t) = c$ and apply the same techniques to show that the Bayes risk is bounded from above by $2^{2/3} \cdot 3^{3/2} \cdot \pi^{1/3}c^{2/3}$.

Acknowledgements. The authors wish to thank Professor Herman Chernoff for many helpful suggestions. This research was supported in part by the Office of Naval Research under contract N00014-75-C-0555 (NR-609-001).

REFERENCES

[1] BICKEL, P. J. and YAHAV, J. A. (1967). On testing sequentially the mean of a normal distribution. Technical Report No. 26, Nov. 10, 1967, Dept. of Statist., Stanford University, Stanford, California.

[2] CHERNOFF, H. (1961). Sequential tests for the mean of a normal distribution. *Proc. Fourth Berkeley Symp. Math. Statist. Probab.* **1** 79–91. University of California Press, Berkeley.

[3] CHERNOFF, H. (1965). Sequential tests for the mean of a normal distribution III (Small t). *Ann. Math. Statist.* **36** 28–54.

[4] CHERNOFF, H. (1983). Private communication.

STATISTICS CENTER
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
77 MASSACHUSETTS AVENUE, RM. E40-111
CAMBRIDGE, MASSACHUSETTS 02139