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RECTANGULAR REGIONS OF MAXIMUM PROBABILITY CONTENT

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The problem of characterizing the rectangular region of maximum probability content among the rectangles of fixed area is considered. The rectangles are assumed to have their sides parallel to the axes and the bivariate distribution is assumed to have density ellipitically contoured and unimodal.

1. Introduction. Let $\mathbf{X}$ be a bivariate random vector with probability distribution which is absolutely continuous with respect to Lebesgue measure, $f(\mathbf{x})$ being the probability density function (pdf) of $\mathbf{X}$. For any measurable subset A of $\mathbb{R}^2$, let $P(A) = P[\mathbf{X} \in A]$ and $\lambda(A)$ be the area of the region A. For a fixed $\lambda > 0$, let us define the following class of subsets of $\mathbb{R}^2$

(1.1) $$\mathbf{R}_\lambda = \{R: R \text{ is a rectangle in } \mathbb{R}^2 \text{ with sides parallel to the axes and } \textstyle\int_R d\mathbf{x} = \lambda\}.$$

The object of this paper is to characterize the region in the above class for which the probability content is maximum. As noted in Rattihalli (1981) the characterization of such regions is useful to find the Bayes regional estimators when (i) the decision space is the class of rectangular regions and (ii) the loss function is a linear combination of the area of the region and the indicator of noncoverage of the region. Further, by using the Bayes regional estimators and the rate of convergence of posterior Bayes risk and the results of Gleser and Kunte (1976), one can find asymptotically pointwise optimal and asymptotically optimal stopping rules.

We will restrict our attention to those random vectors $\mathbf{X}$, for which the pdf $f(\mathbf{x})$ is of the type

(1.2) $$f(\mathbf{x}) = g((\mathbf{x} - \boldsymbol{\mu})\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})'),$$

where $\boldsymbol{\mu} \in \mathbb{R}^2$, $\Sigma^{-1} = (\sigma^{ij})$ is a 2×2 symmetric positive definite matrix, and the function $g: [0, \infty) \to (0, \infty)$ is a strictly decreasing function. For example, bivariate normal and bivariate t variates are of the above kind.

2. Formulation. For $a > 0$, let

$$R(a) = \{(x_1, x_2): |x_1| \le a, |x_2| \le a^{-1}\}.$$

LEMMA 2.1. *Let* $\mathbf{X}$ *be a bivariate random vector with pdf of the form* (1.2), *with* $\boldsymbol{\mu} = \mathbf{0}$, $\Sigma^{-1} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, $|\rho| < 1$. *Then for* $a_1 > a_2 \ge 1$

$$P(R(a_1)) < P(R(a_2)).$$

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PROOF. Let $h(a) = P(R(a))$. Then in order to show that $h(a_1) < h(a_2)$, by deleting the common domain of integration it is enough to show that

$$\int_A f(\mathbf{x})\, d\mathbf{x} > \int_B f(\mathbf{x})\, d\mathbf{x},$$

where A and B are rectangles in $\mathbb{R}^2$ given by

$$A = \{\mathbf{x} = (x_1, x_2)\colon -a_2 < x_1 < a_2,\ a_1^{-1} < x_2 < a_2^{-1}\}$$

$$B = \{\mathbf{x} = (x_1, x_2)\colon a_2 < x_1 < a_1,\ -a_1^{-1} < x_2 < a_1^{-1}\}.$$

Now consider the transformation $y_1 = a_1 a_2 x_2$, $y_2 = x_1/a_1 a_2$. Under this transformation the rectangle A is transferred to B and

$$\int_A f(\mathbf{x})\, d\mathbf{x} = \int_A g(x_1^2 + 2\rho x_1 x_2 + x_2^2)\, dx_1\, dx_2$$

$$= \int_B g\left((a_1 a_2 x_2)^2 + 2\rho x_1 x_2 + \left(\frac{x_1}{a_1 a_2}\right)^2\right) dx_1 dx_2.$$

Hence

$$\int_A f(\mathbf{x})\, d\mathbf{x} - \int_B f(\mathbf{x})\, d\mathbf{x}$$

$$= \int_B \left\{g\left((a_1 a_2 x_2)^2 + 2\rho x_1 x_2 + \left(\frac{x_1}{a_1 a_2}\right)^2\right) - g(x_1^2 + 2\rho x_1 x_2 + x_2^2)\right\} dx_1\, dx_2.$$

But for any point $(x_1, x_2) \in B$ we have

$$(a_1 a_2 x_2)^2 + \left(\frac{x_1}{a_1 a_2}\right)^2 - x_1^2 - x_2^2 = (a_1^2 a_2^2 - 1)\left[\frac{x_2^2 - x_1^2}{a_1^2 a_2^2}\right] < 0.$$

The last inequality follows from the fact that $a_1 a_2 > 1$ and for $(x_1, x_2) \in B$ we have $x_2^2 < a_1^{-2}$, $x_1^2 > a_2^2$. Further, as g is a strictly decreasing function, for any point $(x_1, x_2) \in B$ we have

$$g((a_1 a_2 x_2)^2 + 2\rho x_1 x_2 + (x_1/a_1 a_2)^2) > g(x_1^2 + 2\rho x_1 x_2 + x_2^2)$$

and hence the result. □

We observe that $P(R(a)) = P(R(a^{-1}))$ and hence from Lemma 2.1 we have the following result.

COROLLARY 2.1. *$P(R(a))$ is increasing for $a < 1$ and is decreasing for $a > 1$.* □

It was only for simplicity of notation in the above that we restricted attention to the class of rectangles of area equal to 4. A similar proof also goes through if we fix the area to be equal to any $\lambda > 0$ and we get the following result. Define

$$R_\lambda(a) = \{(x_1, x_2): |x_1| \le a, |x_2| \le \lambda/4a\}.$$

LEMMA 2.2. *Let* $\mathbf{X}$ *be a bivariate random vector with pdf of the form* (1.2), *with* $\mu = \mathbf{0}$, $\Sigma^{-1} = \binom{1\ \rho}{\rho\ 1}$, $|\rho| < 1$. *Then* $P(R_\lambda((a)))$ *is increasing for* $a < \sqrt{\lambda}/2$ *and is decreasing for* $a > \sqrt{\lambda}/2$. □

LEMMA 2.3. *Let* $\mathbf{X}$ *be a bivariate random vector with pdf of the form* (1.2), *with* $\mu = \mathbf{0}$. *Then* $P(R_\lambda(a))$ *is increasing for* $a < c$ *and decreasing for* $a > c$, *where*

$$c = [(\lambda/4)(\sigma^{22}/\sigma^{11})^{1/2}]^{1/2}. \tag{2.1}$$

PROOF. Let $y_1 = \sqrt{\sigma^{11}}x_1$ and $y_2 = \sqrt{\sigma^{22}}x_2$. Now the result follows from Lemma 2.2. □

By using Anderson's Theorem (1955) and Lemma 2.3, we get the following result regarding the characterization of the rectangular region in R_λ which has maximum probability content.

THEOREM 2.1. *Let* $\mathbf{X}$ *be a bivariate random vector with pdf of the form* (1.2). *Then for any fixed* $\lambda > 0$ *the maximal set in the class* $\mathbf{R}_\lambda$ *is*

$$\{(x_1, x_2): |x_1 - \mu_1| < c, |x_2 - \mu_2| < \lambda/4c\},$$

with c given by (2.1). □

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