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A SHARP NECESSARY AND SUFFICIENT CONDITION FOR INADMISSIBILITY OF ESTIMATORS IN A CONTROL PROBLEM[1]

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Suppose $\mathbf{x} = (x_1, \cdots, x_m)^t$ is an m-variate normal random variable with mean vector $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_m)^t$ and identity dispersion matrix. We consider the control problem which, in canonical form, is the problem of estimating $\boldsymbol{\theta}$ with respect to the loss

$$L(\boldsymbol{\theta}, \boldsymbol{\delta}) = (1 - \boldsymbol{\theta}^t\boldsymbol{\delta})^2,$$

where $\boldsymbol{\delta}(x) = (\delta_1(x), \cdots, \delta_m(x))^t$. A necessary and suffcient condition for the admissibility of spherically symmetric generalized Bayes $\delta(x)$ is given in terms of a Dirichlet problem. This condition is also equivalent to recurrence of a diffusion process and insolubility of certain elliptic boundary value problems.

1. Introduction. The control problem, which arises in economics, deals with the choice of the levels of certain input factors in a system so that the "yield" (or output) of the system is at the desired control level. Basu (1974) and Zaman (1981) consider a standard normal model of the control problem, in which the output, Y, occurs as a linear function

$$Y = \boldsymbol{\theta}^t z + \varepsilon$$

where $\boldsymbol{\theta}$ is an m-vector of unknown coefficients (factor levels) of the system and $\mathbf{z}$ is an m-vector of nonstochastic control variables to be chosen so as to achieve some desired output Y^*. Suppose the loss in achieving output Y is measured by $(Y - Y^*)^2$ and an estimate $\delta(\mathbf{x}) = (\delta_1(\mathbf{x}), \cdots, \delta_m(\mathbf{x}))^t$ of θ is available from past multivariate normal data $\mathbf{x} = (x_1, \cdots, x_m)^t$, then this problem can be transformed (in the simplest situation) into a problem of estimating the inverse of the mean of an m-variate normal random variable with unknown mean $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_m)^t$ and identity dispersion matrix, in which it is desired to estimate $\boldsymbol{\theta}$ with respect to the loss

$$L(\boldsymbol{\theta}, \boldsymbol{\delta}) = (1 - \boldsymbol{\theta}^t\boldsymbol{\delta})^2. \tag{1.1}$$

The appropriate parameter space for this problem turns out to be $\boldsymbol{\Theta}^* = R^m - \{0\}$. See Berger et al. (1982) for a decision theoretic explanation. Also, it seems natural to exclude $\theta = 0$ because it would correspond to the case where the input factors have no effect on the yield, thus reducing the original problem to a meaningless situation.

An estimator $\boldsymbol{\delta}$, as usual, will be evaluated by its risk function $R(\theta, \delta)$ and will

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be called *inadmissible* if there exists an estimator δ^*, such that $R(\theta, \delta^*) \le R(\theta, \delta)$ for all θ with strict inequality for some θ. Otherwise, δ is admissible. Of course, the range of a decision rule $\delta(\mathbf{x})$ is $\bar{\Theta}^* = R^m$, the closure of Θ^* under the natural topology.

In this paper we confine our attention to orthogonally invariant nonrandomized decision rules. It is shown in Berger et al. (1982) that nonrandomized rules form a complete class for this problem. It is well known that spherically symmetric nonrandomized rules can be written as

$$\delta(\mathbf{x}) = \phi(|x|)\mathbf{x}/|x|. \tag{1.2}$$

In particular, the generalized Bayes estimators with respect to spherically symmetric priors have the representation (1.2).

The main results of this paper are concerning the admissibility (or inadmissibility) of estimators of the form (1.2). Subject to mild growth conditions on the prior, we obtain a necessary and sufficient condition for the admissibility of a generalized Bayes estimator with respect to a spherically symmetric prior. See Theorems 3.1 and 4.1. The first theorem gives the necessary condition and the second the sufficient condition. The necessary and sufficient condition (modulo the growth conditions on the prior) is analogous to the one given by the fundamental paper of L. Brown (1971) (also see Srinivasan, 1981) for estimating the mean of multivariate normal distribution under quadratic loss. It is appropriate here to mention that there are two important papers which deal with admissibility of spherically symmetric estimators in a control problem. Berger et al. (1982) have obtained a necessary condition for the admissibility of an estimator of the form (1.2). Their approach is different and their conditions are relatively harder to verify. Berliner (1980), in a notable work on the control problem, has obtained among other results, sufficient conditions for the admissibility of a generalized Bayes estimator with respect to an absolutely continuous spherical-symmetric prior. His conditions are more stringent than ours. For a comparison of our results with that of Berger and Zaman, and Berliner, see Section 5.

As mentioned earlier, the necessary and sufficient condition for admissibility given here is similar to the one given by Brown (1971) in estimating θ with respect to the quadratic loss (i.e. $L(\theta, t) = |\theta - t|^2$). One might wonder whether the similarity is due to the fact that both problems have the same probability model, the multivariate normal distribution. We want to point out that this is not the reason for the similarity and, on the contrary, the common underlying model has very little role to play. The explanation lies in the fact that for $\theta = (\theta_1, 0, \cdots, 0)$ with θ large, and for $\delta_0(x) = X + \gamma(x)$ where $\gamma(x)$ is bounded

$$L(\theta, \delta) = \left(1 - \frac{\theta_1}{X_1 + \gamma_1(x)}\right)^2 = \frac{(X_1 + \gamma_1 - \theta_1)^2}{(X_1 + \gamma_1(x))^2} \approx \frac{|\delta_0(x) - \theta|^2}{|\theta|^2}.$$

Thus for large θ, $L(\theta, \delta)$ is a weighted quadratic loss. Brown, in his heuristics paper (Brown, 1979), argued that in regular problems the admissibility (or inadmissibility) of an estimator is governed by the loss function and not by the underlying probability model, and two distinct estimation problems would exhibit similar admissibility phenomena if their loss functions are similar. These heuristics of Brown explain the similarity between results on control problem and normal estimation problem.

2. Preliminaries. Let $x^t = (x_1, \cdots, x_m)$ be an $m \times 1$ random vector distributed according to an m-dimensional normal distribution with mean vector $\theta^t = (\theta_1, \cdots, \theta_m)$ and dispersion matrix identity. The corresponding normal density will be denoted by $p_\theta(x)$. Let $\Theta^* = R^m - \{0\}$ and F be any σ-finite measure on Θ^* such that $f(x) = \exp(½ \mid x \mid^2) \int p_\theta(x) F(d\theta) < \infty$ for all $x \in R^m$. Here $\mid x \mid = (\sum x_i^2)^{1/2}$, i.e. the usual Euclidean norm. Let $\nabla f(x)$ and $\nabla^2 f(x)$ denote the first derivative vector and the second derivative matrix respectively. Then the generalized Bayes estimator of θ with respect to F under the loss (1.1) is given by $\delta_F(x) = (\nabla^2 f(x))^{-1} \nabla f(x)$ (assuming $\nabla^2 f(x)$ is invertible). In the case of spherically symmetric F, $\delta_F(x)$ can be written as $\phi_F(\mid x \mid)(x/\mid x \mid)$ where

$$\phi_F(\mid x \mid) = \frac{\int_0^\infty (1/\nu)\sinh \nu \mid x \mid \mu_F(d\nu)}{\int_0^\infty \cosh \nu \mid x \mid \mu_F(d\nu)}. \tag{2.1}$$

Here

$$\mu_F(d\nu) = \nu^2 \exp(-½ \nu^2) F_1(d\nu)$$

and

$$F_1(d\nu) = \int \exp(-½ \textstyle\sum_2^m \theta_i^2) F(d\nu, d\theta).$$

See Zaman (1981) for details. In the same paper, Zaman has shown that any admissible spherically symmetric estimator has the above representation (2.1) for some finite measure μ on $[0, \infty)$.

Since we are concerned only with spherically symmetric estimators, in what follows the measure F will be orthogonally invariant. Also, because priors with compact support trivially yield admissible procedures, we will assume the support of F is unbounded.

The proofs of our main results obviously depend on the fundamental Stein-LeCam theorem which gives necessary and sufficient condition for admissibility. It is so well known that we will not pause to state it. However, to fix our ideas, we will use the version stated and proved in the appendix of Berger et al. (1982). For other references see Farrell (1968).

The approximating sequence of finite measures given by Stein-LeCam theorem will be denoted by G_n. The necessity part of Stein-LeCam theorem, moreover, says that there exists a compact set $K \subset \Theta^*$ such that $G_n(K) \geq 1$ for all n. We will take, without loss of generality, K to be a compact subset of $S_1 = \{\theta: \mid \theta \mid \leq 1\}$. Note that since F is spherically symmetric it suffices to assume that so are G_n's. Also, without loss of generality, assume $F(S_1 \cap \Theta^*) > 0$ where $S_1 = \{x: \mid x \mid \leq 1\}$ (since F is a nontrivial measure on Θ^*, there exists $r > 0$ such that $F(S_r \cap \Theta^*) > 0$ and the proofs of the results in this paper go through for any r; so we will take r to be 1.)

As indicated in the introduction, our necessary and sufficient conditions given in Theorems 3.1 and 4.1 are in terms of a variational problem similar to the one considered by Brown (1971), Srinivasan (1981) and Johnstone (1981). Naturally, our proofs (especially the sufficiency part) depend on smooth minimizing functions of this variational problem and certain technical lemmas involving the multivariate normal density. The needed technical results are available in Brown (1971) and Srinivasan (1981). For the sake of completeness, we state them below without proofs. Through the rest of this paper let $\hat{f}(x) = \int p_\theta(x)F(d\theta)$ and define the set J to be

$$J = \begin{cases} \text{the set of nonnegative real valued piecewise differentiable} \\ \text{functions } j(x) \text{ defined on } R^m \text{ such that} \\ j(x) \to 0 \text{ as } |x| \to \infty \text{ and } j(x) \geq 1 \text{ for } |x| \leq 1. \end{cases}$$

Moreover, since the proof of the sufficiency part is in many respects similar to the arguments in the main theorems of Brown (1971) and Srinivasan (1981), we will not give all the details. Instead, we will carry out the computations up to a stage from where one can complete the proof of sufficiency by following the arguments of the main theorems of Brown or Srinivasan.

For the proofs of the following three results (Lemmas 2.1, 2.2 and 2.3) see Brown (1971).

LEMMA 2.1. *Let F be a nonnegative measure on R^m such that the closed convex hull of the support of F is R^m. Then for every measure G with compact support in R^m*

$$\lim_{|x|\to\infty} \frac{\hat{g}(x)}{\hat{f}(x)} = 0.$$

LEMMA 2.2. *Let F be a measure satisfying the condition $|\nabla \log \hat{f}(x)| < B$ for all x in R^m. Then given a constant $K > 0$ there exist constants $K_1 > 0$, $K_2 > 0$ (depending only on m, K and B) such that*

i) $$\int e^{K|x-\theta|} p_\theta(x)F(d\theta) \leq K_1 \hat{f}(x) \quad \textit{for all} \quad x \in R^m$$

ii) $$\int e^{-K|x-\theta|} p_\theta(x)F(d\theta) \geq K_2 \hat{f}(x) \quad \textit{for all} \quad x \in R^m.$$

We also need the following consequences of Lemmas 2.2 and 2.3. For a proof of the next result see Srinivasan (1981).

LEMMA 2.3. *Let F be a measure on R^m satisfying the condition $|\nabla \log \hat{f}(x)| < B$ for all $x \in R^m$. Then there exists a constant $B_1 > 0$ (depending only on B and the dimension m) such that*

$$|\Delta \log f(x)| < B_1 \quad \textit{for all} \quad x \in R^m$$

where $f(x) = \exp(\frac{1}{2}|x|^2)\hat{f}(x)$ and "Δ" is the Laplacian.

The following deep result, due to Brown (1971), plays a crucial role in the proof of the sufficiency part (Theorem 4.1). One can give an alternative proof to Theorem 4.1 using Poincare inequalities. However, we have chosen to use Brown's result because the alternate proof is not any easier than the present approach. The version of Brown's result given below is for the spherically symmetric case and is tailored to our needs.

LEMMA 2.4. *Let F be a spherically symmetric measure such that $|\nabla \log \hat{f}(|x|)| < B$ for all $x \in R^m$. Suppose*

$$\inf_{j \in J} \int \left| \nabla j(x) \right|^2 \frac{1}{1 + |x|^2} \hat{f}(|x|)\, dx = 0. \tag{2.2}$$

Then there exists a constant $B_1 > 0$ (depending only on B and m) such that for a given $\varepsilon > 0$ there exists a spherically symmetric function $h \in J$ satisfying

(i) $\displaystyle\int \left| \nabla h(x) \right|^2 \frac{1}{1 + |x|^2} \hat{f}(|x|)\, dx < \varepsilon$

(ii) $h(x) \le B_1 h(y) \exp(B_1 |x - y|)$

(iii) *there exist a family of probability densities $\{u(x, y): x \in R^m\}$ such that $u(x, y) < B_1$ for x, y in R^m and*

$$h(x) = \int h(y) u(x, y)\, dy$$

Moreover, $u(x, y) = 0$ if $|x - y| > 1$.

(iv) $\displaystyle\int h^2(\theta) F(d\theta) < \infty.$

There is a slight difference between Lemma 2.4 as stated in this paper and the version stated and proved in Brown (1971; Lemma 5.4.1). The difference lies in the appearance of the term $1/(1 + |x|^2)$ in the integral

$$\int \left| \nabla j(x) \right|^2 \frac{1}{1 + |x|^2} \hat{f}(|x|)\, dx,$$

but this does not cause any problems and the proof of Brown goes through.

LEMMA 2.5. *Assume F is a measure satisfying the conditions of Lemma 2.4. Let $h \in J$ be the spherically symmetric function given by Lemma 2.4 for some $\varepsilon > 0$. Then there exists a constant $C > 0$ (depending only on B and m) such that*

$$\frac{g''(|x|)}{g(|x|)} > C \frac{f''(|x|)}{f(|x|)}$$

where

$$f(|x|) = \int e^{\theta_1|x|}e^{-1/2|\theta|^2}F(d\theta),$$

$$g(|x|) = \int e^{\theta_1|x|}e^{-1/2|\theta|^2}h^2(\theta)F(d\theta)$$

and, f'' and g'' are the second derivatives of f and g respectively.

The proof of this lemma depends on Lemmas 2.1 through 2.4 and is given in the appendix.

3. Necessary condition for admissibility. In this section we present the necessary condition in terms of a variational problem involving the function $\hat{f}(x)$.

THEOREM 3.1. *Suppose* $|\nabla \log \hat{f}(x)| < B \ \forall x \in R^m$. *Then a necessary condition for the admissibility of* $\delta_F(x)$ *is*

$$\inf_{j\in J} \int_{|x|\geq 1} |\nabla j(x)^2| \frac{1}{1+|x|^2}\hat{f}(x)\,dx = 0. \tag{3.1}$$

Before we proceed to prove the theorem, we would like to make a remark about the condition $|\nabla \log \hat{f}(x)| < B$. This condition is same as the one in Brown (1971). We are not aware of any statistical interpretation of this condition in the present problem. The only interpretation we can give here is that it is a growth condition on the measure F. In what follows we will drop the subscript F in δ_F.

PROOF. Suppose $\delta(x)$ is admissible. Then by Stein-LeCam theorem (see Berger, Zaman and Berliner, 1981) there exist a sequence of spherically symmetric finite measures $\{G_n\}$ with compact supports on Θ^* such that

(i) $\int p_\theta(x)G_n(d\theta) \geq 1$ for $|x| \leq 1$, for all n.

(ii) $R(G_n, \delta) - R(G_n, \delta_n) = \int (R(\theta, \delta) - R(\theta, \delta_n))G_n(d\theta) \to 0$ as $n \to \infty$, where δ_n is the Bayes estimator with respect to the measure G_n.

Now, formula (2.1) and routine calculation yields

$$\begin{aligned} R(G_n, \delta) - R(G_n, \delta_n) &= \int (\delta(x) - \delta_n(x))^t\left(\nabla^2 \int e^{\theta x}e^{-1/2|\theta|^2}G_n(d\theta)\right) \\ &\qquad \cdot (\delta(x) - \delta_n(x))e^{-1/2|x|^2}\,dx \\ &= \int \left|\frac{f(|x|)}{f'(|x|)} - \frac{g_n(|x|)}{g_n'(|x|)}\right|^2 g_n'(|x|)e^{-(1/2)|x|^2}\,dx \end{aligned} \tag{3.2}$$

where

$$f(|x|) = \int_0^\infty \left(\frac{1}{\nu} \sinh \nu |x|\right) \nu^2 e^{-1/2\nu^2} F_1(d\nu).$$

$f'(|x|) = (d/d|x|)f(|x|)$ and, $g_n(|x|)$ and $g_n'(|x|)$ are similarly defined. The rest of the proof involves constructing a smooth sequence of functions which give lower bounds of (3.2). Towards this construction, define a sequence of nonnegative piecewise differentiable functions $\{q_n(z)\}$ as follows. Let $C > 1$ be a constant such that $g_n'(z)/g_n(z) < C(f'(z))/f(z)$ for $z \le 1$. Set $q_n(z) = g_n(z)$ for $z \le 1$ and define

$$\log q_n(z) = \int_1^z \min\left\{\frac{d}{dy} \log g_n(y),\ C \frac{d}{dy} \log f(y)\right\} dy + \log g_n(1) \quad \text{for } z \ge 1. \tag{3.3}$$

Then q_n has the following properties.

(i) Since $(d/dz) \log q_n(z) \le (d/dz) \log g_n(z)$ for $z > 1$,

$$\log q_n(z) - \log q_n(1) \le \log g_n(z) - \log g_n(1) \quad \text{for} \quad z > 1, \tag{3.4}$$

and so $q_n(z) \le C_1 g_n(z)$ for all z for some $C_1 > 1$.

(ii) $\dfrac{g_n'(z)/g_n(z)}{q_n'(z)/q_n(z)} \ge 1$ for $z > 0$,

(iii) $\dfrac{q_n(z)}{q_n'(z)} = \max\left(\dfrac{g_n(z)}{g_n'(z)}, \dfrac{1}{C}\dfrac{f(z)}{f'(z)}\right)$,

(iv) $\left|\dfrac{q_n(z)}{q_n'(z)} - \dfrac{f(z)}{f'(z)}\right|^2 \le \left|\dfrac{g_n(z)}{g_n'(z)} - \dfrac{f(z)}{f'(z)}\right|^2.$

Denoting $|x|$ by r, we thus have from (3.2) and (3.4) (iv)),

$$\int \left|\frac{f(r)}{f'(r)} - \frac{g_n(r)}{g_n'(r)}\right|^2 g_n'(r) e^{-(1/2)r^2}\, dx \ge \int \left|\frac{q_n(r)}{q_n'(r)} - \frac{f(r)}{f'(r)}\right|^2 g_n'(r) e^{-(1/2)r^2}\, dx. \tag{3.5}$$

Now, let $j_n(r) = q_n(r)/f(r)$. We can assume without loss of generality that

$$j_n(r) = q_n(r)/f(r) \ge 1 \quad \text{for} \quad r \le 1.$$

Moreover j_n satisfies the following conditions.

(3.6)

a) $j_n(r) \le g_n(r)/f(r)$ for $r \ge 1$;

b) By Lemma 2.1, $j_n(r) \to 0$ as $r \to \infty$ for all n.

Substituting $j_n(r)$ for $q_n(r)/f(r)$ and using polar coordinates, we get

$$\int \left| \frac{q_n(r)}{q_n'(r)} - \frac{f(r)}{f'(r)} \right|^2 g_n'(r) e^{-(1/2)r^2} \, dx$$

$$= s_1 \int_0^\infty | q_n(r) f'(r) - q_n'(r) f(r) |^2 \frac{g_n'(r)}{(q_n'(r))^2 (f'(r))^2} e^{-(1/2)r^2} r^{m-1} \, dr$$

$$(3.7) \quad = s_1 \int_0^\infty | j_n(r) f'(r) f(r) - (j_n(r) f(r))' f(r) |^2 \frac{g_n'(r)}{(q_n'(r))^2 (f'(r))^2} e^{-(1/2)r^2} r^{m-1} \, dr$$

$$= s_1 \int_0^\infty | j_n'(r) |^2 \frac{f^4(r) g_n'(r)}{(q_n'(r))^2 (f'(r))^2} e^{-(1/2)r^2} r^{m-1} \, dr,$$

where s_1 is a constant. Invoking now the property (ii), $g_n'(r) q_n(r)/g_n(r) q_n'(r) \geq 1$, of (3.4) we have therefore

$$(3.8) \qquad (3.7) \geq \frac{s_j}{C_1} \int | j_n'(r) |^2 \frac{f^4(r)}{(f'(r))^2} \frac{1}{q_n'(r)} e^{-(1/2)r^2} r^{m-1} \, dr.$$

Appealing now to property (iii), $q_n(r)/q_n'(r) \geq 1/C \, (f(r)/f'(r))$, of (3.4)

$$(3.9) \qquad \begin{aligned} (3.8) &\geq C_3 \int | j_n'(r) |^2 \frac{1}{j_n(r)} \frac{f^4(r)}{(f'(r))^3} e^{-(1/2)r^2} r^{m-1} \, dr \\ &= C_3 \int | h_n'(r) |^2 \frac{f^4(r)}{(f'(r))^3} e^{-(1/2)r^2} r^{m-1} \, dr \end{aligned}$$

where $h_n(r) = j_n^{1/2}(r)$. Recalling the definition of $f(|x|)$, observe that

$$f(|x|) = \int_{-\infty}^\infty \nu e^{\nu |x|} e^{-(1/2)\nu^2} F_1(d\nu) = \frac{d}{d|x|} \int_{-\infty}^\infty e^{\nu|x|} e^{-(1/2)\nu^2} F_1(d\nu).$$

Therefore, setting $\tilde{f}(|x|) = \int_{-\infty}^\infty e^{\nu|x|} e^{-(1/2)\nu^2} F_1(d\nu)$, it is easy to see that the assumption $| \nabla \log \hat{f}(x) | < B$ is equivalent to

$$| f(|x|)/\tilde{f}(|x|) - |x| | < B.$$

Moreover, by Schwartz inequality

$$f'(|x|)/\tilde{f}(|x|) \geq (f(|x|)/\tilde{f}(|x|))^2.$$

Combining these facts along with Lemma 2.3 we have, for all large $r > 0$

$$(3.10) \qquad \frac{f^4(r)}{(f'(r))^2} = (\tilde{f}(r))^2 \left(\frac{f(r)}{\tilde{f}(r)} \right)^4 \left(\frac{\tilde{f}(r)}{f'(r)} \right)^2 \geq C_4 (\tilde{f}(r))^2.$$

Therefore, using the estimate (3.10) in (3.9) and by Lemma 2.3 we get

$$
\begin{aligned}
(3.9) &\geq C_5 \int |h_n'(r)|^2 \frac{1}{(f'(r)/\tilde{f}(r))} \tilde{f}(r) e^{-(1/2)r^2} r^{m-1}\, dr \\
&\geq C_5 \int |h_n'(r)|^2 \frac{1}{C + (f(r)/\tilde{f}(r))^2} \tilde{f}(r) e^{-(1/2)r^2} r^{m-1}\, dr.
\end{aligned} \tag{3.11}
$$

Finally, recall that $|f(|x|)/\tilde{f}(|x|)|^2 \leq 2|x|^2 + 2B$ and, note j_n and hence $h_n = j_n^{1/2}$ belong to the class J. Therefore,

$$
R(G_n, \delta_F) - R(G_n, \delta_n) \geq C \inf_{j\in J} \int_{|x|\geq 1} (j'(|x|))^2 \frac{1}{1 + |x|^2} \hat{f}(x)\, dx \tag{3.12}
$$

for all n. Appealing to the Stein-LeCam Theorem now, we have that δ_F is admissible implies the right side of (3.12) is zero. This completes the proof.

A few comments are in order at this stage. A necessary condition for the admissibility of δ_F can be obtained without the growth condition $|\nabla \log \hat{f}(x)| < B$. We stated our Theorem 3.1 with this condition mainly because it gives an elegant as well as easily verifiable necessary condition for admissibility (what can be easier than convoluting a measure with the normal density and obtaining estimates for its tail behavior!). Below, we give a version of Theorem 3.1 without any assumption.

Let $\delta_F(x) = \phi(|x|)(x/|x|)$ be the generalized Bayes procedure with respect to a spherically symmetric measure F with unbounded support. Then we have the following result.

THEOREM 3.2. *A necessary condition for the admissibility of* $\delta_F(x) = \phi(|x|)(x/|x|)$ *is*

$$
\inf_{j\in\tilde{J}} \int |j'(r)|^2 \phi^3(r) \exp\left(-\int \left(r - \frac{1}{\phi(r)}\right) dr\right) r^{m-1}\, dr = 0
$$

where, $\int (r - 1/\phi(r))\, dr$ *is the usual indefinite integral and* $\tilde{J} = \{j \in J: j$ *is spherically symmetric*$\}$.

PROOF. The proof follows from the Stein-LeCam theorem, step (3.9) of the proof of Theorem 3.1 and the fact that $\phi(|x|) = f(|x|)/f'(|x|)$.

We conclude this section with a result which gives a necessary condition for the admissibility of a spherically symmetric procedure which is not generalized Bayes. Many of the procedures proposed for the control problem in the past are not generalized Bayes (they are, however, approximately generalized Bayes; see Berger, Zaman and Berliner (1982) for details). Indeed, the complete class obtained by Zaman (1981) contains, in addition to all the generalized Bayes procedures, a plethora of nongeneralized Bayes procedures. In view of this, it is important to know whether a given spherically symmetric procedure (procedure belonging to Zaman's complete class) is inadmissible. The following theorem gives a necessary condition for admissibility of such estimators.

THEOREM 3.3. *Let $\delta(x) = \phi(|x|)(x/|x|)$ be a spherically symmetric estimator. Let $\tilde{J}$ be as defined in Theorem 3.2. Assume, for every $K > 0$*

$$(*)\qquad \lim_{r\to\infty}\exp\left(Kr - \int \frac{1}{\phi(r)}\,dr\right) = 0.$$

Then a necessary condition for the admissibility of $\delta(x)$ is

$$\inf_{j\in\tilde{J}} \int_{|x|\geq 1} |j'(r)|^2\phi^3(r)\left(\exp\left(\int \frac{1}{\phi(r)}\,dr - r^2/2\right)\right)r^{m-1}\,dr = 0.$$

PROOF. Suppose $\delta(x) = \phi(|x|)(x/|x|)$ is admissible. Then, by the complete class theorem of Zaman (1980), $\phi(r) = f(r)/f'(r)$ where $f(r) = \int_0^\infty 1/\eta \sinh \eta r \mu(d\eta)$ for some finite measure μ. Therefore,

$$\delta(x) = \frac{f(|x|)}{f'(|x|)}\frac{x}{|x|} \quad \text{where} \quad f(r) = \int \frac{1}{\phi(r)}\,dr.$$

Also, for any finite spherically symmetric measure G with compact support, $g(r) \leq Ke^{rK}$ for some $K > 0$ where $g(|x|) = \int \theta_1 e^{\theta_1|x|}e^{-(1/2)|\theta|^2}G(d\theta)$. Combining the above two facts with the assumption $(*)$, it follows that $\lim_{r\to\infty}(g(r)/f(r)) = 0$ for every $g(r)$ given by a finite measure G with compact support. With this observation, one can easily adapt the proof of Theorem 3.1 to the present situation and the above theorem would follow from step (3.9) of Theorem 3.1.

4. Sufficient condition.

THEOREM 4.1. *Let $\delta_F(x)$ be a generalized Bayes estimator with respect to a spherically symmetric prior F. Assume*

$$(4.1)\qquad |\nabla \log \hat{f}(x)| < B.$$

Then a sufficient condition for δ_F to be admissible is

$$(*)\qquad \inf_{j\in J} \int_{|x|\geq 1} |\nabla j(x)|^2 \frac{1}{1 + |x|^2}\hat{f}(x)\,dx = 0.$$

PROOF. Assume $(*)$ holds. We shall exhibit, for a given $\varepsilon > 0$, a finite measure G on Θ^* such that

$$(4.2)\qquad \int (R(\theta, \delta_F) - R(\theta, \delta_G))G(d\theta) < \varepsilon.$$

Then it would follow from Stein-LeCam theorem, since ε is arbitrary, that δ_F is admissible. Towards this, let $\eta > 0$ (to be chosen later) be fixed. Then, by Lemma 2.4 (note the conditions of the lemma are satisfied here) there exists a function

$j \in J$ such that

$$\int_{|x|\ge 1} |\nabla j(x)|^2 \frac{1}{1+|x|^2} \hat{f}(x)\, dx < \eta$$

and possessing the properties (ii), (iii), and (iv) listed here. We can take, without loss of generality, $j(x)$ to be spherically symmetric. Define the measure G by setting $G(d\theta) = j^2(|\theta|)F(d\theta)$. Plainly, G is a spherically symmetric finite measure. Set $\hat{g}(|x|) = \int p_\theta(x)G(d\theta)$ and $g(|x|) = e^{1/2|x|^2}\hat{g}(|x|)$. Let $g'(|x|)$ and $g''(|x|)$ denote the first and second derivatives of g. Define $\tilde{g}'(|x|)$ and $\tilde{g}''(|x|)$ by setting

$$\tilde{g}'(y) = \frac{g'(y)}{g(y)} = \int \theta_1 e^{\theta_1 y} e^{-(1/2)|\theta|^2} G(d\theta)/g(y)$$

$$\tilde{g}''(y) = \frac{g''(y)}{g(y)} = \int \theta_1^2 e^{\theta_1 y} e^{-(1/2)|\theta|^2} G(d\theta)/g(y).$$

$\tilde{f}'(|x|)$ and $\tilde{f}''(|x|)$ are similarly defined. For the rest of the proof, x appearing as an argument of a function should be interpreted as $|x|$. Thus $f(x) = f(|x|)$, $f'(x) = f'(|x|)$ and so on. Also, the constants that appear in the proof that follows are absolute constants depending only on the dimension m and the constant B in (4.1). Then it is easy to show that

$$R(G, \delta_F) - R(G, \delta_G) = \int (R(\theta, \delta_F) - R(\theta, \delta_G))G(d\theta)$$

$$= \int \left| \frac{\tilde{g}'(x)}{\tilde{g}''(x)} - \frac{\tilde{f}'(x)}{\tilde{f}''(x)} \right|^2 \tilde{g}''(x) g(x) p_0(x)\, dx \tag{4.3}$$

$$\le 2 \int \left| \frac{\tilde{g}'(x)}{\tilde{g}''(x)} - \frac{\tilde{f}'(x)}{\tilde{g}''(x)} \right|^2 \tilde{g}''(x) g(x) p_0(x)\, dx$$

$$+ 2 \int \left| \frac{\tilde{f}'(x)}{\tilde{g}''(x)} - \frac{\tilde{f}'(x)}{\tilde{f}''(x)} \right|^2 \tilde{g}''(x) g(x) p_0(x)\, dx \tag{4.4}$$

where $p_0(x) = e^{-(1/2)|x|^2}$. We shall obtain below upper bounds for the right side of (4.4). Consider the first term. By Lemma 2.5,

$$\int \left| \frac{\tilde{g}'(x)}{\tilde{g}''(x)} - \frac{\tilde{f}'(x)}{\tilde{g}''(x)} \right|^2 \tilde{g}''(x) g(x) p_0(x)\, dx$$

$$= \int |\tilde{g}'(x) - \tilde{f}'(x)|^2 \frac{1}{\tilde{g}''(x)} g(x) p_0(x)\, dx$$

$$\le C \int |\tilde{g}'(x) - \tilde{f}'(x)|^2 \frac{1}{\tilde{f}''(x)} g(x) p_0(x)\, dx \tag{4.5}$$

$$\leq C \int \frac{1}{\tilde{f}''} \int (j(\theta) - j(x))^2(\theta_1 - \tilde{f}'(x))^2 p_\theta(x)F(d\theta)\, dx \tag{4.6}$$

by Schwartz inequality. Now consider the 2nd term of (4.4). By Lemma 2.5,

$$\int \left| \frac{1}{\tilde{g}''(x)} - \frac{1}{\tilde{f}''(x)} \right|^2 (\tilde{f}'(x))^2 \tilde{g}''(x) g(x) p_0(x)\, dx$$

$$= \int |\tilde{g}''(x) - \tilde{f}''(x)|^2 \frac{(\tilde{f}'(x))^2}{(\tilde{f}''(x))^2} \frac{1}{\tilde{g}''(x)} g(x) p_0(x)\, dx \tag{4.7}$$

$$\leq C \int |\tilde{g}''(x) - \tilde{f}''(x)|^2 \frac{(\tilde{f}'(x))^2}{(\tilde{f}''(x))^3} g(x) p_0(x)\, dx.$$

Observe now,

$$\tilde{g}''(x) - \tilde{f}''(x) = \int (\theta_1^2 - \tilde{f}''(x))(j^2(\theta) - j^2(x)) p_\theta(x) F(d\theta)/\hat{g}(x) \tag{4.8}$$

and therefore

$$\begin{aligned} (4.7) \leq C \int \left| \int (\theta_1^2 - \tilde{f}''(x))(j^2(\theta) - j^2(x)) \frac{P_\theta(x)F(d\theta)}{\hat{f}(x)} \right|^2 \\ \cdot \frac{(\tilde{f}'(x))^2}{(\tilde{f}''(x))^3} \frac{f^2(x)}{g(x)} p_0(x)\, dx, \end{aligned} \tag{4.9}$$

since $\hat{g}(x) = p_0(x)g(x)$ and $\hat{f}(x) = p_0(x)f(x)$. It is easy to see, by assumption (4.1) and Lemma 2.3, $\tilde{f}''(x) - (\tilde{f}'(x))^2 < B_1$. Using this estimate, we have

$$\begin{aligned} (4.9) \leq C \int \left| \int (\theta_1^2 - (\tilde{f}'(x))^2)(j^2(\theta) - j^2(x)) p_\theta(x) \frac{F(d\theta)}{\hat{f}(x)} \right|^2 \\ \cdot \frac{(\tilde{f}'(x))^2}{(\tilde{f}''(x))^3} \frac{f^2(x)}{g(x)} p_0(x)\, dx \\ + C \int \left| \int (j^2(\theta) - j^2(x)) p_\theta(x) \frac{F(d\theta)}{\hat{f}(x)} \right|^2 \\ \cdot \frac{(\tilde{f}'(x))^2}{(\tilde{f}''(x))^3} \frac{f^2(x)}{g(x)} p_0(x)\, dx; \end{aligned} \tag{4.10}$$

call the two terms in the right side of (4.10) as I and II respectively. The term II can easily be bounded as follows. Using Schwartz inequality

$$\begin{aligned} \mathrm{II} \leq \int \left(\int (j(\theta) + j(x))^2 \frac{p_\theta(x)F(d\theta)}{\hat{f}(x)} \right) \\ \cdot \left(\int (j(\theta) - j(x))^2 \frac{p_\theta(x)F(d\theta)}{\hat{f}(x)} \right) \frac{(\tilde{f}'(x))^2}{(\tilde{f}''(x))^3} \frac{f^2(x)}{g(x)} p_0(x)\, dx. \end{aligned} \tag{4.11}$$

Lemma 2.4 (ii) along with Lemma 2.2 implies

$$\int (j(\theta) + j(x))^2 p_\theta(x) \frac{F(d\theta)}{\hat{f}(x)} \le Cj^2(x)$$

and

$$f(x)/g(x) \ge Cj^2(x).$$

Combining these two facts along with (4.11), we have

$$\text{II} \le C \int \int (j(\theta) - j(x))^2 p_\theta(x) F(d\theta) \frac{1}{(\tilde{f}''(x))^2}\, dx \tag{4.12}$$

because, by Schwartz inequality, $\tilde{f}''(x) - (\tilde{f}'(x))^2 \ge 0$. Consider now the term I on the right side of (4.10); writing $\theta_1^2 - (\tilde{f}'(x))^2$ as $(\theta_1 + \tilde{f}'(x))(\theta_1 - \tilde{f}'(x))$ and $j^2(\theta) - j^2(x)$ as $(j(\theta) + j(x))(j(\theta) - j(x))$ and using Schwartz inequality we have

$$\begin{aligned} &\left(\int (\theta_1^2 - (\tilde{f}'(x))^2)(j^2(\theta) - j^2(x)) \frac{p_\theta(x)F(d\theta)}{\hat{f}(x)}\right)^2 \\ &\le \left(\int (\theta_1 + \tilde{f}'(x))^2 (j(\theta) + j(x))^2 \frac{p_\theta(x)F(d\theta)}{\hat{f}(x)}\right) \\ &\times \left(\int (\theta_1 - \tilde{f}'(x))^2 (j(\theta) - j(x))^2 \frac{p_\theta(x)F(d\theta)}{\hat{f}(x)}\right). \end{aligned} \tag{4.13}$$

Now recall that $x = (|x|, 0, \cdots, 0)$ and therefore

$$\begin{aligned} (\theta_1 + \tilde{f}'(x))^2 &= (\theta_1 - |x| + |x| + \tilde{f}'(x))^2 \\ &\le 2|\theta - x|^2 + 2(|x| + \tilde{f}'(x))^2. \end{aligned} \tag{4.14}$$

But, by assumption (4.1), $|\tilde{f}'(x) + |x|| \le B + 2\ |\tilde{f}'(x)|$ and hence

$$(\theta_1 + \tilde{f}'(x))^2 \le 2|x - \theta|^2 + 2B^2 + 4|\tilde{f}'(x)|^2. \tag{4.15}$$

Using this estimate along with Lemma 2.2 and Lemma 2.4 we can bound the first term on the right side of (4.13) as follows:

$$\begin{aligned} &\int (\theta_1 + \tilde{f}'(x))^2 (j(\theta) + j(x))^2 \frac{p_\theta(x)F(d\theta)}{\hat{f}(x)} \\ &\le Cj^2(x)(1 + (\tilde{f}'(x))^2) \int e^{C|x-\theta|} \frac{p_\theta(x)F(d\theta)}{\hat{f}(x)}. \end{aligned} \tag{4.16}$$

Combining (4.16) and (4.13), and using Lemma 2.2 and the argument leading to (4.12) we have

$$\begin{aligned} I \le C \int (1 + \tilde{f}'(x))^2 \frac{(\tilde{f}'(x))^2}{(\tilde{f}''(x))^3} \int (j(\theta) - j(x))^2 \\ \cdot (\theta_1 - \tilde{f}'(x))^2 p_\theta(x) F(d\theta)\, dx \end{aligned} \tag{4.17}$$

$$(4.18) \qquad \leq C \int \frac{1}{\tilde{f}''(x)} \int (j(\theta) - j(x))^2 (\theta_1 - \tilde{f}'(x))^2 p_\theta(x) F(d\theta)\, dx.$$

Thus (4.7) can be bounded by (4.12) and (4.18) via (4.8) and therefore we have

$$(4.19) \qquad (4.7) \leq C \int \frac{1}{\tilde{f}''(x)} \int [(\theta_1 - \tilde{f}'(x))^2 + C]\, (j(\theta) - j(x))^2 p_\theta(x) F(d\theta)\, dx.$$

In obtaining (4.19), we have used the fact $\tilde{f}'(x) > 1$ if $|x| > B + 1$ by (4.1) and therefore $\inf \tilde{f}''(x) > 1/C > 0$. Now, going back to (4.6) and combining it with (4.17) we therefore have

$$(4.20) \qquad R(G, \delta_F) - R(G, \delta_G) \leq C \int \frac{1}{\tilde{f}''(x)} \int [(\theta_1 - \tilde{f}'(x))^2 + C] \cdot (j(\theta) - j(x))^2 p_\theta(x) F(d\theta)\, dx$$

$$(4.21) \qquad \leq C \int \frac{1}{\tilde{f}''(x)} \int e^{C|x-\theta|} (j(\theta) - j(x))^2 \cdot p_\theta(x) F(d\theta)\, dx$$

since $|\theta_1 - \tilde{f}'(x)| \leq |\theta - x| + |x - \tilde{f}'(x)| \leq |\theta - x| + B$. Observe now that by Lemma 2.2 and the assumption (4.1), we have for $|x| > B + 1$,

$$(4.22) \qquad \frac{\tilde{f}''(\theta)}{\tilde{f}''(x)} \leq \frac{C + |\theta|^2}{|x|^2 - B} \leq \frac{C + 2|x|^2 + 2|x - \theta|^2}{x^2 - B} \leq \frac{C + 2(B+1)}{B} + 2|x - \theta|^2 \leq C + C|x - \theta|^2$$

and therefore, using (4.22) in (4.21), we get

$$(4.23) \qquad (4.21) \leq C \int \frac{1}{\tilde{f}''(\theta)} \int e^{C|x-\theta|} (j(\theta) - j(x))^2 p_\theta(x)\, dx\, F(d\theta).$$

The inner integral in 4.23, using the property (iii) of j listed in Lemma 2.2 and following the argument of Brown (1971), can be shown to be bounded by

$$(4.24) \qquad C \int_{|W| \leq C+1} \int \int_{|\xi| \leq C+1} |\nabla j(x)|^2 e^{-(1/2)|x+\xi+W-\theta|^2}\, dx\, d\xi\, dW$$

(for details see Brown, 1971, page 895). Substituting (4.24) in (4.23) and using (4.22) again, one gets

$$(4.25) \qquad (4.21) \leq C \int |\nabla j(x)|^2 \frac{1}{\tilde{f}''(x)} \int_{|\xi_1| \leq C} \int_{|\xi_2| \leq C} \int_{|\xi_3| \leq C} \times p_\theta(x + \xi_1 + \xi_2 + \xi_3) F(d\theta)\, dx\, d\xi_1\, d\xi_2\, d\xi_3$$

$$\leq C \int |\nabla j(x)|^2 \frac{1}{\hat{f}''(x)} \hat{f}(x)\, dx \tag{4.26}$$

$$\leq C \int |\nabla j(x)|^2 \frac{1}{1 + |x|^2} \hat{f}(x)\, dx. \tag{4.27}$$

In obtaining (4.26) and (4.27) we have used (4.1) repeatedly. Thus we have shown

$$R(G, \delta_F) - R(G, \delta_G) \leq C \int |\nabla j(x)|^2 \frac{1}{1 + |x|^2} \hat{f}(x)\, dx \leq c\eta$$

where C is an absolute constant. Finally, by choosing $\eta = \varepsilon/C$ the proof is completed.

REMARK. The above theorem can be generalized by weakening the condition (4.1) along the lines of Srinivasan (1981). Such a weakening of condition (4.1) would result in a weakened version of the Harnack inequality, Lemma 2.4 (ii) (i.e. the constant B, which appears in Lemma 2.4 (ii) would depend on $f(x)$ in a nontrivial way) and mean value property (Lemma 2.4 (iii)).

5. Applications. A look at the statements of Theorems 3.1 and 4.1 clearly indicates that, modulo certain growth conditions on the spherically symmetric prior, the necessary and sufficient condition for admissibility of a generalized Bayes estimator is that the infinum of a certain variational problem be zero. It is well known (see Brown 1971, Srinivasan 1981) that this latter condition is equivalent to certain nice behaviour of the tail of $\hat{f}(x) = \int p_\theta(x)F(d\theta)$. The following theorem, stated without proof, summarizes this fact.

THEOREM 5.1. *Assume the condition* $|\nabla \log \hat{f}(x)| < B$. *Then a necessary and sufficient condition for the admissibility of* δ_F *is*

$$\int_1^\infty \frac{1 + r^2}{r^{m-1} f(r)}\, dr = \infty. \tag{5.1}$$

The conditions of the above theorem are easily verified for all reasonable generalized priors. In a given situation, all one has to do is to compute or obtain estimates for $\hat{f}(x)$ and its first two derivatives. As an example, we consider below the class of spherically symmetric priors given by the density $|\theta|^{c-1}\, d\theta$, $c > 1 - m$. This family of priors was treated by Berger and Zaman (1980) and they have shown that the corresponding generalized Bayes estimator is given by $\delta_c(x) = (|x|^2 + c)^{-1}x + |x|^{-4}W(|x|)x$ where $W(|x|) = O(1)$ (as $|x| \to \infty$) and this class of generalized Bayes estimators contains most of the estimators proposed for the control problem.

Let $\hat{f}_c(x) = \int p_\theta(x)\, |\theta|^{c-1}\, d\theta$. For $c > 1 - m$, $\hat{f}_c(x)$ is finite for all x and, since $|(d/dr) \log r^{-1}| < B$ for large r, it follows $|\nabla \log \hat{f}_c(x)| < B$ for all x and $c > 1 - m$. Moreover, $\hat{f}_c(x) = O(|x|^{c-1})$ for large $|x|$. Therefore an application of

Theorem 5.1 would imply $\delta_c(x)$ is admissible if and only if

$$\int_1^\infty \frac{1 + |x|^2}{|x|^{m-1}|x|^{c-1}} \, d|x| = \infty,$$

i.e. $\delta_c(x)$, is admissible if and only if $m + c - 4 \leq 1$ or $m + c \leq 5$. In particular, the generalized Bayes estimator with respect to the Lebesgue measure $\delta(x) = x/(1 + |x|^2)$, which is obtained by setting $c = 1$ above, is admissible if and only if $m \leq 4$. It is appropriate to point out here that the inadmissibility part of the above example (i.e. δ_c is inadmissible if $m + c > 5$) was first obtained by Berger and Zaman (1980) by a different argument. Stein and Zaman (1980) proved the admissibility of $\delta(x)$ for dimension $m = 4$ and its inadmissibility for $m = 5$. There have been other spherically symmetric procedures, which are not generalized Bayes, proposed for the control problem. Takeuchi (1968) considered procedures of the form $\delta^c(x) = (1/(c + |x|^2))x$, for $c > 0$ and has obtained certain asymptotic efficiency results for them. These procedures are not generalized Bayes. The following result, a consequence of Theorem 3.3, shows Takeuchi's procedures are inadmissible for dimension $m > 5$.

THEOREM 5.2. *The spherically symmetric procedure* $\delta_\beta^\alpha(x) = (1/(\alpha + \beta|x|^2))x$, $\alpha > 0$, $\beta > 0$ *is inadmissible if* $\beta > 1$ *or* $\beta = 1$ *and* $\alpha + m > 5$.

PROOF. It is easy to check that Theorem 3.3 is applicable here. Therefore, to prove the result, it suffices to show

$$\int_1^\infty \frac{1}{(\phi(r))^3} \exp\left(\frac{r^2}{2} - \int \frac{1}{\phi(r)} \, dr\right) \frac{1}{r^{m-1}} \, dr \tag{5.2}$$

is finite, where $\phi(r) = r/(\alpha + \beta r^2)$. Since $\int (1/\phi(r)) \, dr = \alpha \log r + \beta(r^2/2)$,

$$(5.2) < \int_1^\infty \frac{1}{r^3} \frac{1}{r^{m-1-\alpha}} e^{-(\beta-1)(r^2/2)} \, dr < \infty \tag{5.3}$$

if $\beta > 1$ or $\beta = 1$ and $\alpha + m > 5$. Hence, $\delta_\beta^\alpha(x)$ is inadmissible when $\beta > 1$ or $\beta = 1$ and $\alpha + m > 5$.

There is yet another spherically symmetric estimator which has been treated in great detail by econometricians (Anderson and Taylor, 1976; Basu, 1974). It is the so-called certainty equivalence estimator given by $\delta_0(x) = x/|x|^2$. For obvious reasons, $\delta_0(x)$ is the most natural estimator to be considered for a control problem. $\delta_0(x)$ has some very nice asymptotic properties including asymptotic normal distribution. Since $\delta_0(x)$ has infinite risk in low dimensions (for $m \leq 2$), a truncated version of it, $\delta_K(x) = \min(K, 1/|x|)(x/|x|)$ has been recommended by Anderson and Taylor (1976). It follows from Theorem 5.2 that either of these estimators is inadmissible if $m \geq 6$.

We conclude this section with a few remarks of general nature. One could propose scores of nonspherically symmetric procedures (particularly, generalized Bayes procedures) for a control problem. The interesting ones would be radial inverses of the known admissible estimator of θ, especially the inverses of (i) linear shrinkers considered by Cohen (1966), (ii) the linear estimators treated by Lindley and Smith (1971) and the ones developed and recommended by Rao (1976) from compound decision theoretic considerations. It would be of great interest to know whether any of the above procedures are admissible, and if not, how far are they from being admissible. Though we do not have answers to all these questions, we have a generalization of Theorem 4.1 to general priors. As for an analog of Theorem 3.1, we have only partial results. These generalizations and their applications, hopefully, will appear elsewhere.

APPENDIX

PROOF OF LEMMA 2.5. Let $j \in J$ be the spherically symmetric function given by Lemma 2.4 satisfying the Harnack inequality $j(x)/j(y) \le ce^{c|x-y|}$. Let $G(d\theta) = j^2(\theta)F(d\theta)$ and $\hat{g}(x) = \int p_\theta(x)G(d\theta)$. Assume, without loss of generality $x = (|x|, 0 \cdots 0)$. Then, by Lemma 2.2

$$\hat{g}(x) \le c^2 j^2(x) \int e^{2c|x-\theta|} p_\theta(x)F(d\theta) \le c_1 j^2(x)\hat{f}(x) \tag{A1}$$

where, of course, $\hat{f}(x) = \int p_\theta(x)F(d\theta)$. Now, setting $\tilde{f}''(x) = f''(x)/f(x)$, $\tilde{f}'(x) = f'(x)/f(x)$, it follows from Schwartz inequality and Lemma 2.3

$$0 < \tilde{f}''(x) - (\tilde{f}'(x))^2 \le c \quad \text{and} \quad |\tilde{f}'(x+\xi) - \tilde{f}'(x)| \le c|x - \xi|. \tag{A2}$$

Therefore, for any ξ such that $|\xi| < B_1$ and $|x|$ large (say $|x| > (B_1 + B + 1)^2$), we have

$$\begin{aligned} \frac{\tilde{f}''(x+\xi)}{\tilde{f}''(x)} &\le \frac{c + |\tilde{f}'(x+\xi)|^2}{(\tilde{f}'(x))^2} \\ &< \frac{c + c^2\,|x-\xi|^2 + (|x| + B)^2}{(|x| - B)^2} < B_1 < \infty. \end{aligned} \tag{A3}$$

Consider now $g''(x)/g(x)$. By the Harnack inequality and (A1), it is easy to see

$$\frac{g''(x)}{g(x)} = \frac{\int \theta_1^2 p_\theta(x) j^2(\theta)F(d\theta)}{\hat{g}(x)} \ge c_2 \frac{\int \theta_1^2 p_\theta(x)e^{-2c|x-\theta|}F(d\theta)}{\hat{f}(x)}. \tag{A4}$$

Now, by (A3), Lemma 2.2 is applicable for the measure $\mu(d\theta) = \theta_1^2 F(d\theta)$ and therefore we have

$$\int \theta_1^2 p_\theta(x)e^{-2c|x-\theta|}F(d\theta) \ge c_3 \int \theta_1^2 p_\theta(x)F(d\theta)$$

for all x. Using this fact in (A4) we get

$$\frac{g''(x)}{g(x)} \ge c_4 \frac{\int \theta_1^2 p_\theta(x)F(d\theta)}{\hat{f}(x)} = c_4 \frac{f''(x)}{f(x)}.$$

This completes the proof of the Lemma.

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