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PROPER ACTION IN STEPS, WITH APPLICATION TO DENSITY RATIOS OF MAXIMAL INVARIANTS[1]

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Let G be a locally compact group acting continuously on the left of a locally compact space $\mathscr{X}$. It is assumed that $G = HK$ where H and K are closed subgroups. It is shown that if K acts properly on $\mathscr{X}$ and H acts properly on $\mathscr{X}/K$, then G acts properly on $\mathscr{X}$. Under a mild additional condition the converse is also true. Several examples are given to show how these results can help decide the properness of composite actions. Proper action can be used to justify the representation of the density ratio of a maximal invariant as a ratio of integrals over the group.

1. Introduction. Comparison of proper action and Cartan property. Let X be a random variable taking values in a locally compact space $\mathscr{X}$ on which a locally compact group G acts continuously on the left. (In applications $\mathscr{X}$ will usually be a subset of Euclidean space.) We adopt in this paper Bourbaki's definition of "locally compact," which includes the requirement that the space be Hausdorff ([5] I Section 9.7, Definition 4). Let μ_G be left Haar measure on G, i.e., $\mu_G(g_1\, dg) = \mu_G(dg)$ for every $g_1 \in G$. We shall also assume that there is on $\mathscr{X}$ a measure λ (usually Lebesgue measure) that is relatively invariant with multiplier χ; i.e., there is a positive continuous function χ on G such that $\lambda(g\, dx) = \chi(g)\lambda(dx)$, $g \in G$. Consider only distributions P of X that have a density p with respect to λ. If $t(X)$ is a maximal invariant and p_1, p_2 two densities of X, then the following formula for the density ratio p_2^I/p_1^I of $t(X)$ has proved itself very useful:

$$\frac{p_2^I}{p_1^I}(t(x)) = \frac{\int p_2(gx)\chi(g)\mu_G(dg)}{\int p_1(gx)\chi(g)\mu_G(dg)}. \tag{1.1}$$

Such "integration over the group" methods in statistical problems were initiated by Charles Stein (1956) and several early applications in specific problems are known (for references see, e.g., [11]). Examples of more recent use of (1.1) include Andersson and Perlman (1984), Eaton and Kariya (1983), Kariya (1978), Wijsman (1972).

There is, however, a nontrivial question under what general conditions (1.1) is valid. In Wijsman (1967) (1.1) was derived first of all under the restrictions that G is a Lie group, $\mathscr{X}$ a subset of Euclidean space, and the action of G on $\mathscr{X}$ linear. In Wijsman (1972, Section 7) this was extended to affine action. These

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restrictions do not pose much of a limitation in practice since all known examples seem to be of that nature. However, there is another condition that was needed in [11] to derive (1.1): $\mathscr{X}$ was required to be a *Cartan* G-space. That means that every $x \in \mathscr{X}$ has a *thin* neighborhood, i.e., a neighborhood V_x such that $((V_x, V_x))$ has compact closure; cf. Palais (1961). We have used here Palais' notation: for any $A, B \subset \mathscr{X}$, define

$$(1.2) \qquad ((A, B)) = \{g \in G: gA \cap B \neq \varnothing\}.$$

Some sufficient conditions for $(\mathscr{X}, G)$ to be Cartan were presented in Wijsman (1972, Section 7).

More recently, the Copenhagen school has demonstrated the usefulness of the Bourbaki notions of *proper action* and *quotient measure* in problems concerning distributions of maximal invariants; among the main papers in this area are Andersson (1982) and Andersson, Brøns, and Jensen (1983). The general definition of proper action ([5] III Section 4.1, Definition 1) is that the mapping $\theta: G \times \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ given by $(g, x) \to (x, gx)$ be proper ([5] I Section 10.1, Definition 1). However, for G and $\mathscr{X}$ locally compact there is a criterion that is easier to apply, stated below for future reference as a lemma. It is essentially a restatement of [5] III Section 4.5, Theorem 1(c) (note that our $((A, B))$ defined by (1.2) is denoted $P(A, B)$ in [5]).

LEMMA 1.1 *If $\mathscr{X}$ and G are locally compact and G acts continuously on $\mathscr{X}$, then the action is proper if, for every pair A and B of compact subsets of $\mathscr{X}$, $((A, B))$ has compact closure.*

This follows from the fact that a mapping from one locally compact space into another is proper if and only if the inverse image of every compact set is compact ([5] I Section 10.3, Proposition 7). Now $((A, B)) = \mathrm{pr}_1\, \theta^{-1}(A \times B)$, where pr_1 is the projection of $G \times \mathscr{X}$ onto G, and $\theta^{-1}(A \times B)$ is closed and contained in $(\mathrm{pr}_1\, \theta^{-1}(A \times B)) \times A$. Therefore, $((A, B))$ is compact if and only if $\theta^{-1}(A \times B)$ is compact. Hence, the action is proper if and only if for every pair of compact subsets A and B of $\mathscr{X}$, $((A, B))$ is compact. Since in any case for continuous, but not necessarily proper, action the set $((A, B))$ is closed ([5] III Section 4.5, Theorem 1(a)), it suffices to check that $((A, B))$ has compact closure.

For $\mathscr{X}$ and G locally compact, the following useful equivalence is stated in [5] III Section 4.4, Proposition 7: *the action of G is proper if and only if for every $x, y \in \mathscr{X}$ there exist neighborhoods V_x and V_y such that $((V_x, V_y))$ has compact closure.* By taking $x = y$ it is seen that if G acts properly on $\mathscr{X}$, then $\mathscr{X}$ is a Cartan G-space. It is not known whether the converse is true so that the notion of proper action may be slightly stronger than the Cartan property. However, to this writer no example is known where the latter holds and the former does not. Therefore, it looks as though for all practical purposes there is no real loss in generality by requiring the action to be proper as compared to requiring the Cartan property. Assume proper action has the advantage that quotient measure becomes available; cf. Bourbaki (1963) VII Section 2. With the aid of quotient measure it is shown by Andersson (1982) that (1.1) is valid.

Thus, the main purpose of this paper is to develop useful tools for proving the action of G on $\mathscr{X}$ proper. Usually G is built up from several subgroups, and each of these may already be known to act properly. Under what conditions can G itself be concluded to act properly? This situation is analogous to the problem of construction of a maximal invariant for G out of maximal invariants for the subgroups. There it is well-known (cf. Lehmann, 1959, Chapter 6, Theorem 2) that a maximal invariant can be found in steps if at each stage the action of the next group induces an action on the maximal invariant obtained in the present stage. The analogue for the notion of proper action turns out to be true and is a very convenient way of establishing the properness of action of complicated groups. This will be shown in Section 2. In Section 3 several applications of the theorems in Section 2 will be given that together with some elementary rules help decide the properness of combined actions.

2. Proper action in steps. *Assumptions and notation:* $\mathscr{X}$ and G are locally compact; G acts continuously on the left of $\mathscr{X}$. The *G-orbit* of $x \in \mathscr{X}$ is Gx; the space of G-orbits in $\mathscr{X}$ is denoted $\mathscr{X}/G$ and is endowed with the quotient topology ([5] I Section 3.4). The symbol $((A, B))$ is defined in (1.2). Let $GL(p) =$ general linear group of all $p \times p$ real nonsingular matrices, $PD(p) =$ all real $p \times p$ positive definite matrices, $O(p) =$ group of all $p \times p$ orthogonal matrices, $M(p, q) =$ set of all real $p \times q$ matrices. If $M(p, q)$ is used as a group it will be understood that group multiplication is matrix addition.

For studying proper action in steps, it suffices to consider two steps. Let G have closed subgroups H and K such that $G = HK$, meaning that any $g \in G$ can be represented as $g = hk$, $h \in H$, $k \in K$. The assumption that H and K are closed implies that both subgroups are locally compact. Consider the K-orbits Kx, $x \in \mathscr{X}$, and let π be the orbit projection $\mathscr{X} \to \mathscr{X}/K$ given by $\pi(x) = Kx$. Suppose that for each $g \in G$ and $x \in \mathscr{X}$, $gKx = Kgx$; i.e., g maps each K-orbit into a K-orbit. Thus, the action of G on $\mathscr{X}$ induces an action of G on $\mathscr{X}/K$: $g\pi(x)$ is defined as $\pi(gx)$. We shall, for short, simply say that G acts on $\mathscr{X}/K$ if this action is induced by the action of G on $\mathscr{X}$. From the structure $G = HK$ it follows that G acts on $\mathscr{X}/K$ if and only if H does, and then a maximal invariant for G may be obtained in steps. If G acts on $\mathscr{X}/K$, then it follows from an argument given by Andersson (1982), relations (21) and (22), that the continuity of the action of G on $\mathscr{X}$ implies the continuity of the action of G on $\mathscr{X}/K$, and therefore of H on $\mathscr{X}/K$.

It is plausible to expect that if K acts properly on $\mathscr{X}$ and H properly on $\mathscr{X}/K$, then G acts properly on $\mathscr{X}$. This is indeed true and will be proved in Theorem 2.1. The converse is true under an additional mild assumption on the structure of HK and will be proved in Theorem 2.2.

THEOREM 2.1. *Let G have closed subgroups H, K, such that $G = HK$ and such that H acts on $\mathscr{X}/K$. If K acts properly on $\mathscr{X}$ and H acts properly on $\mathscr{X}/K$, then G acts properly on $\mathscr{X}$.*

PROOF. Let A, B be two arbitrary compact subsets of $\mathscr{X}$. Using Lemma 1.1, it suffices to show that there exists compact $C_1 \subset H$ and compact $C_2 \subset K$ such that $((A, B)) \subset C_1C_2$ which is compact since C_1C_2 is the image of $C_1 \times C_2$ under the continuous mapping $(h, k) \to hk$. Let π: $\mathscr{X} \to \mathscr{X}/K$ be the orbit projection $x \to Kx$ and put $\bar{A} = \pi(A)$, $\bar{B} = \pi(B)$. Then $\bar{A}$ and $\bar{B}$ are compact in $\mathscr{X}/K$ since π is continuous. Define $C_3 = \{h \in H: hkA \cap B \neq \varnothing$ for some $k \in K\}$, then C_3 can also be written $C_3 = \{h \in H: h\bar{A} \cap \bar{B} \neq \varnothing\}$. It follows from the properness of the action of K on $\mathscr{X}$ that $\mathscr{X}/K$ is locally compact, using [5] III Section 4.5, Proposition 11. Then by Lemma 1.1, $C_3 \subset C_1$ compact in H. Next, compute

$$\begin{aligned}((A, B)) &= \{hk: hkA \cap B \neq \varnothing\} \subset C_1\{k \in K: hkA \cap B \neq \varnothing \text{ for some } h \in C_1\}\\ &= C_1\{k \in K: kA \cap h^{-1}B \neq \varnothing \text{ for some } h \in C_1\}\\ &\subset C_1\{k \in K: kA \cap C_1^{-1}B \neq \varnothing\} \subset C_1C_2,\end{aligned}$$

where $\{k \in K: kA \cap C_1^{-1}B \neq \varnothing\} \subset C_2$ compact in K since $C_1^{-1}B$ is compact and the action of K on $\mathscr{X}$ proper. □

The next theorem needs an additional assumption on the structure of $G = HK$. Let f: $H \times K \to G$ be defined by $f(h, k) = hk$ so that f is continuous. Then we shall assume that f is a *proper mapping* ([5] I Section 10.1). This condition is certainly fulfilled if G is a product group $H \times K$, for then f is the identity map. In cases where G is not a product group the properness of f may be concluded if the decomposition $g = hk$ is unique (for instance, if G is a semidirect product) and G is second countable. This follows by using a device of Bourbaki (1963), Chapter VII Section 2.9, which consists of letting $H \times K$ act on the left of G by $g \to hkg^{-1}$. This action is continuous and transitive, and the isotropy subgroup at $g = e$ is trivial as a consequence of the uniqueness of the decomposition $g = hk$. It follows that the function φ: $H \times K \to G$ defined by $\varphi(h, k) = hk^{-1}$ is one-to-one. Then φ is a homeomorphism since G, and therefore $H \times K$, is second countable ([4] page 97, Lemme 2). Let ψ be the homeomorphism $(h, k) \to (h, k^{-1})$. Then $f = \varphi \circ \psi$ so that f is a homeomorphism and therefore proper.

THEOREM 2.2. *Let $G = HK$ where H and K are closed subgroups of G and assume that f: $H \times K \to G$ defined by $f(h, k) = hk$ is proper. Suppose that G acts properly on $\mathscr{X}$ and that H acts on $\mathscr{X}/K$. Then the actions of K on $\mathscr{X}$ and of H on $\mathscr{X}/K$ are proper.*

PROOF. Since G acts properly on $\mathscr{X}$ and K is a closed subgroup of G, it follows that K acts properly on $\mathscr{X}$ ([5] III Section 4.1, Example 1). It will now be shown that H acts properly on $\mathscr{X}/K$. First observe that $\mathscr{X}/K$ is locally compact as in the proof of Theorem 2.1. Then, using Lemma 1.1, we have to show that for any compact $\bar{A}, \bar{B} \subset \mathscr{X}/K$, $((\bar{A}, \bar{B}))$ as a subset of H has compact closure. Let π be as in the proof of Theorem 2.1. By [5] Section 4.5, Proposition 10 there exist compact subsets A and B of $\mathscr{X}$ such that $\pi(A) = \bar{A}$, $\pi(B) = \bar{B}$. First consider the subset C_1 of $H \times K$ defined by $C_1 = \{(h, k): khA \cap B \neq \varnothing\}$, then $C_1 = f^{-1}C_2$, where $C_2 = ((A, B))$. Since G acts properly by hypothesis, C_2

has compact closure and therefore so does C_1 since f is proper. Now $((\bar{A}, \bar{B})) = \{h \in H: h\bar{A} \cap \bar{B} \neq \varnothing\} = \{h \in H: \exists k \in K \ni hkA \cap B \neq \varnothing\} = \mathrm{pr}_1 C_1$, where pr_1 is the projection of $H \times K$ onto H. Since pr_1 is continuous and C_1 has compact closure, so does $((\bar{A}, \bar{B}))$. □

REMARK 2.3. In Theorems 2.1 and 2.2 the subgroups H and K are assumed to be closed. This is a natural assumption and guarantees from the outset that H and K are locally compact. Strictly speaking, it is not necessary to make these assumptions since the closedness of H and K is implied in Theorem 2.1 by the properness of the actions and in Theorem 2.2 by the properness of f.

REMARK 2.4. A simple counterexample shows that the condition in the hypothesis of Theorem 2.2 that f be proper cannot be relaxed. Take G to be noncompact (e.g., R under addition) and $H = K = \mathscr{X} = G$. Then G acts properly on itself (see Subsection 3.1). However G/K consists of one point and H does not act properly on G/K since H is not compact.

3. Examples of proper actions and the use of Theorems 2.1 and 2.2. Several examples will be given to illustrate how Theorems 2.1 and 2.2 can be combined with a few elementary rules to prove properness of composite actions. Subsections 3.1–3.3 list elementary rules that are used repeatedly. Subsections 3.4 and 3.5 prove properness of some simple actions that occur often in statistics. Subsections 3.6–3.10 deal with composite actions and the use of Theorems 2.1 and 2.2, in conjunction with numbers 3.1–3.5 following.

3.1. If $\mathscr{X} = G$, then G acts properly since the mapping $G \times G \to G \times G$ defined by $(g_1, g_2) \to (g_2, g_1 g_2)$ is a homeomorphism and therefore proper.

3.2. If G acts properly on $\mathscr{X}$ and G acts continuously on $\mathscr{Y}$, then G acts properly on $\mathscr{X} \times \mathscr{Y}$. This follows rather trivially from an application of Lemma 1.1.

3.3. If $G = G_1 \times G_2$, $\mathscr{X} = \mathscr{X}_1 \times \mathscr{X}_2$, and G_i acts properly on $\mathscr{X}_i$, $i = 1, 2$, then G acts properly on $\mathscr{X}$. This follows easily by an application of Lemma 1.1. It is even true without any restrictions on the groups and spaces by the general Definition 1 in [5] III Section 4.1 and I Section 10.1 Proposition 4.

3.4. Let G be a closed subgroup of $GL(p)$ and let $\mathscr{X} = M(p, n)$ restricted to matrices of rank p. Let the action be defined by $X \to CX$ (matrix multiplication), $X \in \mathscr{X}$, $C \in G$. It will be shown that G acts properly on $\mathscr{X}$, and it will be sufficient to do this for $G = GL(p)$, using [5] III Section 4.1, Example 1. If $n = p$, then $\mathscr{X} = G$ and G acts properly by number 3.1. If $n > p$, then each $X \in \mathscr{X}$ has $m = \binom{n}{p}$ submatrices of order $p \times p$. Denote these by $X_1, \cdots, X_m$. At least one of the X_j must be nonsingular. If X_j is nonsingular, then so is CX_j for every $C \in G$. Thus, $\mathscr{X} = \mathscr{X}_1 \cup \cdots \cup \mathscr{X}_m$, where $\mathscr{X}_j$ is invariant and consists of all X with X_j nonsingular. It suffices to show that G acts properly on each $\mathscr{X}_j$. Now $\mathscr{X}_j$ can be written in the form $\mathscr{Y}_1 \times \mathscr{Y}_2$, where $\mathscr{Y}_1$ consists of the submatrices X_j of X. Therefore, $\mathscr{Y}_1 = G$. Hence, G acts properly on $\mathscr{Y}_1$, by number 3.1, and G acts continuously on $\mathscr{Y}_2$. Then use number 3.2.

3.5. Let $\mathscr{X}$ be as in number 3.4 and take $G = M(p, r)$, where $r \leq p$. Let $L: r \times n$ of rank r be a fixed matrix and define the action of G by: $X \to X + BL$, $B \in G$. Then the action is proper. To show this, first postmultiply the matrices X and L by a fixed nonsingular matrix (which may be chosen orthogonal) such that the new L is of the form $[C, 0]$, with $C: r \times r$ nonsingular. This replaces $\mathscr{X}$ by a homeomorphic image and does not change the problem. Next, replace the matrices B by BC^{-1}, which amounts to a homeomorphism of G with itself. For the new $\mathscr{X}$ the action is $X \to X + [B, 0]$, $B \in G$. Write $X = [X_1, X_2]$ with $X_1: p \times r$ and write $\mathscr{X} = \mathscr{X}_1 \times \mathscr{X}_2$, then $\mathscr{X}_1 = G$ so G acts properly on $\mathscr{X}_1$ by number 3.1, and G acts trivially on $\mathscr{X}_2$. Then use number 3.2.

3.6. *Properness of an affine action.* Let $G = HK$, $H = GL(p)$, $K = M(p, r)$ $(r \leq p)$ with group multiplication $(C_1, B_1)(C_2, B_2) = (C_1C_2, B_2 + C_2^{-1}B_1)$ if $C_i \in H$, $B_i \in K$. Let $\mathscr{X} = M(p, n)$, $n \geq p + r$, restricted to matrices of rank p. Let L be as in number 3.5 and define $P_L = L'(LL')^{-1}L$, which is the orthogonal projection onto the row space of L. Let $\mathscr{X} = M(p, n)$, $n \geq p + r$, restricted to matrices X such that $X[I_n - P_L] \equiv X^*$ is of rank p. Let the action of G be $X \to C(X + BL)$, $C \in H$, $B \in K$. A maximal invariant under K is X^*, which is homeomorphic to $\mathscr{X}/K$. This can be seen by postmultiplying X and L by an orthogonal matrix Γ whose first r columns span the row space of L. Then the action of K on the new X is (as in number 3.5) $[X_1, X_2] \to [X_1 + B, X_2]$ and a topological maximal invariant is obviously $[0, X_2]$. Transforming back with Γ' establishes the claim. Next, H acts on $\mathscr{X}/K$ by $X^* \to CX^*$. The action of K on $\mathscr{X}$ was shown to be proper in number 3.5, and the action of H on $\mathscr{X}/K$ is proper by number 3.4. Then use Theorem 2.1 to conclude that the affine action of G on $\mathscr{X}$ is proper.

Special choices of L lead to well-known examples in statistics. Take $r = 1$, $L = [1, \cdots, 1]$, and write x_j for the jth column of X. Then the action is $x_j \to C(x_j + b)$, $C \in GL(p)$, $b \in M(p, 1)$. This is relevant if the columns of X constitute a sample of size n from a p-variate population, and is also important in sequential problems. In Wijsman (1972, Theorem 7.1) the Cartan property was proved for this kind of action. As a second example of affine action, take the canonical form of MANOVA. Of n independent random p-vectors (here represented as column vectors), the means of the first r vectors are unspecified both under the model and under the linear hypothesis. Partition the observation matrix $X = [X_1, X_2]$: $p \times n$, with $X_1: p \times r$, and G acts on X according to $X_1 \to C(X_1 + B)$, $X_2 \to CX_2$. This corresponds to $L = [I_r, 0]$.

3.7. Let $G = HK$ be as in Section 2. Suppose that K acts properly on $\mathscr{X}$ and that H is compact and acts on $\mathscr{X}/K$. Since H is compact, it acts properly on $\mathscr{X}/K$. Then by Theorem 2.1, G acts properly on $\mathscr{X}$. As an application, suppose

$\mathscr{X} = M(p, n)$, restricted to matrices of rank p. Let $H = O(n)$, $K = GL(p)$, $G = H \times K$ with action defined by $X \to CX\Gamma'$, $\Gamma \in H$, $C \in K$. The action of K on $\mathscr{X}$ is proper by number 3.4. Since H is compact we conclude that G acts properly.

3.8. *Properness of the action.*

$$S \to CSC', \quad S \in PD(p), \quad C \in G, \tag{3.1}$$

where G is a closed subgroup of $GL(p)$.

This represents the usual action on covariance matrices, and many invariance groups in multivariate problems contain (3.1) in all or part of their action. A recent example is the paper by Andersson, Brøns, and Jensen (1983). The properness of the action (3.1) is well-known and follows for instance from a general result stated in [5], page 308, Exercise III Section 4, number 3. Andersson (1982) states that Tolver Jensen in a 1971 dissertation derived the properness by showing the inverse image of a bounded set to be bounded. Essentially the same method was used in Wijsman (1967) in the proof of the Cartan property. Here we shall give a totally different proof by using Theorems 2.1 and 2.2. As in number 3.4 we may suppose $G = GL(p)$.

Let $\mathscr{X}$ be as in number 3.7, with $p = n$, but interchange the definitions of H and K. That is, $G = H \times K$, $H = GL(p)$, $K = O(p)$, and the action is $X \to CX\Gamma'$, $C \in H$, $\Gamma \in K$. A maximal invariant under the action of K is $S = XX'$. We have to show that S is a topological representation of $\mathscr{X}/K$. Let L be the group of $p \times p$ lower triangular matrices with positive diagonal elements. Then $L \times K$ acts continuously and transitively on the left of $\mathscr{X}$ by $X \to TX\Gamma'$, $T \in L$, $\Gamma \in K$. The isotropy subgroup of $L \times K$ at $X = I_p$ is trivial. Therefore, there is a one-to-one correspondence between $\mathscr{X}$ and $L \times K$ given by $X = T\Gamma'$ (this also follows from a Gram-Schmidt decomposition). That this correspondence is a homeomorphism follows, for instance, from [4], page 97, Lemme 2. The action of K on $\mathscr{X}$ induces an action of K on $L \times K$ which is transitive on K and trivial on L, so that L is an obvious topological maximum invariant. Finally, there is a homeomorphic correspondence between $T \in L$ and $S = TT' \in PD(p)$.

By number 3.7 the action of G on $\mathscr{X}$ is proper. Theorem 2.2 applies because the function f in that theorem is proper since $G = H \times K$ is a product group. The action of H on $\mathscr{X}/K$ is (3.1), and by Theorem 2.2 it is proper.

3.9. *Canonical correlations.* Let n observations be taken on a p-variate population. If the observations are represented by row vectors, then the observation matrix is X: $n \times p$. Partition it as $X = [X_1, X_2]$, X_i: $n \times p_i$, $p_1 + p_2 = p$, where it may be assumed that rank $X_i = p_i$, $i = 1, 2$. To obtain the canonical correlations as a maximal invariant the relevant group is $G = O(n) \times GL(p_1) \times GL(p_2)$ and the action is defined by $X_i \to \Gamma X_i C_i'$, $\Gamma \in O(n)$, $C_i \in GL(p_i)$, $i = 1, 2$. Here $GL(p_1) \times GL(p_2)$ acts properly on $\mathscr{X}$ by numbers 3.3 and 3.4, and therefore G acts properly on $\mathscr{X}$ by number 3.7.

3.10. *GMANOVA* (cf. Kariya, 1978). In its canonical form $\mathscr{X}$ consists of pairs (X, S), $X \in M(m, p)$, $S \in PD(p)$, and X is partitioned as $X = (X_{ij})$, $i = 1, 2, j = 1, 2, 3$, X_{ij}: $m_i \times p_j$ with $m_1 + m_2 = m$, $p_1 + p_2 + p_3 = p$. The group G consists of three subgroups: G_1 consists of all $m \times p$ matrices F partitioned in the same way as X, with $F_{12} = 0$, $F_{13} = 0$, $F_{23} = 0$, and action $X \to X + F$. G_2 consists of $p \times p$ block-upper triangular nonsingular matrices A with action $X \to XA'$, $S \to ASA'$, and G_3 consists of block-diagonal orthogonal matrices $\Gamma = \mathrm{diag}(\Gamma_1, \Gamma_2)$, with Γ_i: $m_i \times m_i$, acting by $X \to \Gamma X$. It is readily seen that $G = (G_2 \times G_3)G_1$, with G_1 normal in G. The action of G_1 is proper by a combination of numbers 3.1 (or 3.5), 3.2, and 3.3. A topological maximal invariant under G_1 is (X^*, S), where X^* is obtained from X by setting X_{11}, X_{21}, and X_{22} equal to 0. Then G_2 and G_3 act on (X^*, S) as they did on (X, S). The group G_2 is a closed subgroup of $GL(p)$ and therefore the action (3.1) with C replaced by $A \in G_2$ is proper. Thus, G_2 acts properly on (X^*, S) by number 3.2, and $G_2 \times G_3$ acts properly by number 3.7. By Theorem 2.1, G acts properly on $\mathscr{X}$.

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