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THE LIKELIHOOD RATIO DETECTOR FOR NON-GAUSSIAN INFINITELY DIVISIBLE, AND LINEAR STOCHASTIC PROCESSES

BY PATRICK L. BROCKETT[1]

The University of Texas at Austin

We consider the problem of determining absolute continuity, and the distribution of the likelihood ratio (Radon-Nikodym derivative) of the measures induced on function space by two infinitely divisible stochastic processes. The results are applied to linear processes, which are shown to be infinitely divisible.

1. Introduction. The problem of optimum detection of signals in stochastic noise has been solved only in a relatively few cases. Most investigators assume the signal or the noise are Gaussian; however, in many very important practical situations (e.g., radar, sonar, or satellite transmission), this assumption does not hold.

Lugannani and Thomas (1967) developed the class of linear processes, as a potential model for noise, and showed that the class was closed under linear transformations, a desirable property for modeling purposes. For a specialized type of linear process, Eastwood and Lugannani (1977) were able to construct an approximation to the n-dimensional densities of two linear processes evaluated at $(t_1, t_2, \cdots, t_n)$. Consequently they were able to obtain a likelihood ratio test approximation for this special class of processes. It is of some considerable importance that the model of Middleton (1967, 1972a, 1972b, 1976) for acoustical reverberation is a linear process, as are several other models for noise derived from purely physical reasoning.

In this paper we shall show how to identify a linear process as a subclass of infinitely divisible processes (i.e., processes with infinitely divisible finite dimensional marginal distributions). Using the results of Maruyama (1970), Brown (1971), Briggs (1975), Skorokhod (1964) and Veeh (1981), we are then able to explicitly calculate the Radon-Nikodym derivative of the measures determined on function space by two infinitely divisible processes and to find its distribution. This extends the results of several authors and enables us to use the Neyman-Pearson lemma to obtain an optimal detector applicable directly to the sample paths of many stochastic process models.

2. Linear processes and infinitely divisible processes. A linear process

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$Y(t)$ is defined by the stochastic integral

$$Y(t) = \int_a^b f(t, s)\, dX(s) \tag{1}$$

where $X(s)$ is a zero-mean, second order stochastically continuous process with independent increments and $f(t, s)$ is real valued and square integrable with respect to $dV(s) = E\,|\,dX(s)\,|^2$. In brief, Y is an L_2 filtering of an independent increment process. We shall additionally assume f is L_2 continuous (so that $Y(t)$ is stochastically continuous).

Following the method used in Papoulis (1965) for shot noise process, one may determine the finite dimensional characteristic functions of the linear process (1) (cf. Lugannani and Thomas, 1967, or Eastwood and Lugannani, 1977). In the case without Gaussian component, they are given by

$$\Phi_{\mathbf{t}}(\mathbf{u}) = \exp\left\{\int_a^b \int_0^\infty \{\exp(izw) - 1 - izw\}\, M(ds, dz)\right\} \tag{2}$$

where $\mathbf{t} = (t_1, \cdots, t_n)$, $\mathbf{u} = (u_1, \cdots, u_n)$ and $w = u_1 f(t_1, s) + u_2 f(t_2, s) + \cdots + u_n f(t_n, s)$. The measure M is the time-jump measure of the additive process X, i.e., $M((s_1, s_2] \times A)$ is the expected number of jumps (pulses) of the process X during the time interval $(s_1, s_2]$ with the magnitude (amplitude) in the Borel set A. See Gikhman and Skorokhod (1969) for a more detailed explanation of the Lévy measure M and its properties.

Let us now consider the class of infinitely divisible stochastic processes. This class was first studied by Lee (1967), and subsequently studied by Maruyama (1970), Briggs (1975), Wright (1975) and Veeh (1981). A stochastic process is called infinitely divisible if all of its n-dimensional marginal distributions are n-dimensional infinitely divisible random vectors. Gaussian processes are, of course, infinitely divisible and by using a variant of the Kolmogorov representation for second order processes without a Gaussian component, the following representation holds (cf. Lukacs, 1970, page 119). By definition, for every finite subcollection $\lambda = \{t_1, t_2, \cdots, t_n\} \subseteq [a, b]$, there exists a random vector $\mathbf{c}_\lambda$, and an n-dimensional Lévy measure M_λ such that the characteristic function of $(Y(t_1), \cdots, Y(t_n))$ is

$$\ln \Phi_\lambda(\mathbf{u}) = i\mathbf{u}'\mathbf{c}_\lambda + \int \{\exp(i\mathbf{u}'\mathbf{x}) - 1 - iu'x\}\, dM_\lambda(\mathbf{x}). \tag{3}$$

Let $\Lambda = \{\lambda = \{t_1, \cdots, t_n\}\}$ denote the set of all finite subsets of $[a, b]$. The collection $\{(\mathbf{c}_\lambda, M_\lambda), \lambda \in \Lambda\}$ uniquely determines the distribution of an infinitely divisible process Y, and vice versa (Maruyama, 1970, Theorems 1 and 3). Using the partial ordering on Λ by inclusion, we obtain a system of projections $\{P_\lambda, \lambda \in \Lambda\}$ from $\mathbb{R}^{[a,b]}$ onto the coordinate space $\mathbb{R}^\lambda$. The system of Lévy measures $\{M_\lambda, \lambda \in \Lambda\}$ is consistent, and Maruyama shows that a measure Q may be defined on $\mathbb{R}^{[a,b]}$ as the projective limit of the collection $\{M_\lambda, \lambda \in \Lambda\}$. The σ-algebra on $\mathbb{R}^{[a,b]}$ is the usual product σ-algebra.

Thus, corresponding to an infinitely divisible process there is a function $c(t)$, and a measure Q such that for $\lambda = \{t_1, \cdots, t_n\}$, $P_\lambda c = (c(t_1), \cdots, c(t_n)) = \mathbf{c}_\lambda$ and $QP_\lambda^{-1}(A) = M_\lambda(A)$ are the parameters of the infinitely divisible random vector $(Y(t_1), \cdots, Y(t_n))$.

To obtain the appropriate representation of a linear process as an infinitely divisible process, we first manipulate the characteristic function given by (2) into the form of (3). Towards this end we first note that if $\lambda = \{t_1, \cdots, t_n\}$ is given, and M_λ is defined on $\mathbb{R}^\lambda$ via $M_\lambda(A) = M(\{(s, z):(zf(t_1, s), \cdots, zf(t_n, s)) \in A\})$, then M_λ is a Lévy measure on $\mathbb{R}^\lambda$ concentrated on the curve $(zf(t_1, s), \cdots, zf(t_n, s))$, $s \in [a, b]$, $z \in \mathbb{R}$. Moreover, the integral relationship

$$\int h(\mathbf{x})\, dM_\lambda(\mathbf{x}) = \int_a^b \int_{-\infty}^{\infty} h(zf(t_1, s), \cdots, zf(t_n, s))\, M(dz, ds)$$

holds for measurable h. The fact that M_λ is indeed a Lévy measure on $\mathbb{R}^\lambda$ follows (after some calculations) from the square integrability of f with respect to V and from the formula $\int h(s)dV(s) = \int\int z^2h(s)\, M(ds, dz)$ which relates the variance measure to the time-jump measure M.

Now write $\mathbf{f}(s) = P_\lambda f(\cdot, s) = (f(t_1, s), \cdots, f(t_n, s))$, $\mathbf{c}_\lambda = \mathbf{0}$, and observe that

$$\begin{aligned} i\mathbf{u}'\mathbf{c}_\lambda + \int_{\infty}^{\infty} \{\exp(i\mathbf{u}'\mathbf{x}) - 1 - i\mathbf{u}'\mathbf{x}\}\, M_\lambda(dx) \\ = \int\int \{\exp(iz\mathbf{u}'\mathbf{f}_\lambda(s)) - 1 - iz\mathbf{u}'\mathbf{f}_\lambda(s)\}\, M(ds, dz) \\ = \int\int \{\exp(izw) - 1 - izw\}\, M(ds, dz) \end{aligned}$$

where $w = u_1f(t_1, s) + u_2f(t_2, s) + \cdots + u_nf(t_n, s)$ as before. Thus (2) is of the form (3), and hence linear processes are infinitely divisible processes. Moreover, we can determine the projective limit Q of the system of Lévy measures $\{M_\lambda, \lambda \in \Lambda\}$. Namely, if $A \subseteq \mathbb{R}^{[a,b]}$ then $Q(A) = M(\{(s, z):zf(\cdot, s) \in A\})$. This follows since if $B \subseteq \mathbb{R}^\lambda$, then $M_\lambda(B) = M(\{(s, z):(zf(t_1, s), \cdots, zf(t_n, s)) \in B\}) = M(\{(s, z):zP_\lambda f(\cdot, s) \in B\}) = Q(P_\lambda^{-1}(B))$, so that the λth coordinate projection of Q is M_λ.

3. The likelihood ratio for infinitely divisible processes. We begin by sketching the construction of an infinitely divisible process Y as a limit of integrals of Poisson random measures. For details see Gikhman and Skorokhod (1969) or Maruyuma (1970). A similar representation is used by Briggs (1975) and by Akritas and Johnson (1981).

Let π be a Poisson random measure on $\mathbb{R}^{[a,b]}$ which has the corresponding intensity measure Q, i.e., for any set $A \subseteq R^{[a,b]}$ with $Q(A) < \infty$, $\pi(A)$ is a Poisson random variable with expectation $Q(A)$. Moreover, if $A_1, \cdots, A_n$ are disjoint sets, then $\pi(A_1), \cdots, \pi(A_n)$ are independent random variables. See Kallenburg (1976) for details.

The random measure $\pi^*(A) = \pi(A) - Q(A)$ is used to give a pathwise representation of the second order linear process Y, namely we may write

$$(4)\qquad Y(t) = \lim \text{ in Prob.}_{\varepsilon\downarrow 0} \int_{A_\varepsilon} x(t)\pi^*(dx) \quad \text{where} \quad A_\varepsilon = \{x : | x(t) | \geq \varepsilon\}.$$

To see this, note that $\mathbf{Y}_\lambda = (Y(t_1), \cdots, Y(t_n))$ has a characteristic function given by

$$\begin{aligned} \log \phi_\lambda(\mathbf{u}) &= \log E(\exp i \textstyle\sum_{j=1}^n u_j Y(t_j)) \\ &= \int \{\exp(i \textstyle\sum_{j=1}^n u_j x(t_j)) - 1 - i \sum_{j=1}^n u_j x(t_j)\} Q(dx) \\ &= \int \{\exp(i\mathbf{u}' P_\lambda x) - 1 - i\mathbf{u}' P_\lambda x\} Q(dx) \\ &= \int \{\exp(i\mathbf{u}' y) - 1 - i\mathbf{u}' y\} M_\lambda(dy). \end{aligned}$$

Thus, the finite dimensional distributions given by the right-hand side of (4) agree with those of the linear process.

We are now in a position to calculate the likelihood ratio of two infinitely divisible processes without trend functions. The multivariate Lévy measures M_1 and M_2 induce (via projective limits) the measures Q_1 and Q_2 on function space as described earlier. The processes Y_i, $i = 1, 2$ determine measures on function space via $\mu_i(A) = P[Y_i(\cdot) \in A]$, and we wish to determine when $\mu_1 \ll \mu_2$, and the corresponding density $(d\mu_1/d\mu_2)(x)$, $x \in \mathbb{R}^{[a,b]}$.

The following theorem generalizes the results of Briggs (1975) to include general infinitely divisible processes (not just those with nonatomic projective measures Q). Additionally it generalizes some results of Veeh (1981), Brockett, Hudson, and Tucker (1978) and Akritas and Johnson (1981). Moreover, we are able to substantially reduce the length and complexity of the proof of both the results of Briggs (1975) and of Brockett, Hudson and Tucker (1978).

We now state our first results concerning the case with projective mean measures Q_1 and Q_2 finite.

THEOREM 1 (nonstationary compound Poisson case). *Suppose $Y_1(t)$ and $Y_2(t)$ are two stochastically continuous infinitely divisible processes with corresponding projective limit measures Q_1 and Q_2 finite.*

a) *If $Q_1 \ll Q_2$ and $\int x(Q_1 - Q_2)(dx) = 0$, then $\mu_1 = PY_1^{-1} \ll \mu_2 = PY_2^{-1}$. Moreover, using the representation* (4),

$$\ln \frac{d\mu_1}{d\mu_2}(Y_1(\cdot)) = \int \ln \rho(x)\, \pi_1(dx) + Q_2(\mathbb{R}^{[a,b]}) - Q_1(\mathbb{R}^{[a,b]}) \quad \textit{where} \quad \rho(x) = \frac{dQ_1}{dQ_2}(x).$$

b) *The μ_1 distribution of the* log *likelihood ratio in* (a) *is given via the charac-*

teristic function whose logarithm is

$$\ln \phi(u) = iu[(Q_2 - Q_1)(\mathbb{R}^{[a,b]})] + \int (\exp\{iu \ln \rho(x)\} - 1)\, Q_1(dx).$$

Thus $\ln(d\mu_1/d\mu_2)(Y_1(\cdot,\omega))$ *is a translated compound Poisson random variable on* $\mathbb{R}$ *with intensity measure* $\nu(A) = Q_1(\{x : \ln \rho(x) \in A\})$.

PROOF. The proof of a) will follow immediately from Theorem 1 of Brown (1971) once we notice that according to (4), and the corresponding representation for Poisson point processes as measures on a sequence space, we have $Y(\cdot,\omega) = S(\{x_i(\omega)\})$ where $S(\{x_i\}) = \sum_{i\ge 0} x_i - c_1 = \sum x_i - c_2$ and $\{x_i\}$ is a realization of the point process π. Here $c_1 = \int x\, dQ_1(x) = c_2 = \int x\, dQ_2(x)$ by the assumption $\int x\, d(Q_1 - Q_2)(x) = 0$. Now, under the assumptions of the theorem,

$$\mu_1 = \pi_1 S^{-1} \ll \mu_2 = \pi_2 S^{-1},$$

and by Theorem 1 of Brown (1971)

$$\frac{d\mu_1}{d\mu_2}(Y_1(\cdot)) = \frac{d\pi_1 S^{-1}}{d\pi_2 S^{-1}}(f) = \frac{d\pi_1}{d\pi_2}(S^{-1}f) = \frac{d\pi_1}{d\pi_2}(\{x_i(\omega)\})$$
$$= \exp[Q_1(\mathbb{R}^{[a,b]}) - Q_2(\mathbb{R}^{[a,b]})] \prod_{i=1}^{\pi_1(\mathbb{R}^{[a,b]})} \rho(x_i(\omega)),$$

which is the formula in a) once the product is converted to integral form. Here we have used the fact that if ν and η are two measures on $\mathscr{X}$ with $\nu \ll \eta$, and $S: \mathscr{X} \to \mathscr{Y}$, then $\nu S^{-1} \ll \eta S^{-1}$ and $(d\nu S^{-1}/d\eta S^{-1})(y) = (d\nu/d\eta)(S^{-1}y)$. See Lemma 1 of Brockett, Hudson, and Tucker (1978). Note that (ii) and (iii) of the lemma are obviously satisfied in this finite measure case.

To prove b) we simply notice that, according to the lemma, we are dealing with a Poisson sum (e.g., $\pi_1(\mathbb{R}^{[a,b]})$) of random variables, $\ln \rho(x_i(\omega))$. The characteristic function now follows from routine calculations.

Using the results of Theorem 1, it is now just a short step to obtain the general thoerem.

THEOREM 2. *Suppose* $Y_1(t)$ *and* $Y_2(t)$ *are two stochastically continuous infinitely divisible processes with corresponding projective limit measures* Q_1 *and* Q_2.

a) *If* i) $Q_1 \ll Q_2$ *with* $\rho(x) = (dQ_1/dQ_2)(x)$
   ii) $\int x d(Q_1 - Q_2)(x) = 0$
   iii) $\int (1 - \rho^{1/2}(x))^2\, dQ_2(x) < \infty$,
   Then $\mu_1 = PY_1^{-1} \ll \mu_2 = PY_2^{-1}$.

b) *Under the conditions of* a)

$$\ln \frac{d\mu_1}{d\mu_2}(Y_1(\cdot)) = \int_{B_t^c} \ln \rho(x)\pi_1^*(dx) + \int_{B_t} \ln \rho(x) Q_1(dx)$$
$$+ \int_{B_t^c} [1 - \rho(x) + \ln \rho(x)] Q_1(dx) + \int_{B_t} [1 - \rho(x)] Q_1(dx)$$

where, as before, $\pi_1^* = \pi_1 - Q_1$ *and* $B_t = \{x : |\rho(x) - 1| > t\}$.

c) *The logarithm of the characteristic function of* $\ln(d\mu_1/d\mu_2)$ *is*

$$iu \int \left(1 - \rho(x) + \frac{\ln \rho(x)}{1 + (\ln \rho(x))^2}\right) dQ_1 + \int \left(e^{iuy} - 1 - \frac{iuy}{1 + y^2}\right) Q_1 \circ g^{-1}(dy)$$

where $g = \ln \rho$. *Thus it is the translate of an infinitely divisible random variable with Lévy measure* $M(A) = Q_1(\{x : \ln \rho(x) \in A\})$.

PROOF. Veeh (1981) proves a), or it could be derived from Brown (1971) in the previous manner. The proof of b) is given in Briggs (1975), and can follow from Theorem 1 by using her techniques. It should be noted that she does not explicitly state assumption (ii), although it is used in her proof. An alternative proof for b) can be constructed from Theorem 1 by using the techniques of Brockett, Hudson and Tucker (1978). The distribution in c) is obtained by a limiting argument from Theorem 1 after appropriately centering in a manner analogous to that used in Brockett, Hudson and Tucker (1978).

Let us now turn to a development for the likelihood ratio in the linear process case when both driving functions X_1 and X_2 do have Gaussian components. Our development requires the additional assumption that the same filter f is used on both processes. The key step in the development is a result due to Skorokhod (1969, page 245, Theorem 2). We state this result below since it is of independent interest.

LEMMA 1 (Skorokhod, 1964). *Suppose* $X_1(t)$ *and* $X_2(t)$ *are two stochastic processes inducing measures* $\nu_1 = PX_1^{-1}$ *and* $\nu_2 = PX_2^{-1}$ *on function space. Let* S *be a measurable mapping from function space to function space, and* $Y_1 = SX_1$, $Y_2 = SX_2$ *be two stochastic processes with induced measures* $\mu_1 = PY_1^{-1}$ *and* $\mu_2 = PX_2^{-1}$. *If* $\nu_1 \ll \nu_2$ *then* $\mu_1 \ll \mu_2$ *and* $(d\mu_1/d\mu_2)(Y_2(t)) = E[(d\nu_1/d\nu_2)(X_2(t)) \mid Y_2(t)]$.

Using the result of Skorokhod (1964) and Brockett and Tucker (1977) and Brockett, Hudson and Tucker (1978) we have the following:

THEOREM 3. *Suppose* Y_1 *and* Y_2 *are linear processes given by* (1) *with the same filter function* f *for both* Y_1 *and* Y_2. *If the absolute continuity conditions* (i)–(iv) *of Brockett and Tucker* (1977) *all hold, then* $\mu_1 = PY_1^{-1} \sim \mu_2 = PY_2^{-1}$ *and by Skorokhod* (1969, *page* 245, *Theorem* 2) *we have* $d\mu_1/d\mu_2$, *given by*

$$\frac{d\mu_1}{d\mu_2}(Y_1(t)) = E\left[\frac{d\nu_1}{d\nu_2}(X_1(t)) \mid Y_1(t) = \int f(t, s)\, dX_1(s)\right].$$

The quantity $d\nu_1/d\nu_2$ *is determined explicitly in Brockett, Hudson and Tucker* (1978).

PROOF. By Lemma 1 it is clear that $\nu_1 = PX_1^{-1} \sim \nu_2 = PX_2^{-1}$ and $d\nu_1/d\nu_2$ is as given.

We now use the results in Brockett, Hudson and Tucker (1978) to calculate the general likelihood ratio. Let $S: D[0, T] \to D[0, T]$ be defined by

$$(Sg)(t) = \int f(t, s)\, dg(s) = \lim \sum f(t, s_j)\{g(s_{j+1}) - g(s_j)\}$$

where the limit is as the mesh of the partition converges to zero. By the definition of Y_i in equation (1), and by the results of Brockett and Hudson (1982) it is easily seen that S is defined a.s. with respect to both the measures $\nu_1 = PX_1^{-1}$ and $\nu_2 = PX_2^{-1}$. The stated likelihood ratio now follows from Brockett, Hudson and Tucker (1978).

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DEPARTMENT OF FINANCE AND
APPLIED RESEARCH LABS
UNIVERSITY OF TEXAS AT AUSTIN
AUSTIN, TEXAS 78712