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A ROBUSTIFICATION OF THE SIGN TEST UNDER MIXING CONDITIONS

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A robustified version of the two sample sign test is defined which is insensitive to certain deviations from the assumption of the independence of the observations. These deviations are described in terms of mixing conditions.

The asymptotic value of the power function of this robustified sign test is computed on contiguous alternatives possessing the same dependence structure. This entails the calculation of its asymptotic relative efficiencies with respect to some tests which are optimal on these alternatives in the independent case.

It becomes apparent that in general the relative performance of two tests heavily depends on the structure of dependence of the observations, i.e. it may either increase or decrease.

1. Introduction. Until now the technical term "robustness of statistical procedures" mainly stands for distributional robustness. More precisely, those methods are commonly called robust which are insensitive to slight deviations of the shape of the true underlying distribution from the model. A presentation of this particular topic is given in Huber's (1981) monograph. As Huber points out, much less is known about what happens when the other standard assumptions of statistics are not satisfied and about the appropriate safeguards in such cases.

The present article deals with the consequences occuring when the observations are not necessarily independent but fulfill certain mixing conditions. The concept of mixing variables has gained more and more importance since its introduction into probability theory by Rosenblatt (1956), Ibragimov (1959) and Blum et al. (1963) among others. While there exists a large amount of literature concerning the probabilistic aspects of this concept, little has been done to carry it over into statistics.

We mention some relevant results in the field of test theory: the two sample Wilcoxon-test and the chi-squared goodness-of-fit test were treated by Serfling (1968) and, respectively, by Chanda (1981) under a strongly mixing process. Albers (1978) dealt with Student's t-test given m-dependent observations, and the effects of autoregressive dependence on some tests were studied in several articles by Gastwirth and Rubin (1971), (1975) and by Gastwirth, Rubin and Wolff (1967). For a list of further references we refer to the book by Basawa and Rao (1980).

The above cited results show in which way dependence can change the asymptotic variance of a test statistic. Therefore, in order to get an asymptotically

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robust procedure, one has to "studentize" the usual test procedure. Furthermore, the question suggests itself whether and to what extent the relative performance of two robustified tests is influenced by the dependence of the observations. We shall investigate these problems in the case of the easily computable sign test.

To study the robustified version of the sign test in more detail, we compute its power function on translation alternatives and derive its asymptotic relative efficiency (ARE) with respect to the Neyman-Pearson test. If, in particular, the observations come from a Gaussian process the ARE can be expressed in terms of the correlation coefficients and may either decrease or increase. This result reveals the fact that the ARE of two tests heavily depends on the structure of dependence of the observations.

To highlight this point, we furthermore compare the modified sign test on alternatives of the form $F_\Delta = (1-\Delta)F + \Delta F^2$ with Serfling's (1968) robustified version of the Wilcoxon test. Notice that the Wilcoxon test is optimal on these alternatives when the observations are independent. Two examples show that also in this case the ARE may drastically change. However, whether the relative merits of two tests can even be reversed remains an open question.

The article is organized as follows: In Section 2 we construct the robustified version of the sign test. Its asymptotic performance is studied on the level of efficiency in Sections 3 and 4. To make the article more readable, the proofs are postponed until Section 5.

2. Construction of the robustified sign test. Let T_n, $n \in \mathbb{N}$, denote the usual sign test statistic in the two sample case, i.e.

$$T_n((\mathbf{x}, \mathbf{y})) := n^{-1/2}(2 \sum_{i=1}^{n} 1_{(0,\infty)}(y_i - x_i) - n)$$

where $\mathbf{x} := (x_1, \cdots, x_n)$, $\mathbf{y} := (y_1, \cdots, y_n) \in \mathbb{R}^n$. Further let P and Q be nonatomic probability measures on the real line and define $\mu := (P \times Q)\{\pi_2 > \pi_1\}$. Then the Central Limit Theorem for i.i.d. random variables implies $(P^n \times Q^n) * T_n \Rightarrow N_{(0,1)}$ iff $\mu = ½$.

Here $\Rightarrow$ denotes weak convergence, $P^n = P \times \cdots \times P$ the n-fold independent product of P, $P*T$ the probability measure induced by P and T and π_i the ith projection, i.e. $\pi_i(\mathbf{x}) = x_i$.

Thus, the critical region $C_{n,\alpha} := \{|T_n| > u_{\alpha/2}\}$ for testing $H_0 := \{(P, Q): P = Q\}$ against $H_1 := \{(P, Q): P \neq Q\}$ is of asymptotic level α where $u_{\alpha/2}$ denotes the $(1-\alpha/2)$-quantile of $N_{(0,1)}$.

If the observations are not independent then, in general, $C_{n,\alpha}$ is not of asymptotic level α. However, if e.g. the deviation from independence can be described in terms of mixing conditions the distribution of T_n will still converge to a normal distribution $N_{(0,4\sigma^2)}$, $\sigma^2 > 0$. Thus, if $\hat{\sigma}_n^2$ is a consistent estimator of σ^2 the critical region $\{|T_n| > 2\hat{\sigma}_n u_{\alpha/2}\}$ is again of asymptotic level α.

Henceforth we assume that the observations come from two strictly stationary independent processes $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ which are ψ-mixing according to Definition 2.1 and are defined on the same probability space $(\Omega, \mathscr{A}, P)$.

DEFINITION 2.1 (Blum, Hanson, Koopmans, 1963). A sequence $(Z_i)_{i\in\mathbb{N}}$ of measurable functions on a probability space $(\Omega, \mathscr{A}, P)$ with values in a measurable space $(\Psi, \mathscr{B})$ is called ψ-mixing if there exist $K \in \mathbb{N}$ and a sequence $\psi(n) \to_{n\in\mathbb{N}} 0$ such that

$$|P(A \cap B) - P(A)P(B)| \leq \psi(n)P(A)P(B) \tag{2.2}$$

for all $n \geq K$, $A \in \mathscr{A}_1^k$, $B \in \mathscr{A}_{k+n}^\infty$, $k \in \mathbb{N}$. Here $\mathscr{A}_a^b$ denotes the σ-algebra generated by the random variables $Z_a, \cdots, Z_b$. Note that an m-dependent process is ψ-mixing with $K = m + 1$.

The following lemma is essential to our considerations.

Lemma 2.3. *Let $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ be independent processes which are strictly stationary and ψ-mixing with*

$$\textstyle\sum_{n\geq k}\psi(n)^{1/2} < \infty. \tag{2.4}$$

Then the process $((X_i, Y_i))_{i\in\mathbb{N}}$ which takes values in $(\mathbb{R}^2)^{\mathbb{N}}$ is again strictly stationary and ψ-mixing with (2.4). *Thus, this is also true for the process $(1_{(0,\infty)}(Y_i - X_i))_{i\in\mathbb{N}}$.*

Lemma 2.3 and the Central Limit Theorem for mixing variables, see e.g. Billingsley (1968), Theorem 20.1, imply

$$(P_n \times Q_n)*T_n \Rightarrow N_{(0,4\sigma^2(1/2))} \quad \text{if} \quad \mu = 1/2 \tag{2.5}$$

where

$$P_n := P*(X_i)_{i=1}^n,\ Q_n := P*(Y_i)_{i=1}^n$$

and

$$\sigma^2(\mu) := \mu(1-\mu) + \textstyle\sum_{k\geq 2}\{E(1_{(0,\infty)}(Y_1 - X_1)1_{(0,\infty)}(Y_k - X_k)) - \sigma^2\}.$$

Hereafter we assume $\sigma^2(1/2) > 0$. Next we define a sequence of estimators $\hat\sigma_n^2$, $n \in \mathbb{N}$, of σ^2. Put

$$\begin{aligned}
&\hat\sigma_n^2((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) \\
&:= \{n^{-1}\textstyle\sum_{i=1}^n 1_{(0,\infty)}(y_i - x_i)\}\{1 - n^{-1}\sum_{i=1}^n 1_{(0,\infty)}(y_i - x_i)\} \\
&\quad + 2\textstyle\sum_{k=2}^m [n/k]^{-1}\sum_{r=0}^{[n/k]-1}\{1_{(0,\infty)}(y_{rk+1} - x_{rk+1})(y_{(r+1)k} - x_{(r+1)k}) \\
&\qquad - (n^{-1}\textstyle\sum_{i=1}^n 1_{(0,\infty)}(y_i - x_i))^2\}
\end{aligned} \tag{2.6}$$

where $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, $[x]$ denotes the integral part of x and $m = m(n)$ is such that

$$m^{3+\gamma} = O(n) \quad \text{for some} \quad \gamma > 0. \tag{2.7}$$

The next lemma states the strong consistency of $\hat\sigma_n^2$.

Lemma 2.8. *Under the conditions of Lemma* 2.3

$$\lim_{n\in\mathbb{N}}\hat\sigma_n^2 = \sigma^2 \qquad P_\infty \times Q_\infty\text{—a.e.}$$

where $P_\infty := P*(X_i)_{i\in\mathbb{N}}$ and $Q_\infty := P*(Y_i)_{i\in\mathbb{N}}$.

The robustified version of the sign test is now defined by

$$\tilde{C}_{n,\alpha} := \{|T_n| > 2\hat{\sigma}_n u_{\alpha/2}\}. \tag{2.9}$$

The following Theorem is an immediate consequence of (2.5), Lemma 2.8 and Slutzky's Theorem. It states that $\tilde{C}_{n,\alpha}$, $n \in \mathbb{N}$, is a consistent test sequence which is asymptotically of level α.

THEOREM 2.10. *Let $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ be independent processes which are strictly stationary and ψ-mixing with* (2.4). *Then,*

$$\lim_{n\in\mathbb{N}} (P_n \times Q_n)(\tilde{C}_{n,\alpha}) = \begin{cases} \alpha & \text{if } \mu = 1/2 \\ 1 & \text{if } \mu \neq 1/2. \end{cases}$$

3. Asymptotic relative efficiencies against translation alternatives. The aim of this section is to investigate the performance of the robustified sign test against translation alternatives. Moreover, its asymptotic relative efficiency w.r.t. the Neyman-Pearson test is computed in the special case of Gaussian processes.

Define alternatives $P_{n,n} \times Q_{n,n}$, $n \in \mathbb{N}$, by $P_{n,n} := P*((X_i - cn^{-1/2}))_{i=1}^{n}$ and $Q_{n,n} := P*((Y_i + cn^{-1/2}))_{i=1}^{n}$, $c > 0$. Notice that

$$\mu_n := P*(X_1 - cn^{-1/2},\ Y_1 + cn^{-1/2})\{\pi_2 > \pi_1\} \geq 1/2 \tag{3.1}$$

if $P_1 = Q_1$ is nonatomic. Thus, the appropriate critical region for the one-sided testing problem $H_{0,n} := \{(P*X_1) \times (P*Y_1)\}$ against $H_{1,n} := \{(P*(X_1 - cn^{-1/2})) \times (P*(Y_1 + cn^{-1/2}))\}$ is $\tilde{\tilde{C}}_{n,\alpha} := \{T_n > 2\hat{\sigma}_n u_\alpha\}$, which is obviously better adapted to this situation than $\tilde{C}_{n,\alpha}$.

The asymptotic value of the power function of the robustified sign test statistic is given in the following result.

PROPOSITION 3.2. *Let $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ be independent processes which are strictly stationary and ψ-mixing with* (2.4). *Furthermore we assume $P_1 = Q_1$, i.e. $\mu = 1/2$, and that P_1 has a Lebesgue density f such that $\int f^2(x)\,dx < \infty$. Then,*

$$\lim_{n\in\mathbb{N}} (P_{n,n} \times Q_{n,n})(\tilde{\tilde{C}}_{n,\alpha}) = 1 - \Phi\left(u_\alpha - 2c\sigma^{-1} \int f^2(x)\,dx\right)$$

where Φ denotes the distribution function of $N_{(0,1)}$.

In the following, let $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ be two independent strictly stationary N-dependent Gaussian processes with $P_\infty = Q_\infty$ and $P_1 = Q_1 = N_{(0,1)}$. Note that a ψ-mixing Gaussian process is already m-dependent (see Theorem 5, Chapter IV.2., page 125, in Ibragimov and Rozanov, 1978). Further we assume that the covariance matrix $R_n := (\rho_{ij})_{1\leq i,j\leq n}$ with $\rho_{ij} := \rho_{|i-j|} := E(X_1 X_{|i-j|+1})$ is positive definite and $1 + 2\sum_{1\leq k\leq N} \rho_k > 0$.

According to Ibragimov and Rozanov (1978), Chapter V.6, Theorem 11, these assumptions imply that the spectral density s of $(X_i)_{i\in\mathbb{N}}$ has the form $s(y) = \sum_{k=-N}^{N} \rho_k e^{iky} := A(e^{iy})A(e^{-iy}) = |A(e^{iy})|^2$ where $\rho_{-k} := \rho_k$, $1 \leq k \leq N$, $\rho_0 = 1$ and

$A(z) = \sum_{k=0}^{N} a_k z^{N-k}$ is a polynomial of degree N with real coefficients. We assume that the roots m_k of the equation $A(m) = 0$ fulfill $|m_k| < 1$, $1 \le k \le N$.

Denote by $C^*_{n,\alpha}$ the critical region of level α of Neyman-Pearson type for the testing problem $P_n \times Q_n$ against $P_{n,n} \times Q_{n,n}$. Then, under the above assumptions the following result about the power functions of $C^*_{n,\alpha}$ and $\tilde{\tilde{C}}_{n,\alpha}$ can be obtained.

PROPOSITION 3.3.

(i) $\lim_{n\in\mathbb{N}}(P_{n,n} \times Q_{n,n})(C^*_{n,\alpha}) = 1 - \Phi\{u_\alpha - (2c^2/(1 + 2\sum_{k=1}^{N} \rho_k))^{1/2}\}$ *and*

(ii) $\lim_{n\in\mathbb{N}}(P_{n,n} \times Q_{n,n})(\tilde{\tilde{C}}_{n,\alpha})$

$$= 1 - \Phi\{u_\alpha - 2c/((\pi(1 + 4\pi^{-1} \textstyle\sum_{k=1}^{N} \arcsin(\rho_k)))^{1/2})\}.$$

Our first main result is now immediate from Proposition 3.3.

THEOREM 3.4. *The* ARE *(Pitman-efficiency) of* $\tilde{\tilde{C}}_{n,\alpha}$ *w.r.t.* $C^*_{n,\alpha}$ *is given by*

$\mathrm{ARE}(\tilde{\tilde{C}}_{n,\alpha}: C^*_{n,\alpha} \mid P_{n,n} \times Q_{n,n})$

$$= 2\pi^{-1}(1 + 2\textstyle\sum_{k=1}^{N} \rho_k)/(1 + 4\pi^{-1}\sum_{k=1}^{N} \arcsin(\rho_k)).$$

For example, let $\rho_1 \in (-½, ½)$ and put $\rho_k = 0$, $k > 1$. Then for the ARE $=$ ARE(ρ_1) of $\tilde{\tilde{C}}_{n,\alpha}$ w.r.t. $C^*_{n,\alpha}$ we have ARE $\to (6/5)2\pi^{-1}$ if $\rho_1 \to ½$ and ARE $\to 0$ if $\rho_1 \to -½$. This shows that the ARE may be larger or less than $2\pi^{-1}$ which is its value in the independent case.

REMARK 3.5. Let p_n and q_n be Lebesgue densities of $P_n \times Q_n$ and $P_{n,n} \times Q_{n,n}$, respectively. As is shown in the proof of Proposition 3.3 (i) we have $(P_n \times Q_n)*\log(q_n/p_n) \Rightarrow N_{(-\kappa^2/2,\kappa^2)}$, where $\kappa^2 := 2c^2/(1 + 2\sum_{k=1}^{N} \rho_k) > 0$. Thus, Le Cam's First Lemma (see Hájek and Šidák, 1967, page 204) implies that $(P_{n,n} \times Q_{n,n})_{n\in\mathbb{N}}$ is contiguous to $(P_n \times Q_n)_{n\in\mathbb{N}}$.

4. Asymptotic relative efficiencies against alternatives of the form $F_\Delta = (1 - \Delta)F + \Delta F^2$. In this section we compare the studentized sign test with the robustified Wilcoxon two sample test which is defined by

$$C'_{n,\alpha} := \{n^{1/2}\hat{\tau}_n^{-1} Z_n > u_\alpha\}$$

where $Z_n(\mathbf{u}, \mathbf{v}) := n^{-2}\sum_{i,j=1}^{n} \mathrm{sign}(v_j - u_i)$ and $\hat{\tau}^2_n$ is a consistent estimator of the asymptotic variance τ^2 of $n^{1/2}Z_n$.

Let $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ be two independent stationary ψ-mixing processes on $(\Omega, \mathscr{A}, P)$ fulfilling (2.4). Furthermore we assume $P_\infty = Q_\infty$ and $P_1 = Q_1 = Q$, where Q denotes the uniform distribution on $(0, 1)$.

Let F be a continuous distribution function and denote by F^{-1} its generalized inverse, i.e. $F^{-1}(t) := \inf\{s \in \mathbb{R}: F(s) \ge t\}$, $t \in (0, 1)$. Define $P'_n := P*(F^{-1}(X_i))_{i=1}^{n}$, $Q'_n := P*(F^{-1}(Y_i))_{i=1}^{n}$, $P'_{n,n} := P*(F^{-1}_{\Delta_n}(X_i))_{i=1}^{n}$ and $Q'_{n,n} :=$

$P*(F_{\Delta_n}^{-1}(Y_i))_{i=1}^n$, where $F_\Delta := (1-\Delta)F + \Delta F^2$, $\Delta \in (-1, 1)$ and $\Delta_n := cn^{-1/2}$, $c \in (0, 1)$, $n \in \mathbb{N}$.

Note that $U_i := F^{-1}(X_i)$ and $V_i := F^{-1}(Y_i)$, $i \in \mathbb{N}$, are distributed according to F and again form two independent stationary ψ-mixing processes with (2.4).

Finally we assume contiguity of $P'_{n,n} \times Q'_{n,n}$, $n \in \mathbb{N}$, to $P'_n \times Q'_n$, $n \in \mathbb{N}$, which is true in the independent case and can also be achieved e.g. under suitable Markov properties (see Chapter 2 in Roussas, 1972).

It was proved by Serfling (1968) that $(P'_n \times Q'_n)*(n^{1/2}Z_n) \Rightarrow N_{(0,\tau^2)}$, $\tau^2 = 2/3 + 16\sum_{k=2}^{\infty}(E(X_1X_k) - 1/4)$. The following result specifies the asymptotic values of $C'_{n,\alpha}$ and $\tilde{\tilde{C}}_{n,\alpha}$ under the above alternatives $P'_{n,n} \times Q'_{n,n}$, $n \in \mathbb{N}$.

PROPOSITION 4.1

(i) $\lim_{n\in\mathbb{N}}(P'_{n,n} \times Q'_{n,n})(C'_{n,\alpha})$

$$= 1 - \Phi\{u_\alpha - c/(3/2 + 36\textstyle\sum_{k\geq 2}(E(X_1X_k) - 1/4))^{1/2}\},$$

(ii) $\lim_{n\in\mathbb{N}}(P'_{n,n} \times Q'_{n,n})(\tilde{\tilde{C}}_{n,\alpha})$

$$= 1 - \Phi\{u_\alpha - c/(9/4 + 18\textstyle\sum_{k\geq 2}(P\{Y_1 > X_1,\ Y_k > X_k\} - 1/4))^{1/2}\}.$$

THEOREM 4.2.

$$\mathrm{ARE}(\tilde{\tilde{C}}_{n,\alpha}\colon C'_{n,\alpha} \mid P'_{n,n} \times Q'_{n,n}) = \frac{(2/3)\{1 + 24\sum_{k\geq 2}(E(X_1X_k) - 1/4)\}}{1 + 8\sum_{k\geq 2}(P\{Y_1 > X_1,\ Y_k > X_k\} - 1/4)}$$

which equals 2/3 *in the independent case as is well known.*

Let for example $(W_i)_{i\in\mathbb{N}}$ and $(\tilde{W}_i)_{i\in\mathbb{N}}$ be two independent sequences of independent and uniformly on (0, 1) distributed random variables. Denote by F_c the distribution function of $W_1W_2^c$, $c \in \mathbb{R}$, and define the processes $(X_{c,i})_{i\in\mathbb{N}} := (F_c(W_iW_{i+1}^c))_{i\in\mathbb{N}}$ and $(Y_{c,i})_{i\in\mathbb{N}} := (F_c(\tilde{W}_i\tilde{W}_{i+1}^c))_{i\in\mathbb{N}}$. Straightforward calculations show that for the ARE = ARE(c) of $\tilde{\tilde{C}}_{n,\alpha}$ w.r.t. $C'_{n,\alpha}$ $\sup_{c\in\mathbb{R}}\mathrm{ARE}(c) \geq 7/9$ and $\inf_{c\in\mathbb{R}}\mathrm{ARE}(c) \leq 1/9$. However, the question of whether the ARE may become greater than one under appropriate dependence, i.e. whether the relative performance of two tests can ever be reversed, is still open.

5. Auxiliary results and proofs.

PROOF OF LEMMA 2.3. The stationarity of $((X_i, Y_i))_{i\in\mathbb{N}}$ is immediate from the independence and the stationarity of the components. To establish the mixing property of $((X_i, Y_i))_{i\in\mathbb{N}}$, one proves by means of standard arguments from measure theory that

$$\begin{aligned}|P\{((X_i, Y_i))_{i=1}^n \in A,\ ((X_j, Y_j))_{j\geq n+k} \in B\}\\ - P\{((X_i, Y_i))_{i=1}^n \in A\}P\{((X_j, Y_j))_{j\geq n+k} \in B\}|\\ \leq (2\psi(k) + \psi(k)^2)P\{((X_i, Y_i))_{i=1}^n \in A\}P\{((X_j, Y_j))_{j\geq n+k} \in B\}\end{aligned}$$

for all $A \in (\mathscr{B}^2)^n$, $B \in (\mathscr{B}^2)^{\mathbb{N}}$, $k \geq K$, $n \in \mathbb{N}$, where $\mathscr{B}$ denotes the Borel-σ-algebra over $\mathbb{R}$.

The following exponential inequality together with the Borel-Cantelli Lemma implies Lemma 2.8. The proof (which is omitted) follows the lines of Bennett's (1962) first improvement of the Bernstein inequality by making use of a blocking technique due to Philipp (1977).

LEMMA 5.1. *Let Z_i, $i \in \mathbb{N}$, be a strictly stationary sequence of random variables with $E(Z_1) = 0$, $|Z_1| \leq 1$. Suppose that Z_i, $i \in \mathbb{N}$, is ψ-mixing with $\sum_{k \geq K} \psi(k)^\delta < \infty$ for some $\delta \in (0, 1)$. Then for $\varepsilon > 0$, $\theta \in (\delta, 1)$ and $n \geq K$*

(5.2) $$P\{|n^{-1/2} \sum_{i=1}^n Z_i| \geq \varepsilon\} \leq C_1 \exp\{-\varepsilon^2/(C_2 E(Z_1^2) + 8\varepsilon n^{(\theta-1)/(2\theta+2)})\}$$

where C_1, C_2 are positive constants which only depend on ψ and θ.

PROOF OF PROPOSITION 3.2. We shall prove that

(5.3) $$(P_{n,n} \times Q_{n,n}) * T_n \Rightarrow N_{(4c\int f^2(x)dx, 4\sigma^2)} \quad \text{and}$$

(5.4) $$|\hat{\sigma}_n^2 - \sigma^2| \to_{P_{n,n} \times Q_{n,n}} 0.$$

(5.3) and (5.4) combined with Slutzky's Theorem imply the assertion. Ad (5.3):

$$\begin{aligned}&(P_{n,n} \times Q_{n,n}) * T_n\\ &= P*\{(n^{-1/2}(2 \textstyle\sum_{i=1}^n 1_{(0,\infty)}(Y_i - X_i) - n)) + 2n^{-1/2} \sum_{i=1}^n 1_{(-2cn^{-1/2},0]}(Y_i - X_i)\}\\ &=: (P_n \times Q_n)*(T_n + S_n).\end{aligned}$$

According to (2.5) we have $(P_n \times Q_n) \Rightarrow N_{(0,4\sigma^2)}$. Thus it remains to prove

(5.5) $$S_n \to_{P_n \times Q_n} 4c \int f^2(x)\, dx.$$

The dominated convergence theorem implies $E(S_n) \to_{n \in \mathbb{N}} 4c \int f^2(x)\, dx$ and application of Lemma 5.1 to the sequence

$$Z_i := \{1_{(-2cn^{-1/2},0]}(Y_i - X_i) - E(S_n/2)n^{-1/2}\}/2,$$

$i \in \mathbb{N}$, yields (5.5).

Ad (5.4): Define

$$\mu_n := P\{Y_1 - X_1 + 2cn^{-1/2} > 0\}$$

and

$$\begin{aligned}\sigma_n^2 :=\ & E((1_{(0,\infty)}(Y_1 - X_1 + 2cn^{-1/2}) - \mu_n)^2)\\ &+ 2 \textstyle\sum_{k=2}^\infty \{E(1_{(0,1)}(Y_1 - X_1 + 2cn^{-1/2}) 1_{(0,\infty)}(Y_k - X_k + 2cn^{-1/2})) - \mu_n^2\}.\end{aligned}$$

Due to Lemma 2.3 there exists $L > 0$ such that $\sigma^2, \sigma_n^2 \leq L$, $n \in \mathbb{N}$. Hence for $\varepsilon > 0$, $k_0 = k_0(\varepsilon)$ and $n \geq n_0(\varepsilon)$ we have by the dominated convergence theorem

$$\begin{aligned}&|\sigma_n^2 - \sigma^2|\\ &\leq |E((1_{(0,\infty)}(Y_1 - X_1 + 2cn^{-1/2}) - \mu_n)^2) - E((1_{(0,\infty)}(Y_1 - X_1) - 1/2)^2)\\ &\quad + 2 \textstyle\sum_{k=2}^{k_0} \{E(1_{(0,\infty)}(Y_1 - X_1 + 2cn^{-1/2}) 1_{(0,\infty)}(Y_k - X_k + 2cn^{-1/2})) - \mu_n^2\}\\ &\quad - 2 \textstyle\sum_{k=2}^{k_0} \{E(1_{(0,\infty)}(Y_1 - X_1) 1_{(0,\infty)}(Y_k - X_k)) - 1/4\}| + \varepsilon \leq 2\varepsilon.\end{aligned}$$

Thus $\lim_{n\in\mathbb{N}}\sigma_n^2 = \sigma^2$, whence Lemma 5.1 implies for $n \ge n_0$

$$(P_{n,n} \times Q_{n,n})\{|\hat{\sigma}_n^2 - \sigma^2| \ge \varepsilon\} \le (P_{n,n} \times Q_{n,n})\{|\hat{\sigma}_n^2 - \sigma_n^2| \ge \varepsilon/2\}$$
$$\le D_1 \exp(-\varepsilon^2 D_2 n^{D_3}), \quad D_1, D_2, D_3 > 0.$$

This proves (5.4) and thus, the proof of Proposition 3.2 is complete.

In the proof of Proposition 3.3 the limit distribution of $\log(q_n/p_n)$ is computed where p_n and q_n are Lebesgue densities of $P_n \times Q_n$ and $P_{n,n} \times Q_{n,n}$, $n \in \mathbb{N}$, respectively. To this end we establish the following result concerning covariance matrices of N-dependent stationary Gaussian processes.

LEMMA 5.6. *Let Z_i, $i \in \mathbb{N}$, be a stationary N-dependent Gaussian process on $(\Omega, \mathscr{A}, P)$ such that $E(Z_1) = 0$ and $E(Z_1^2) = 1$. Assume that the covariance matrix*

$$R_n := (\rho_{ij})_{1\le i,j\le n} := (\rho_{|i-j|})_{1\le i,j\le n} := (E(Z_1 Z_{|i-j|+1}))_{1\le i,j\le N} \tag{5.7}$$

is positive definite and that $1 + 2\sum_{1\le k\le N}\rho_k > 0$. Finally, suppose that the roots of the characteristic equation of Z_i, $i \in \mathbb{N}$, lie inside the unit circle. Then for the inverse $R_n^{-1} := (\sigma_{ij})_{1\le i,j\le n}$ of R_n we have

$$\begin{gathered}\exists\, \delta \in (0,1)\ \forall\, \varepsilon > 0\ \exists\, n_0 = n_0(\varepsilon)\ \forall\, n \ge n_0 \\ \left|\textstyle\sum_{j=1}^n \sigma_{ij} - n^{-1}\sum_{i,j=1}^n \sigma_{ij}\right| \le \varepsilon\end{gathered} \tag{5.8}$$

for $\alpha_n \le i \le n - \alpha_n$, where $\alpha_n := [\{(1+\alpha)/(2|\log\delta|)\}\log n]$, $\alpha > 0$,

$$\exists\, M > 0\ \forall\, n \in \mathbb{N}\ \forall\, 1 \le i \le n \quad \left|\textstyle\sum_{j=1}^n \sigma_{ij}\right| \le M, \tag{5.9}$$

$$\lim_{n\in\mathbb{N}} n^{-1}\sum_{i,j=1}^n \sigma_{ij} = \left(1 + 2\sum_{k=1}^N \rho_k\right)^{-1}. \tag{5.10}$$

PROOF. In Whittle (1951), pages 21 and 22, it is shown that the process Z_i, $i \in \mathbb{N}$, can be represented as a moving average scheme, i.e.

$$Z_i = \sum_{k=0}^N a_k \varepsilon_{i-k} \tag{5.11}$$

where ε_j, $j \ge -N+1$, are uncorrelated variables on $(\Omega, \mathscr{A}, P)$ with $E(\varepsilon_j) = 0$ and $E(\varepsilon_j^2) = E(\varepsilon_1^2) > 0$. Hence R_n can be interpreted as the covariance matrix of a stationary moving average scheme. Below we will show that R_n^{-1} can be approximated by the covariance matrix of a suitable stationary autoregressive process. Together with results about such matrices, this will prove the assertion.

First define the symmetric $n \times n$-matrix $W_n := (w_{ij})_{1\le i,j\le n}$ by $w_{ij} := 1$ if $j = i+1$, $w_{n1} = 1$ and zero else. Using that W_n is invertible, that $(W_n)^{-k} = (W_n^k)^t$, $1 \le k \le n$, and that $W_n^n = I_n = (\delta_{ij})_{1\le i,j\le n}$ we have $R_n = \sum_{k=-N}^N \rho_k W_n^k - V_n$, where the symmetric $n \times n$-matrix $V_n := (v_{ij})_{1\le i,j\le n}$ is given by $v_{ij} := 0$, $|i-j| \le n-N$, and $v_{ij} := \rho_{n+i-j}$, $j \ge n-N+i$.

Let S be the spectral function of Z_i, $i \in \mathbb{N}$, i.e. $S(e^{iy}) := s(y)$; hence $S(z) = \sum_{k=-N}^N \rho_k z^k = A(z)A(z^{-1})$. For the sake of a clear presentation, we allow matrices as arguments in S and get $R_n = S(W_n) - V_n$, i.e. $R_n \sim A(W_n)A(W_n^{-1})$, cf. Whittle (1951), (4.276).

In order to approximate R_n^{-1} we now consider a stationary process $\tilde{Z}_i$, $i \in \mathbb{N}$, with spectral density $g(y) := s(y)^{-1} = (A(e^{iy})A(e^{-iy}))^{-1} =: \sum_{k=-\infty}^{\infty} \tilde{\rho}_k e^{iky}$, where

$\tilde{\rho}_k = \tilde{\rho}_{-k}$, $k \in \mathbb{N}$. In Whittle (1951), pages 21, 22 it is proved that $\tilde{Z}_i$, $i \in \mathbb{N}$, is an autoregressive process of order N, i.e. $\sum_{k=0}^{N} a_k \tilde{Z}_{j-k} = \tilde{\varepsilon}_j$, where $\tilde{\varepsilon}_j$, $j \geq N+1$, are uncorrelated variables on $(\Omega, \mathscr{A}, P)$ with $E(\tilde{\varepsilon}_j) = 0$ and $E(\tilde{\varepsilon}_j^2) = E(\tilde{\varepsilon}_1^2) > 0$. Further, a_k, $0 \leq k \leq N$ are the coefficients of A. Stationarity of $\tilde{Z}_i$, $i \in \mathbb{N}$, follows from the equivalent assumption $|m_k| < 1$, $1 \leq k \leq N$, for the roots of $A(m) = 0$, cf. Box and Jenkins (1970), Section 3.2.

According to Whittle (1951), page 35, there exists $\delta \in (0, 1)$ such that

$$(5.12) \qquad |\tilde{\rho}_k| = |E(\tilde{Z}_1 \tilde{Z}_{k+1})| \leq \delta^k, \quad k \in \{0, 1, 2, \cdots\}.$$

Define $S_n := (\tilde{\rho}_{ij})_{1\leq i,j\leq n} = (\tilde{\rho}_{|i-j|})_{1\leq i,j\leq n}$ and the symmetric $n \times n$-matrix $\tilde{V}_n := (\tilde{v}_{ij})_{1\leq i,j\leq n}$ by $\tilde{v}_{ii} := 0$ and $\tilde{v}_{ij} := \tilde{\rho}_{n+1-j}$, $j > i$. Then $S_n = \sum_{k=-n+1}^{n-1} \tilde{\rho}_k W_n^k - \tilde{V}_n = (A(W_n)A(W_n^{-1}))^{-1} - \tilde{V}_n - \sum_{k=n}^{\infty} \tilde{\rho}_k (W_n^k + W_n^{-k})$.

Because of (5.12) the elements of $\sum_{k=n}^{\infty} \tilde{\rho}_k (W_n^k + W_n^{-k})$ are in absolute value not greater than $\delta^n/(1-\delta)$. Hence for $(A(W_n)A(W_n^{-1}))^{-1} = S(W_n) := (s_{ij})_{1\leq i,j\leq n}$ we have

$$(5.13) \qquad \begin{aligned} &|s_{ij} - (\tilde{\rho}_{|i-j|} + \tilde{\rho}_{n-|i-j|})| \leq \delta^n/(1-\delta), \quad 1 \leq i \neq j \leq n, \quad \text{and} \\ &|s_{ii} - \tilde{\rho}_0| \leq \delta^n/(1-\delta), \quad 1 \leq i \leq n. \end{aligned}$$

We approximate R_n^{-1} by $S(W_n)^{-1}$. To this end let $\mathbf{r}_i$, $\mathbf{s}_i$ and $\mathbf{e}_i$ be the ith column vector of R_n^{-1}, $S(W_n)^{-1}$ and I_n, respectively, $1 \leq i \leq n$, and define $\|\mathbf{x}\| := (\sum_{i=1}^{n} x_i^2)^{1/2}$, $\mathbf{x} \in \mathbb{R}^n$, and $\|B\| := \lambda_1^{1/2}$ where B is a real $n \times n$-matrix and λ_1 the largest eigenvalue of $B^t B$. Using $\|B\mathbf{x}\| \leq \|B\| \, \|\mathbf{x}\|$ we get

$$(5.14) \qquad \|\mathbf{r}_i - \mathbf{s}_i\| =: \|R_n^{-1}\mathbf{t}_i\| \leq \|R_n^{-1}\| \, \|\mathbf{t}_i\| \leq \|R_n^{-1}\| \, (2N)^{1/2} K \delta^{\min(i,n-i)}.$$

The approximation (cf. Fuller (1976), pages 133 ff.)

$$(5.15) \qquad |\lambda_j - s(2\pi(j-1)/n)| \leq 4n^{-1} \sum_{k=-N}^{N} |k| \, |\rho_k|$$

for the eigenvalues λ_j, $1 \leq j \leq n$, of R_n and the fact that $s(2\pi(j-1)/n) = \prod_{k=1}^{N} (e^{i2\pi(j-1)/n} - m_k) \prod_{k=1}^{N} (e^{-i2\pi(j-1)/n} - m_k)$ where m_k, $1 \leq k \leq N$, are the roots of $A(m) = 0$, $|m_k| < 1$, $1 \leq k \leq N$, imply for $n \geq n_0$

$$(5.16) \qquad \lambda_j \geq 2^{-1}(\min\{1 - |m_k| : 1 \leq k \leq N\})^{2N} > 0, \quad 1 \leq j \leq n.$$

Hence there exists $L > 0$ such that

$$(5.17) \qquad \|R_n^{-1}\| = \lambda_1^{-1} < L < \infty,$$

which together with (5.14) yields

$$(5.18) \qquad \|\mathbf{r}_i - \mathbf{s}_i\| \leq (2N)^{1/2} K L \, \delta^{\min(i,n-i)}.$$

From (5.13) it is immediate that

$$(5.19) \qquad \begin{aligned} &\forall \varepsilon > 0 \; \exists \, C \in \mathbb{N} \; \exists \, n_1 \in \mathbb{N} \; \forall \, n \geq n_1 \; \forall \, C \leq i \leq n - C \\ &\left| \textstyle\sum_{j=1}^{n} s_{ji} - n^{-1} \sum_{k,j=1}^{n} s_{jk} \right| \leq \varepsilon. \end{aligned}$$

Furthermore, from the definition of α_n in (5.8) we have

$$\exists\ n_2 = n_2(\varepsilon)\ \forall\ n \geq n_2\ \forall\ \alpha_n \leq i \leq n - \alpha_n \quad (2N)^{1/2}KL\ \delta^{\min(i,n-i)}n^{1/2} \leq \varepsilon. \tag{5.20}$$

Now (5.8), (5.9) and (5.10) follow from (5.12), (5.13), (5.18), (5.19) and (5.20) by straightforward calculations.

PROOF OF PROPOSITION 3.3. Proposition 3.2 together with Sheppard's formula, cf. Moran (1968), formula (7.86), implies (ii).

Ad (i): By $R_n := (\rho_{ij})_{1\leq i,j\leq n}$ we denote the covariance matrix of X_i, $i \in \mathbb{N}$, and Y_i, $i \in \mathbb{N}$, i.e. $\rho_{ij} := \rho_{|i-j|}$, $|i-j| \leq N$, and zero else and by $R_n^{-1} =: (\sigma_{ij})_{1\leq i,j\leq n}$ its inverse. With $(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^n)^2$ and $\mathbf{m}_n := (cn^{-1/2}, \cdots, cn^{-1/2}) \in \mathbb{R}^n$ define

$$p_n((\mathbf{x}, \mathbf{y})) := (2\pi)^{-n}(\det R_n)^{-1}\exp(-2^{-1}\mathbf{x}R_n^{-1}\mathbf{x}^t - 2^{-1}\mathbf{y}R_n^{-1}\mathbf{y}^t)$$

and

$$q_n((\mathbf{x}, \mathbf{y})) := (2\pi)^{-n}(\det R_n)^{-1}\exp\{-2^{-1}(\mathbf{x} + \mathbf{m}_n)R_n^{-1}(\mathbf{x} + \mathbf{m}_n)^t - 2^{-1}(\mathbf{y} - \mathbf{m}_n)R_n^{-1}(\mathbf{y} - \mathbf{m}_n)^t\}.$$

Here p_n and q_n are Lebesgue densities of $P_n \times Q_n$ and $P_{n,n} \times Q_{n,n}$, respectively, and

$$\log\{q_n((\mathbf{x}, \mathbf{y}))/p_n((\mathbf{x}, \mathbf{y}))\} = cn^{-1/2} \textstyle\sum_{1\leq i\leq n}(\sum_{1\leq j\leq n}\sigma_{ij})y_i - cn^{-1/2} \sum_{1\leq i\leq n}(\sum_{1\leq j\leq n}\sigma_{ij})x_i - c^2n^{-1} \sum_{1\leq i,j\leq n}\sigma_{ij}.$$

Thus, $(P_n \times Q_n)*\log(q_n/p_n) = N_{(-c^2n^{-1}\sum_{1\leq i,j\leq n}\sigma_{ij},\theta_n)}$. We show $\lim_{n\in\mathbb{N}}\theta_n = 2c^2L$, $L := (1 + 2 \sum_{1\leq k\leq N}\rho_k)^{-1}$.

From Lemma 5.6, the independence of $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ and the Cauchy-Schwarz inequality we obtain for $n \geq n_0(\varepsilon)$

$$\begin{aligned} D_n := |\theta_n - 2c^2L| \leq\ & |2c^2n^{-1}L^2 \textstyle\sum_{i,j=\alpha_n}^{n-\alpha_n} E(Y_iY_j) - 2c^2L| \\ & + c^2n^{-1}(n - 2\alpha_n)2N\varepsilon^2 + c^2n^{-1}M^24\alpha_n^2 \\ & + 2(2c^2n^{-1}L^2 \textstyle\sum_{i,j=\alpha_n}^{n-\alpha_n} E(Y_iY_j))^{1/2}(cn^{-1}\varepsilon^2(n - 2\alpha_n)2N)^{1/2} \\ & + 2(2c^2n^{-1}L^2 \textstyle\sum_{i,j=\alpha_n}^{n-\alpha_n} E(Y_iY_j))^{1/2}(c^2n^{-1}M^24\alpha_n^2)^{1/2} \\ & + 2(c^2n^{-1}\varepsilon^2(n - 2\alpha_n)2N)^{1/2}(c^2n^{-1}M^24\alpha_n^2)^{1/2}. \end{aligned}$$

Because of the stationary of $(Y_i)_{i\in\mathbb{N}}$ we have $\lim_{n\in\mathbb{N}}n^{-1} \sum_{i,j=\alpha_n}^{n-\alpha_n} E(Y_iY_j) = L^{-1}$. Thus, $\lim\sup_{n\in\mathbb{N}}D_n \leq c^2\varepsilon^22N + 2(2c^2L)^{1/2}(c^2\varepsilon^22N)^{1/2}$ and, since $\varepsilon > 0$ was arbitrary, $\lim_{n\in\mathbb{N}}\theta_n = 2c^2L$.

Altogether we have $N^2_{(\mathbf{0},R_n)}*\log(q_n/p_n) \Rightarrow N_{(-c^2L,2c^2L)}$, i.e. $P_{n,n} \times Q_{n,n}$, $n \in \mathbb{N}$, is contiguous to $P_n \times Q_n$, $n \in \mathbb{N}$, cf. Remark 3.5.

A most powerful test of level α for testing $N^2_{(\mathbf{0},R_n)}$ against $N_{(-\mathbf{m}_n,R_n)} \times N_{(\mathbf{m}_n,R_n)}$

has according to the Neyman-Pearson Lemma a critical region of the form $C^*_{n,\alpha} = \{\log(q_n/p_n) > a_n\}$ where $\lim_{n\in\mathbb{N}} a_n = (2c^2L)^{1/2}u_\alpha - c^2L$. Since the limit distribution of $\log(q_n/p_n)$ under the contiguous alternatives $P_{n,n} \times Q_{n,n}$, $n \in \mathbb{N}$, is $N_{(c^2L,2c^2L)}$ we have $\lim_{n\in\mathbb{N}}(P_{n,n} \times Q_{n,n})(C^*_{n,\alpha}) = 1 - \Phi\{u_\alpha - (2c^2L)^{1/2}\}$, which is (i).

PROOF OF PROPOSITION 4.1. Ad (i): Serfling (1968) has shown that under suitable mixing conditions being fulfilled here, the sequence

$$W_n((U_i)_{i=1}^n, (V_i)_{i=1}^n)$$
$$:= n^{-1/2} \sum_{i=1}^n (1 - 2F(U_i)) + n^{-1/2} \sum_{i=1}^n (2F(V_i) - 1)$$
$$= n^{-1/2} \sum_{i=1}^n (1 - 2X_i) + n^{-1/2} \sum_{i=1}^n (2Y_i - 1), \quad n \in \mathbb{N},$$

is asymptotically equivalent to $S_n((U_i)_{i=1}^n, (V_i)_{i=1}^n)$, $S_n := n^{1/2}Z_n$.

With $\Delta_n = cn^{-1/2}$, $c \in (0, 1)$, $n \in \mathbb{N}$, define $U_{i,n} := F^{-1}_{-\Delta_n}(X_i)$, $V_{i,n} := F^{-1}_{\Delta_n}(Y_i)$, $i \in \mathbb{N}$.

Under the assumptions of Proposition 4.1, $(P*(U_{i,n})_{i=1}^n) \times (P*(V_{i,n})_{i=1}^n) = P'_{n,n} \times Q'_{n,n}$, $n \in \mathbb{N}$, is contiguous to $P'_n \times Q'_n$, $n \in \mathbb{N}$. This implies the asymptotic equivalence of $W_n((U_{i,n})_{i=1}^n, (V_{i,n})_{i=1}^n)$, $n \in \mathbb{N}$, and $S_n((U_{i,n})_{i=1}^n, (V_{i,n})_{i=1}^n)$, $n \in \mathbb{N}$. Hence $\lim_{n\in\mathbb{N}}(P'_{n,n} \times Q'_{n,n})(C'_{n,\alpha}) = \lim_{n\in\mathbb{N}}((P'_{n,n} \times Q'_{n,n})*\tau^{-1}W_n)((u_\alpha, \infty))$, if the last limit exists. Therefore, (i) follows if we show

$$(5.21)\quad (P'_{n,n} \times Q'_{n,n})*W_n = P*\{n^{-1/2} \sum_{i=1}^n (1 - 2F(F^{-1}_{-\Delta_n}(X_i))) + n^{-1/2} \sum_{i=1}^n (2F(F^{-1}_{\Delta_n}(Y_i) - 1))\} \Rightarrow N_{(2c/3,\tau^2)}.$$

Since $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ are independent, it suffices to prove

$$(5.22)\quad P*\{n^{-1/2} \sum_{i=1}^n (1 - 2F(F^{-1}_{-\Delta_n}(X_i)))\} \Rightarrow N_{(c/3,\tau^2/2)} \quad \text{and}$$

$$(5.23)\quad P*\{n^{-1/2} \sum_{i=1}^n (2F(F^{-1}_{\Delta_n}(Y_i)) - 1)\} \Rightarrow N_{(c/3,\tau^2/2)}.$$

We only treat (5.22) since (5.23) is completely analoguous and show

$$(5.24)\quad E\{n^{-1/2} \sum_{i=1}^n (1 - 2F(F^{-1}_{-\Delta_n}(X_i)))\} = c/3 \quad \text{and}$$

$$(5.25)\quad \lim_{n\in\mathbb{N}} E\{(n^{-1/2} \sum_{i=1}^n (1 - 2F(F^{-1}_{-\Delta_n}(X_i))) - c/3)^2\} = \tau^2/2.$$

From (5.24), (5.25) and Corollary 1, page 1102, in Withers (1975), assertion (5.22) is immediate. Wither's condition (f): $\sum_{i=1}^j i^2\psi(i) \le Cj^r$, $1 \le j \le n$, $r < 3/2$, is fulfilled here since w.l.o.g. one can assume $\psi(k) \downarrow$ and thus (2.4) implies $\psi(k) = O(k^{-2})$.

(5.24) follows from $E(F(F^{-1}_{-\Delta_n}(X_1))) = ½ - \Delta_n/6$.

Ad (5.25):

$$E\{(n^{-1/2} \sum_{i=1}^n (1 - 2F(F^{-1}_{-\Delta_n}(X_i))) - c/3)^2\}$$
$$= 1 - 4(½ - c/(6n^{1/2})) + 4E(F(U_{1,n})^2) - c^2/(9n)$$
$$+ 2 \sum_{i=1}^{n-1} \{E((1 - 2F(U_{1,n}))(1 - 2F(U_{i+1,n}))) - c^2/(9n)\}$$
$$- 2n^{-1} \sum_{i=1}^{n-1} i\{E((1 - 2F(U_{1,n}))(1 - 2F(U_{i+1,n}))) - c^2/(9n)\}$$
$$=: A_n + B_n + C_n.$$

First from (2.4) and a bound analoguous to that in formula (20.43) in Billingsley (1968), page 173,

$$\lim_{n\in\mathbb{N}} C_n = 0. \tag{5.26}$$

Further,

$$\lim_{n\in\mathbb{N}} A_n = 1/3 \quad \text{and} \tag{5.27}$$

$$\lim_{n\in\mathbb{N}} B_n = 8 \textstyle\sum_{k=2}^{\infty} (E(X_1 X_k) - 1/4). \tag{5.28}$$

(5.27) follows from $E(F(U_{1,n})^2) = 1/3 - \Delta_n/6$.

Ad (5.28): By (2.4) we have $\sum_{k\ge k_0}\psi(k) < \varepsilon/2$ for k_0 large enough and thus by the inequality $|E(X_1X_k) - E(X_1)E(X_k)| \le \psi(k-1)E(X_1)E(X_k)$ (cf. Billingsley, 1968, Lemma 1, Chapter 20)

$$\begin{aligned} &|B_n - 8 \textstyle\sum_{k\ge 2}(E(X_1X_k) - 1/4)| \\ &\quad \le |2 \textstyle\sum_{k=2}^{k_0-1} \{1 - 4(1/2 - c/(6n^{1/2})) + 4E(F(U_{1,n})F(U_{k,n})) \\ &\qquad\qquad - c^2/(9n) - 4E(X_1X_k) + 1\}| + 3\varepsilon \\ &\quad = 8 |\textstyle\sum_{k=2}^{k_0-1} \{E(F(U_{1,n})F(U_{k,n})) - E(X_1X_k)\}| + O(n^{-1/2}) + 3\varepsilon. \end{aligned}$$

Since $\lim_{n\in\mathbb{N}} F(F^{-1}_{-\Delta_n}) = \mathrm{id}_{(0,1)}$ we have $\lim_{n\in\mathbb{N}} E(F(U_{1,n})F(U_{k,n})) = E(X_1X_k)$ which proves (5.28).

Ad (ii): We show

$$\begin{aligned} &(P'_{n,n} \times Q'_{n,n}) * T_n \Rightarrow N_{(2c/3, \tilde\tau^2)} \quad \text{with} \\ &\tilde\tau := 1 + 8 \textstyle\sum_{k\ge 2}(P\{Y_1 > X_1,\ Y_k > X_k\} - 1/4). \end{aligned} \tag{5.29}$$

Since $P'_{n,n} \times Q'_{n,n}$, $n \in \mathbb{N}$, is contiguous to $P'_n \times Q'_n$, $n \in \mathbb{N}$, we have $\hat\sigma_n^2 \to \tilde\tau^2/4$ under $P'_{n,n} \times Q'_{n,n}$, $n \in \mathbb{N}$. Hence, together with (5.29)

$$\lim_{n\in\mathbb{N}} (P'_{n,n} \times Q'_{n,n})(\tilde{\tilde{C}}_{n,\alpha}) = 1 - \Phi(u_\alpha - 2c(3\tilde\tau)^{-1}). \tag{5.30}$$

Ad (5.29): This follows in analogy to the proof of (5.22) from

$$E\{n^{-1/2}(2 \textstyle\sum_{i=1}^{n} 1_{(0,\infty)}(V_{i,n} - U_{i,n}) - n)\} = 2c/3 \quad \text{and} \tag{5.31}$$

$$\lim_{n\in\mathbb{N}} E(\{n^{-1/2}(2 \textstyle\sum_{i=1}^{n} 1_{(0,\infty)}(V_{i,n} - U_{i,n}) - n) - 2c/3\}^2) = \tilde\tau^2. \tag{5.32}$$

Ad (5.32): By the stationarity and independence of $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ we have

$$\begin{aligned} &E(\{n^{-1/2}(2 \textstyle\sum_{i=1}^{n} 1_{(0,\infty)}(V_{i,n} - U_{i,n}) - n) - 2c/3\}^2) \\ &\quad = 1 - 4c^2/(9n) + 8n^{-1} \textstyle\sum_{k=1}^{n-1} (n-k)\{P\{V_{1,n} > U_{1,n},\ V_{k+1,n} > U_{k+1,n}\} \\ &\qquad\qquad - (1/2 + c/(3n^{1/2}))^2\}. \end{aligned}$$

For $2 \le k \le n$ we have

$$\begin{aligned} |P\{V_{1,n} > U_{1,n},\ V_{k,n} > U_{k,n}\} - P\{Y_1 > X_1,\ Y_k > X_k\}| \\ &\le |P\{V_{1,n} > U_{1,n},\ V_{k,n} > U_{k,n}\} - P\{V_{1,n} > U_1,\ V_{k,n} > U_k\}| \\ &\quad + |P\{V_{1,n} > U_1,\ V_{k,n} > U_k\} - P\{Y_1 > X_1,\ Y_k > X_k\}| \\ &=: A_{n,k} + B_{n,k} \end{aligned}$$

and by definition (note that $F_{\Delta_n} = F - \Delta_n F(1-F) \le F$ and $X_1 = F(U_1)$):

$$\begin{aligned} B_{n,k} &= P\{Y_1 > F_{\Delta_n}(U_1),\ Y_k > F_{\Delta_n}(U_k)\} - P\{Y_1 > F(U_1),\ Y_k > F(U_k)\} \\ &= P\{F(U_1) \ge Y_1 > F_{\Delta_n}(U_1),\ Y_k > F_{\Delta_n}(U_k)\} \\ &\quad + P\{Y_1 > F(U_1),\ F(U_k) \ge Y_k > F_{\Delta_n}(U_k)\} \\ &\le 2P\{F(U_1) \ge Y_1 > F_{\Delta_n}(U_1)\} \le c/(2n^{1/2}). \end{aligned}$$

Furthermore, by definition $F_{-\Delta_n} - \Delta_n/2 \le F_{\Delta_n} \le F$, and hence,

$$\begin{aligned} A_{n,k} &= P\{F^{-1}(X_1) \ge F_{\Delta_n}^{-1}(Y_1) > F_{-\Delta_n}^{-1}(X_1),\ F_{\Delta_n}^{-1}(Y_k) > F_{-\Delta_n}^{-1}(X_k)\} \\ &\quad + P\{F_{\Delta_n}^{-1}(Y_1) > F^{-1}(X_1),\ F^{-1}(X_k) \ge F_{\Delta_n}^{-1}(Y_k) > F_{-\Delta_n}^{-1}(X_k)\} \\ &\le 2P\{F^{-1}(X_1) \ge F_{\Delta_n}^{-1}(Y_1) > F_{-\Delta_n}^{-1}(X_1)\} \\ &\le 2P\{X_1 \ge Y_1 > X_1 - \Delta_n/2\} \le c/n^{1/2}. \end{aligned}$$

In analogy to the proof of (5.25) we have

$$\begin{aligned} |E(\{n^{-1/2}(2 \textstyle\sum_{i=1}^n 1_{(0,\infty)}(V_{i,n} - U_{i,n}) - n) - 2c/3\}^2) \\ - (1 + 8 \textstyle\sum_{k\ge 2}(P\{Y_1 > X_1,\ Y_k > X_k\} - \tfrac{1}{4}))| = O(n^{-1/2}) + 3\varepsilon. \end{aligned}$$

This implies (5.32) and hence (5.30). Thus, the proof of Proposition 4.1 is complete.

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