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ASYMPTOTIC BEHAVIOR OF TWO-SAMPLE RANK TESTS IN THE PRESENCE OF RANDOM CENSORING[1]

By Sue Leurgans

University of Wisconsin–Madison

Two samples, $\{X_{ji}, 1 \le i \le n(j)\}$ $(j = 1, 2)$ are assumed to be composed of iid random variables with survival functions $(1 - F_j)(1 - H_j)$, where H is the cdf of the "censoring times" and F is the cdf of the "true lifetimes." A unified derivation of the Pitman efficiencies of a class of rank statistics for censored samples is presented. The conditions under which the result holds do not require contiguous alternatives, since convergence to normality is shown to hold uniformly in equicontinuous (F_1, F_2, H_1, H_2) with bounded hazard rates. The uniformity is obtained by studying a convenient joint representation of several counting processes. The results are applied to the translated exponential distributions, a noncontiguous family of alternatives.

1. Introduction. Spurred by practical problems in clinical trials, many statisticians have proposed two-sample rank tests which can accommodate right-censoring. This paper studies the large sample behavior of a large class of linear rank tests for censored data under fixed alternatives. As one application, Pitman efficiencies of these tests under general, not necessarily contiguous, alternatives are derived.

Efficiency calculations summarize the behavior of sequences of statistics S_n and T_n under a sequence of alternatives indexed by $\theta(n) = \theta_0 + cn^{-1/2} + o(n^{-1/2})$. If, under the sequence of alternatives, $n^{1/2}(T_n - \mu(\theta(n)))/\sigma(\theta(n))$ is asymptotically standard normal and the functions μ and σ are, respectively, continuously differentiable and continuous at θ_0, then the efficacy of the sequence T_n at θ_0 is given by the limit of $(\mu'(\theta_0)/\sigma(\theta_0))^2$. (Some refer to the square root of this quantity as the efficacy of T_n.) Consistent estimation of $\sigma(\theta_0)$ does not change the efficacy of T_n. The Pitman relative efficiency of the sequence T_n, with respect to the sequence S_n, is then the ratio of the efficacy of T_n to the efficacy of S_n. The asymptotic normality of the statistics under the sequence of alternatives is usually obtained from LeCam's third lemma, which implies that the joint asymptotic normality under θ_0 of the sequence of statistics and the log likelihood ratio of $\theta_{(n)}$ to θ_0, if the ratio of the limiting mean of the log likelihood ratio to the limiting variance is -2, is a sufficient condition for the asymptotic normality of the sequence of test statistics under the sequence of alternatives. (See Hájek and Sidák, 1967, page 208.)

For noncontiguous alternatives (such as the translated exponentials of Section 5) the log likelihood ratio can fail to be asymptotically normal. (For the translated

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exponentials, the log likelihood ratio is $n\theta(n)$, which diverges for the usual sequences of alternatives.) However, the sequence of centered and scaled test statistics may nonetheless converge weakly to a normal distribution under every fixed alternative. Noether (1955) proved that in this case it suffices, but is not necessary, to show that the weak convergence of $n^{1/2}(T_n - \mu(\theta))/\sigma(\theta)$ is uniform in θ for θ in a neighborhood of θ_0. Section 3 contains a proof of the uniformity of the convergence in distribution of appropriately centered and scaled statistics, since this uniformity is of independent interest.

The proof in Section 3 is based on the construction of a convenient version of the sequences of statistics. This construction and some deterministic lemmas are presented in Section 2. In the fourth section, the theorems of Section 3 are applied to local sequences and formulas for Pitman efficiencies are derived. The formulas are applied in Section 5 to particular tests to verify that this approach gives known efficiencies and to obtain efficiencies not heretofore calculated.

This section concludes with a description of the details of the testing problem and the class of statistics to be considered.

Suppose that the true lifetimes $Y_{1,1}, \cdots, Y_{1,n(1)}$ form a sample from F_1 and that $Y_{2,1}, \cdots, Y_{2,n(2)}$ form an independent sample from F_2. The null hypothesis being entertained is that F_1 and F_2 are equal. If the lifetimes are randomly censored, then $C_{1,1}, \cdots, C_{1,n(1)}$, and $C_{2,1}, \cdots, C_{2,n(2)}$ will be samples independent of the Y's, from censoring distributions H_1 and H_2, respectively. The two censoring distributions can differ. The observed lifetimes $X_{ji} = \min(Y_{ji}, C_{ji})$ and the indicators $\delta_{ji} = [X_{ji} = Y_{ji}]$ are the variables on which test statistics must be based. (Throughout this paper, the indicator function of the set A will be denoted by [A].) Define the random functions $R^*_{j,n(j)}(t) = \sum_{i=1}^{n(j)} [X_{ji} \geq t]$, the number of individuals in the jth population "at risk at time t," and $D^*_{j,n(j)}(t) = \sum_{i=1}^{n(j)} [X_{ji} \leq t][\delta_{ji} = 1]$, the number of "deaths" in the jth population before t. The second subscript of $R^*_{j,n(j)}$ and $D^*_{j,n(j)}$ will be omitted when no confusion will result. Set $R^*_+(t) = R^*_1(t) + R^*_2(t)$, $D^*_+(t) = D^*_1(t) + D^*_2(t)$ and $n = n(1) + n(2)$. The cumulative hazard functions $\Lambda_j(t) = -\ln(1 - F_j(t))$ are estimated by

$$\hat{\Lambda}^*_j(t) = \int^t \frac{dD^*_j(s)}{R^*_j(s)} = \sum_{i=1}^{n(j)} \frac{[X_{ji} \leq t][\delta_{ji} = 1]}{R^*_j(X_{ji})}.$$

The cumulative hazards of the censoring distributions will be $\Lambda_{j+2}(t) = -\ln(1 - H_j(t))$. The derivative of Λ at t will be denoted by λ.

The class of statistics

$$T^*_n = \int K^*_n(s)[d\hat{\Lambda}^*_1(s) - d\hat{\Lambda}^*_2(s)], \tag{1.1}$$

where $K^*_n(t)$ is a possibly random function determined by $\{D^*_j(s), R^*_j(s), s \leq t, j = 1, 2\}$, includes many of the standard censored data rank tests. This class has been studied by Gill (1980) and the related k-sample tests are discussed in Andersen, Borgan, Gill and Keiding (1982). This class of statistics includes the two-sample form of the linear rank tests proposed in Prentice (1978), if a natural condition is imposed on the weights. (See Prentice and Marek, 1979, expression

(9).) Mehrotra, Michalek and Mihalko (1982) show that this condition is satisfied by the members of Prentice's class that he proposed as locally most powerful tests. If $K_n R_+^*/(R_1^* R_2^*) = j(R_+^*/n)$, where j is a nonrandom function defined on [0, 1], T_n is a member of the class studied by Tarone and Ware (1977). If $K_n/(R_1^* R_2^*) = L(R_+)$ for some function L, T_n is one of the tests Aalen (1978, page 720) proposed as a linear rank test. If $K_n R_+^*/(R_1^* R_2^*)$ is a power of the pooled product-limit estimator, the resulting test is a member of the class proposed by Harrington and Fleming (1982). Cox (1972) proposed partial likelihood score statistics for testing $\beta = 0$ in the model $\lambda_\beta(t) = e^{\beta j(t)}\lambda_0(t)$, where j is a deterministic function. If K_n is taken to be $jR_1^* R_2^*/R_+^*$, the class (1.1) includes Cox's tests. For specific examples, see Section 5.

2. Notation and representation. This section contains further notation, some assumptions, statements of deterministic lemmas, and the representation of the counting processes to which the deterministic lemmas apply.

The notation introduced above is extended to define $D_{j+2,n(j)}^*(t) = \Sigma[X_{ji} \le t] \cdot [\delta_{ji} = 0]$, a process that counts censored observations. Set $R_{j+2}^* = R_j^*$ and $n(j+2) = n(j)$. If all the random variables are continuous, the probability that no two of the D^*-processes jump simultaneously and that $R_1^*(t) = n(1) - D_1^*(t-) - D_3^*(-)$ is one. The process D_j^* has cumulative intensity process $n(j)\gamma_{j,n(j)}^*$, where

$$\gamma_{j,n(j)}^*(t) = \int_0^t \frac{R_{j,n(j)}^*(s)}{n(j)} \lambda_j(s)\, ds.$$

Therefore, the following processes are orthogonal martingales:

$$M_{j,n(j)}^*(t) = D_{j,n(j)}^*(t) - n(j)\gamma_{j,n(j)}^*(t), \quad j = 1, \cdots, 4.$$

See Aalen (1978) or Gill (1980) for details.

The weight function K_n^* is assumed to be a predictable process such that $K_n^* J_1^* J_2^* = K_n^*$, where $J_{j,n(j)}^*(t) = [R_{j,n(j)}^*(t) > 0]$. With this assumption, K_n^*/R_j^* will be taken to be zero whenever $R_j^* = 0$.

A bar above a distribution function will denote the corresponding survival function. The product limit estimator of the survival function $\bar{F}_j$ will be denoted by $\hat{\bar{F}}_{j,n(j)}$. The following deterministic functions will appear below:

$$\rho_j(t) = P\{X_{ji} \ge t\} = \bar{H}_j(t)\bar{F}_j(t), \qquad \rho_{j+2} = \rho_j, \quad j = 1, 2,$$

$$\gamma_j(t) = \int_0^t \rho_j(s)\lambda_j(s)\, ds, \qquad j = 1, \cdots, 4,$$

$$\tau_j(t) = \int_0^t \frac{\lambda_j(s)}{\rho_j(s)}\, ds = \int_0^t \frac{f_j(s)\, ds}{\bar{F}_j^2(s)\bar{H}_j(s)}, \quad \tau_{j+2} = \tau_j, \quad j = 1, 2.$$

The four distribution functions F_1, F_2, H_1 and H_2 determine the distribution of T_n^*. The class of quadruples of distribution functions under consideration will be denoted by $\mathscr{F}$. The induced set of quadruples of functions $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ will be denoted by Γ.

A convenient version of the processes described above will now be constructed. The use of this version will be indicated by the omission of the asterisk (*). In the manner of Breslow and Crowley (1974) and Pyke and Shorack (1968), the pointwise convergence of the sequence of random variables based on these versions to a proper random variable will imply the weak convergence of the corresponding random variables based on the original processes to the distribution of the limit of the versions.

The version is based on $\{Z_{j,n(j)},\ n(j) \geq 1\}$, four independent sequences of standard Poisson counting processes defined on a common space (Ω, B). Theorem 3.3 of Billingsley (1971) implies that the sequences can be chosen so that the sequence of normalized sample paths

$$G_{j,n(j)}(t) = n_j^{1/2}(Z_{j,n(j)}(n(j)t)/n(j) - t)$$

converges, for every ω in Ω, to independent standard Wiener processes G_j uniformly on bounded intervals as $n(j)$ diverges. Kurtz (1982) shows that processes D_j and R_j can be constructed on this space such that

$$D_{j,n(j)} = Z_{j,n(j)}\{n(j)\gamma_{j,n(j)}(t)\}, \qquad j = 1, \cdots, 4$$

$$(2.1) \qquad R_j(t) = n(j) - D_j(t-) - D_{j+2}(t-); \quad R_{j+2} = R_j, \quad j = 1, 2,$$

$$\gamma_{j,n(j)} = \int_0^t \frac{R_{j,n(j)}(s)}{n(j)} \lambda_j(s)\, ds, \qquad j = 1, \cdots, 4$$

and the vector processes $D_{j,n(j)}$ and $D^*_{j,n(j)}$ have the same distribution. The processes γ_j are predictable, because they are continuous. The versions $D_{j,n(j)}$ are thus randomly time-changed standard Poisson processes, with the same standard Poisson processes for every member of $\mathscr{F}$.

Since normalized statistics based on the version (2.1) will converge pointwise, the deterministic lemmas below will apply.

LEMMA 2.1. *Let $\mathscr{G}$ be a collection of functions g_α on $[0, \infty)$ satisfying* L1. *If the sequences of signed measures $\nu_{n,\alpha}$ satisfy* L2 *and* L3, *then*

$$\lim_{n\to\infty}\sup_\alpha \left| \int_0^\infty g_\alpha\, d\nu_{\alpha,n} - \int_0^\infty g_\alpha\, d\nu_\alpha \right| = 0.$$

L1. *$\mathscr{G}$ is an equicontinuous family of uniformly bounded functions.*

L2. *For each α, the signed measures $\nu_{\alpha,n}$ converge weakly to a continuous signed measure ν_α, and this convergence is uniform in α in the sense that*

$$\lim_{n\to\infty}\sup_\alpha | \nu_{\alpha,n}([0, x]) - \nu_\alpha([0, x]) | = 0$$

for all finite x.

L3. *The total variation of $\nu_{\alpha,n}$ is bounded uniformly in α and n.*

This lemma is an extension of a result of Ranga Rao (1962), given as problem 2.8 of Billingsley (1968).

LEMMA 2.2. *If* $\mathscr{G}$ *is a set of functions satisfying* L1, *if the signed measures* $\nu_{\alpha,n}$ *satisfy* L2 *and* L3, *and if the functions* $g_{\alpha,n}$ *satisfy* L4; *then*

$$\lim_{n\to\infty}\sup_\alpha \left| \int^\infty g_{\alpha,n}\, d\nu_{\alpha,n} - \int_0^\infty g_\alpha\, d\nu_\alpha \right| = 0.$$

L4.

$$\lim_{n\to\infty}\sup_\alpha\sup_{0\le x<\infty} | g_{\alpha,n}(x) - g_\alpha(x) | = 0.$$

Since L3 implies that $\nu_{\alpha,n}([0, \infty))$ is uniformly bounded, this lemma is a trivial consequence of the preceding lemma.

LEMMA 2.3. *If* $\{u_\alpha, \alpha \in \mathscr{A}\}$ *is a uniformly equicontinuous family of monotone functions on* $[0, \infty)$ *such that*

L5. $$\lim_{x\to\infty}\sup_{\alpha\in\mathscr{A}} | u_\alpha(x) - u_\alpha(\infty) | = 0,$$

if $\{u_{\alpha,n}, n \ge 1, \alpha \in \mathscr{A}\}$ *is a family of uniformly bounded monotone functions, and if*

L6. $$\lim_{n\to\infty}\sup_{\alpha\in\mathscr{A}} | u_{\alpha,n}(x) - u_\alpha(x) | = 0 \quad \textit{for every} \quad 0 \le x < \infty,$$

then

$$\lim_{n\to\infty}\sup_{\alpha\in\mathscr{A}}\sup_{0\le x<\infty} | u_{\alpha,n}(x) - u_\alpha(x) | = 0.$$

3. The fundamental theorems. In this section, the limiting behavior of statistics based on the version described above is obtained. Since this section uses the specially constructed versions (2.1), it will be sufficient to establish convergence for these versions. All limits in this section are pointwise for the special versions (2.1). Weak convergence for the original processes will then follow immediately. The first lemma shows that the random time changes converge uniformly to deterministic functions. The main theorem is stated with fairly complicated conditions; the corollaries and lemmas indicate how these conditions can be obtained.

The lemmas are preceded by a list of the conditions which will be imposed. The behavior of the weight function is specified in T1, T1′, T5, and T5′. These conditions are most easily verified by inspection. Corollary 3.1 gives one set of sufficient conditions. Regularity conditions on $\mathscr{F}$ are in T3.

T1. The total version of $n(j)K_n/R_{j,n(j)}$ is bounded uniformly in n and K is a deterministic function such that

$$\lim_{n\to\infty}\sup_{0\le t<\infty} \left| \frac{n(j)K_n(t)}{R_{j,n(j)}(t)} - \frac{K(t)}{\rho_j(t)} \right| = 0.$$

T1′. The total variation of $n(j)K_n/R_{j,n(j)}$ is bounded uniformly in n and in $\mathscr{F}$, and there is a deterministic function K such that

$$\lim_{n\to\infty}\sup_{\mathscr{F}}\sup_{0\le t<\infty} \left| \frac{n(j)K_n(t)}{R_{j,n(j)}(t)} - \frac{K(t)}{\rho_j(t)} \right| = 0.$$

T2. $$\lim_{n\to\infty}(n(j)/n) = p_j \in (0, 1).$$

T3. The hazard functions $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ induced by $\mathscr{F}$ are uniformly bounded and

$$\lim_{t\to\infty}\sup_{\mathscr{F}} \rho_j(t) = 0, \quad j = 1(1)4.$$

T4. The constant μ defined by

$$\mu = \int K(\lambda_1 - \lambda_2)$$

is finite.

T5. There is a function δ satisfying

$$\int |\delta| \, |\lambda_1 - \lambda_2| < \infty$$

and

$$\lim_{n\to\infty}\sup_t \left| \frac{n^{1/2}(K_n(t) - K(t)) - W(t)}{\delta(t)} \right| = 0,$$

where (W, G_1, G_2) is a vector Gaussian process and the process W has (almost surely) continuous paths and its covariance function r satisfies

$$\int r^{1/2}(s, s) \, |\lambda_1(s) - \lambda_2(s)| \, ds < \infty.$$

T5′. The expansion in T5 holds, with

$$\sup_{\mathscr{F}} \int |\delta| \, |\lambda_1 - \lambda_2| \le B < \infty.$$

LEMMA 3.1. *Under the model described in Section* 1, *if* $n(j)$ *diverges to infinity*,

$$\lim_{n\to\infty}\sup_{\mathscr{F}}\sup_{0\le x<\infty} \left| \frac{R_{j,n(j)}(x)}{n(j)} - \rho_j(x) \right| = 0, \quad j = 1, 2 \tag{3.1}$$

and

$$\lim_{n\to\infty}\sup_{0\le x<\infty} | \gamma_{j,n(j)}(x) - \gamma_j(x) | = 0. \tag{3.2}$$

If T3 *also holds, then*

$$\lim_{n\to\infty}\sup_{\mathscr{F}}\sup_{0\le x<\infty} | \gamma_{j,n(j)}(x) - \gamma_j(x) | = 0. \tag{3.3}$$

PROOF. Set $X_{j,n(j)} = R_{j,n(j)}/n(j) - \rho_j$. The lemma will be proven in four steps. First a process $\varepsilon_{j,n(j)}(t)$ will be defined satisfying

$$X_{j,n(j)}(t) + \int_0^t X_{j,n(j)}(s)(\lambda_j(s) + \lambda_{j+2}(s)) \, ds = \varepsilon_{j,n(j)}(t), \quad j = 1, 2.$$

The second step shows that the $\varepsilon_{j,n(j)}$ processes are bounded by random variables

$\varepsilon_{j,n(j)}$ which do not depend on γ and which converge to zero for every element of the sample space. Assertion (3.1) is then established by showing that if ε is a positive number such that

$$\text{(3.4)} \qquad \sup_{\mathscr{I}} \sup_{0 \le t < \infty} | X_{j,n(j)}(t) | > 2\varepsilon,$$

then $\varepsilon_{j,n(j)} > \varepsilon$. Since this inequality can hold for only finitely many $n(j)$, (3.4) can hold for at most finitely many $n(j)$, and so (3.1) holds. The concluding paragraph explains why (3.2) and (3.3) follow.

The definition of $R_{j,n(j)}$ and the representation (2.1) imply that

$$X_{j,n(j)}(t) = 1 - \rho_j(t) - \frac{D_{j,n(j)}(t-)}{n(j)} - \frac{D_{j+2,n(j)}(t-)}{n(j)}$$

$$= \gamma_j(t) + \gamma_{j+2}(t) - \left(\frac{Z_{j,n(j)}(n(j)\gamma_{j,n(j)}(t-))}{n(j)} + \frac{Z_{j+2,n(j)}(n(j)\gamma_{j,n(j)}(t-))}{n(j)} \right)$$

$$j = 1, 2.$$

Adding and subtracting $\gamma_{j,n(j)} + \gamma_{j+2,n(j)}$, rewriting in terms of $G_{j,n(j)}$, and substituting the definitions of $\gamma_{j,n(j)}$ and of γ_j gives

$$X_{j,n(j)} = - \frac{G_{j,n(j)}(\gamma_{j,n(j)}(t-))}{n(j)^{1/2}} - \frac{G_{j+2,n(j)}(\gamma_{j+2,n(j)}(t-))}{n(j)^{1/2}}$$

$$- \int_0^t \left(\frac{R_{j,n(j)}}{n(j)} (s) - \rho_j(s) \right) (\lambda_j(s) + \lambda_{j+2}(s))\, ds, \quad j = 1, 2.$$

Therefore (3.1) holds with

$$\varepsilon_{j,n(j)}(t) = - \frac{(G_{j,n(j)}(\gamma_{j,n(j)}(t-)) + G_{j+2,n(j)}(\gamma_{j+2,n(j)}(t-)))}{n(j)^{1/2}}.$$

The next step is the construction of $\varepsilon_{j,n(j)}$. Because each of the $n(j)$ individuals in the jth sample can die at most once, no sample path of $D_{j,n(j)}$ has more than $n(j)$ jumps and $n(j)\gamma_{j,n(j)}(\infty) \le T_j(n(j))$, the time at which $Z_{j,n(j)}$ hits $n(j)$.

Therefore $| \varepsilon_{j,n(j)}(t) | \le \varepsilon_{j,n(j)}$, where

$$\varepsilon_{j,n(j)} = \sup_{0 \le x \le T_j(n(j))/n(j)} \frac{| G_{j,n(j)}(x) + G_{j+2,n(j)}(x) |}{n(j)^{1/2}}.$$

The assumption that $G_{j,n(j)}(x) = G_j(x) + o(1)$ on compact intervals as $n(j)$ increases and the relation $Z_{j,n(j)}(n(j)t)/n(j) = n(j)^{-1/2}G_{j,n(j)}(t) + t$ imply that for every element of the sample space $1 = T_j(n(j))/n(j) + n(j)^{-1/2}G_j(T_j(n(j))/n(j)) + o(n(j)^{-1/2})$. Since the second two terms become small compared to the first term, the limit of $T_j(n(j))/n(j)$ must exist and equal one. (The ordinary strong law of large numbers does not apply to $T_j(n(j))/n(j)$, because of the special choice of Poisson processes $Z_{j,n(j)}$.) Because of the uniform convergence of $G_{j,n(j)}$

to G_j, it follows that

$$\limsup_{n(j)\to\infty} n(j)^{1/2}\varepsilon_{j,n(j)} \le \sup_{0\le x\le 1} |G_j(x) + G_{j+2}(x)|, \quad j = 1, 2,$$

and that $\varepsilon_{j,n(j)}$ converges to zero.

Now take $\varepsilon > 0$ and suppose that for some $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ there is a t such that $X_{j,n(j)}(t) > 2\varepsilon$. Because $X_{j,n(j)}$ is continuous except for jumps of size $-1/n(j)$, there is a random variable $T_{j,n(j)} < t$ such that $X_{j,n(j)}(T_{j,n(j)}) \le \varepsilon$ and $X_{j,n(j)}(s) > \varepsilon$ for s in $(T_{j,n(j)}, t]$. It follows from (3.1) that

$$X_{j,n(j)}(t) - X_{j,n(j)}(T_{j,n(j)}) + \int_{T_n}^{t} X_{j,n(j)}(s)(\lambda_j(s) + \lambda_{j+2}(s))\, ds$$
$$= \varepsilon_{j,n(j)}(t) - \varepsilon_{j,n(j)}(T_{j,n(j)}) < \varepsilon_{j,n(j)}.$$

The definition of $T_{j,n(j)}$ implies that the left-hand side of the equation above is greater than $\varepsilon(1 + \Lambda_j(t) + \Lambda_{j+2}(t) - \Lambda_j(T_{j,n(j)}) - \Lambda_{j+2}(T_{j,n(j)}))$ and hence that $\varepsilon_{j,n(j)} > \varepsilon$. A similar argument also gives $\varepsilon_{j,n(j)} < \varepsilon$ if $X_{j,n(j)}(t) < -2\varepsilon$. As explained above, (3.1) follows.

It remains to establish (3.2) and (3.3). If $\Lambda_j(t) < \infty$, then $|\gamma_{j,n(j)}(t) - \gamma_j(t)| \le \varepsilon_{j,n(j)}\Lambda_j(t)$, and hence converges to zero as $n(j)$ diverges. Since both $\gamma_{j,n(j)}$ and γ_j are continuous bounded monotone functions, $\gamma_{j,n(j)} - \gamma_j$ converges to zero uniformly on the closure of the set of t's such that $\Lambda_j(t) < \infty$. If $\Lambda_j(t) = \infty$, both R_j and ρ_j are zero. Consequently $\gamma_{j,n(j)}$ and γ_j are constant on this set, and (3.2) follows. If T3 holds, the uniform bound on the hazard rates implies that the functions γ_j are uniformly equicontinuous and that the $\gamma_{j,n(j)}$'s and the γ_j's satisfy L6. The tail condition on the ρ_j's implies that L5 holds for the functions γ_j. Therefore (3.3) follows from Lemma 2.3. □

The random variable

$$\mu_n = \int (K_n - K)(\lambda_1 - \lambda_2)$$

provides a natural random centering for T_n. Theorem 3.1 examines the limit of $n^{1/2}(T_n - \mu_n)$ and Lemma 3.2 gives conditions under which $n^{1/2}(\mu_n - \mu)$ has a tractable limit.

THEOREM 3.1. *If* T1 *and* T2 *hold, then*

$$\lim_{n\to\infty} n^{1/2}(T_n - \mu_n) = p_1^{-1/2} \int G_1(\gamma_1)\, d\left(\frac{K}{\rho_1}\right) + p_2^{-1/2} \int G_2(\gamma_2)\, d\left(\frac{K}{\rho_2}\right). \tag{3.5}$$

If T1′ *and* T3 *also hold, then the convergence* (3.5) *is uniform in* $\mathscr{F}$.

PROOF. As in Gill (1980, page 46),

$$n^{1/2}(T_n - \mu_n) = n^{1/2}\left[\int \frac{K_n}{R_1}\, dM_1 - \int \frac{K_n}{R_2}\, dM_2\right].$$

Integrating by parts (and recalling that $M_1(0) = (K_n/R_n)(\infty) = 0$) and substituting the representation (2.1) gives the following chain of equations:

$$n^{1/2}\int \frac{K_n}{R_1}\,dM_1 = n^{1/2}\int [D_{1,n(1)}(s) - n_{(1)}\gamma_{1,n(1)}(s)]\,d\left[\frac{K_n(s)}{R_{1,n(1)}(s)}\right]$$
$$= \left(\frac{n}{n(1)}\right)^{1/2}\int G_{1,n(1)}(\gamma_{1,n(1)}(s))\,d\left[\frac{n(1)K_n(s)}{R_{1,n(1)}(s)}\right].$$

By Lemma 3.1, $\gamma_{1,n(1)}$ converges (uniformly in s and in Γ, under T3) to the continuous subdistribution function γ_j. Since each sample path of G_j is uniformly continuous on [0, 1], which contains all the intervals $[0, \gamma_j(\infty)]$, $G_{1,n(1)}\circ\gamma_{1,n(1)}$ converges uniformly (in s and in Γ) to $G_1 \circ \gamma_1$. Since T1 (T1′ and T4) ensures that $n(1)K_n/R_{1,n(1)}$ satisfies the conditions on $\nu_{\alpha,n}$ for Lemma 2.1 (2.2),

$$\lim_{n\to\infty} n^{1/2}\int \frac{K_n}{R_{1,n(1)}}\,dM_{1,n(1)} = p_1^{-1/2}\int G_1(\gamma_1)\,d\left(\frac{K}{\rho_1}\right).$$

The convergence of the other integral follows by symmetry.

LEMMA 3.2. *If* T4 *and* T5 *hold, then*

$$\lim_{n\to\infty} n^{1/2}(\mu_n - \mu) = \int W(\lambda_1 - \lambda_2). \tag{3.6}$$

If T5′ *also holds, then the convergence is uniform in* $\mathscr{F}$.

PROOF. The lemma follows immediately from the assumptions, since the integrability condition on the covariance function r implies that the integral (3.6) can be interpreted as an ordinary Lebesgue integral. (See Sections 5.3 and 5.4 of Cramer and Leadbetter, 1967.) □

COROLLARY 3.1. *If* $K_n = Q(R_{1,n(1)}/n_{(1)}, R_{2,n(2)}/n_{(2)})$ *and* Q *satisfies* C1, C2 *and* C3, *then conditions* T1′ *and* T5 *hold. If* C4 *is added, then* T5′ *holds.*

C1. *The two functions* $Q_j(x_1, x_2) = Q(x_1, x_2)/x_j$ $(j = 1, 2)$ *are continuous and have bounded variation on the unit square.*

C2. *A Taylor expansion* $Q(\mathbf{x}) = Q(\mathbf{x}_0) + q^T(\mathbf{x})(\mathbf{x} - \mathbf{x}_0) + o(q^T(\mathbf{x})(\mathbf{x} - \mathbf{x}_0))$ *holds uniformly in the unit square.*

C3. *The vector function* q *is bounded and continuous on the unit square and its component functions satisfy*

$$\int |q_j|\,\rho_j^{1/2}\,|\lambda_1 - \lambda_2| < \infty \quad \textit{and} \quad \int |q_j|\,|\lambda_1 - \lambda_2| < \infty.$$

C4. *The integrals in* C3 *are bounded uniformly in* $\mathscr{F}$.

PROOF. The total variation of $n(j)K_n/R_{j,n(j)}$ is automatically less than the total variation of Q_j, which is uniformly bounded. The uniform convergence to $K/\rho_j = Q_j(\rho_1, \rho_2)$ is a consequence of the continuous mapping theorem, since the functions Q_j are uniformly continuous and $R_{j,n(j)}/n(j)$ converges uniformly to ρ_j. This establishes T1′.

Condition C2 implies that

$$n^{1/2}(K_n - K) = q^T(\rho_1, \rho_2)n^{1/2}\begin{pmatrix} R_{1,n(1)}/n(1) - \rho_1 \\ R_{2,n(2)}/n(2) - \rho_2 \end{pmatrix} + o\left(n^{1/2}q^T(\rho_1, \rho_2)\begin{pmatrix} R_{1,n(1)}/n(1) - \rho_1 \\ R_{2,n(2)}/n(2) - \rho_2 \end{pmatrix}\right).$$

For the version here, $n^{1/2}(R_{j,n(j)}/n_{(j)} - \rho_j)$ converges uniformly in t to the Gaussian process $V_j = p_j^{-1/2}(G_j \circ \gamma_j + G_{j+2} \circ \gamma_{j+2})$. The boundedness of q and the uniformity of the remainder term therefore imply that $n^{1/2}(K_n - K)$ converges uniformly to a Gaussian process $W = p_1^{-1/2}q_1(\rho_1, \rho_2)V_1 + p_2^{-1/2}q_2(\rho_1, \rho_2)V_2$. Since q is continuous and the sample paths of the Brownian motion G are continuous, W has continuous paths. The variance function of W is $p_1 q_1^2 \rho_1(1 - \rho_1) + p_2 q_2^2 \rho_2(1 - \rho_2)$, which is dominated by $q_1^2\rho_1 + q_2^2\rho_2$. Therefore the integral conditions of C3 will imply the remaining part of T5. Clearly C4 will ensure T5′.

The following corollary is an immediate consequence of Theorem 3.1 and Lemma 3.2:

COROLLARY 3.2. *When* T1, T2, T3, T4 *and* T5 *hold,*

$$\lim_{n\to\infty} n^{1/2}(T_n - \mu) = p_1^{-1/2}\int G_1(\gamma_1)\, d\left(\frac{K}{\rho_1}\right) + p_2^{-1/2}\int G_2(\gamma_2)\, d\left(\frac{K}{\rho_2}\right) + \int W(\lambda_1 - \lambda_2).$$

If T2′ *and* T5′ *also hold, the convergence is uniform in* $\mathscr{F}$.

This limit will have a mean zero normal distribution. The variance is difficult to obtain, because the third term is correlated with the other two. When $\lambda_1 = \lambda_2$, however, the simple expressions of Corollary 4.1 apply.

4. Pitman efficiencies and asymptotically distribution-free tests. In this section, the results of Section 3 are applied to local alternatives. The conditions used here result in formulas for Pitman efficiencies which coincide with those of Gill (1980) (formula (5.2.15), page 105) for contiguous alternatives.

The notation for the local alternatives in Sections 4 and 5 differs from the notation used above. The role of the subscript is changing from population number to parameter, and the parameter for the first population will be suppressed. Thus the distribution function of the first sample, heretofore F_1, will be denoted by F, which is presumed to be F_0, a member of the parametric family $\{F_\theta, \theta \in \Theta\}$ to which the distribution function of the second sample, formerly denoted F_2, is restricted. The symbols λ and f denote the hazard rate and density corresponding to F; λ_θ will be the hazard rate for F_θ. The censoring distributions are H and H_θ, with hazard rates η and η_θ. It is *not* assumed that $H = H_0$. Note that ρ will not equal ρ_0 and γ will not equal γ_0 whenever H and H_0 differ.

The following assumptions apply to the local alternatives indexed by $\theta(n)$:

A1. The deterministic quantities below are finite and the function v is continuous at 0.

$$v(\theta) = \int \frac{K_\theta^2}{\rho_\theta} \lambda_\theta, \quad v = \int \frac{K^2}{\rho} \lambda, \quad \mu(\theta(n)) = \int K_{\theta(n)}(\lambda - \lambda_{\theta(n)}).$$

A2. The sequence $\theta(n)$ converges to zero at a rate such that $\rho_{\theta(n)}$ converges to ρ_0 and the random variables μ_n satisfy

$$\lim_{n\to\infty} n^{1/2}(\mu_n - \mu(\theta(n))) = 0.$$

A3. The functions $\{\lambda_\theta, \eta, \eta_\theta, |\theta| < \varepsilon\}$ are uniformly bounded for some positive ε.

Not all contiguous alternatives satisfy Assumption A3, but some noncontiguous sequences of local alternatives do satisfy A3. One such sequence is the exponential location family discussed in Section 5.

COROLLARY 4.1. *Let* $\mathscr{F} = \{(F, H; F_\theta, H_\theta), \theta \in \Theta\}$. *If* A1, A2, A3, T1′ *and* T2 *hold, then*

$$n^{1/2}(T_n^* - \mu(\theta(n))) \to_d N\left(0, \frac{v}{p_1} + \frac{v(0)}{p_2}\right).$$

If, in addition, $H = H_0$, *the limiting variance is* $v/(p_1 p_2)$.

PROOF. The assumption A2 implies that the corollary can be proved by showing that for the special versions based on (2.1),

$$\text{(4.1)} \quad \lim_{n\to\infty} n^{1/2}(T_n - \mu_n) = p_1^{-1/2} \int G_1 \circ \gamma \, d\left(\frac{K}{\rho}\right) + p_2^{-1/2} \int G_2 \circ \gamma_0 \, d\left(\frac{K}{\rho_0}\right).$$

Since G_1 and G_2 are independent Gaussian processes, the limit of (4.1) has a normal distribution with variance $v/p_1 + v(0)/p_2$. Take $\mathscr{F}$ to be $\{H, H_\theta, F_\theta, |\theta| < \varepsilon\}$. The assumption that $\theta(n)$ converges to zero implies that for n sufficiently large, F_θ and H_θ are in $\mathscr{F}$. The rest of A2 implies that T3 holds for $\mathscr{F}$. Since T1′ and T2 are assumed to hold, Theorem 3.1 implies that the following limit is uniform in $\mathscr{F}$:

$$\text{(4.2)} \quad \begin{aligned} &\lim_{n\to\infty} n^{1/2}(T_n - \mu_n(\theta)) \\ &\quad = p_1^{-1/2} \int G_1 \circ \gamma \, d\left(\frac{K_\theta}{\rho}\right) + p_2^{-1/2} \int G_2 \circ \gamma_\theta \, d\left(\frac{K_\theta}{\rho_\theta}\right). \end{aligned}$$

The uniformity of (4.1) implies that the limit of $n^{1/2}(T_n - \mu_n)$ must be the right-hand side of (4.2) with θ replaced by 0. The first conclusion follows. The second conclusion results from the equality of v and $v(0)$ when $H = H_0$.

The following corollary is a consequence of Noether (1955).

COROLLARY 4.2. *Assume* A1, A3, T1′ *and* T2 *hold.*

1. *If* A2 *holds for the sequence of parameters* $\theta(n) = \theta n^{-1/2}$, *then the efficacy of* T_n^* *is*

$$\frac{\mu'(0)}{\sigma(0)} = \frac{\lim_{\theta\downarrow 0} \int K_\theta((\lambda - \lambda_\theta)/\theta)}{\{\int K_0^2\, ((1/(p_1\rho)) + (1/(p_2\rho_0)))\}^{1/2}}.$$

2. *Furthermore, if* $H = H_0$, *if the limit and the integral in the numerator above can be interchanged, and if*

$$\dot{\lambda}(s) = \frac{\partial}{\partial\theta}\lambda_\theta(s)\Big|_{\theta=0}, \quad \textit{then} \quad \frac{\mu'(0)}{\sigma(0)} = \frac{-(p_1p_2)^{1/2}\int K_0\dot{\lambda}}{\{\int (K_0^2\lambda/\rho)\}^{1/2}}.$$

When performing hypothesis tests based on T_n, the null hypothesis variance must be estimated, unless the null hypothesis variance is independent of both F and H. In particular, the variance σ^2 under any pair of censoring distributions must equal the variance under no censoring, which can happen generally only if the integrands in the variance formulas are equal. If κ_0 is the limiting weight function in the absence of censoring and L_0 is the null hypothesis limit of the weight function of the corresponding asymptotically distribution-free test, the following equation must hold:

$$L_0^2\lambda\left[\frac{1}{p_1\rho} + \frac{1}{p_2\rho_0}\right] = (\kappa_0)^2\lambda\left[\frac{1}{p_1\bar{F}} + \frac{1}{p_2\bar{F}}\right].$$

This equation reduces to the following requirement for L_0:

$$L_0 = \kappa_0\left(\frac{\bar{H}\bar{H}_0}{p_1\bar{H} + p_2\bar{H}_0}\right)^{1/2}.$$

If a weight function satisfies this requirement, the null hypothesis variance is

$$\sigma^2 = (p_1p_2)^{-1}\int\left(\frac{K_0}{\bar{F}}\right)^2 f.$$

If κ_0 is a function of F, the value of this integral will be the same for all continuous densities f. The product limit estimators $\hat{\bar{F}}_1$ and $\hat{\bar{F}}_2$ can be used to construct consistent estimators of κ and thus to generate test statistics whose limiting variances are distribution free. Fleming and Harrington (1982) have exploited this idea. The examples in the next section show that the efficacies of such tests depend on the censoring pattern and may be less than the efficacies of other tests whose distributions depend on the censoring. This discussion is summarized by the following corollary:

COROLLARY 4.3. *If* J *is a continuous function on* [0, 1] *such that*

$$V = \int_0^1 \left(\frac{J(x)}{x}\right)^2 dx < \infty$$

and if the weight function of a test statistic of the form (1.1) *is* L_n, *where* L_n *meets* *the conditions of Corollary* 4.2 *and its limiting value* L_θ *satisfies*

$$L_0(t) = J(\bar{F}(t))\left(\frac{\bar{H}(t)\bar{H}_0(t)}{p_1\bar{H}(t) + p_2\bar{H}_0(t)}\right)^{1/2},$$

then $n^{1/2}T_n^*$ *converges weakly under the null hypothesis to a normal distribution with mean* 0 *and variance* $V/(p_1p_2)$.

5. Examples and comments. Formal calculations for extensions of two common two-sample rank tests are given under the familiar exponential scale family ($F_\theta(t) = 1 - \exp(-(1 + \theta)t)$ for t positive) and under the less familiar exponential location family ($F_\theta(t) = 1 - \exp(-(t - \theta))$ for t greater than θ). This latter family is noncontiguous, although covered by the corollaries of the preceding section and of some practical interest. (See Zelen, 1966.) The regularity conditions are easy to check for the location family for all the weight functions considered here. For the scale family, conditions on the censoring distribution are required for some of the tests. However, in all cases, the formulas are correct if the integrals on $[0, \infty)$ in statistics and formulas are replaced by $[0, T]$.

Computation of $\mu'(0)$ is easy in both cases. For the scale family,

$$(\lambda(t) - \lambda_\theta(t))/\theta = 1,$$

and $\mu'(0)$ is the integral of K_0. For the location family,

$$\mu(\theta) = \int_0^\theta K_\theta(t)\, dt.$$

If $K_\theta(t)$ is jointly continuous in t and θ at the origin, $\mu'(0)$ will be $K_0(0)$. The $\theta = 0$ member of each family is the standard exponential distribution, so $\sigma^2(0)$ is the same for both families.

EXAMPLE 5.1 *The logrank test.* The weight function for the logrank test is

$$K_n^{(0)}(t) = \frac{R_1(t)}{n_1}\,\frac{R_2(t)}{n_2}\,\frac{n}{R_+(t)}.$$

In the absence of censoring, this test reduces to Savage's two-sample rank test. This test is closely related to tests proposed by Cochran (1954), Mantel and Haenszel (1959), Peto and Peto (1972) and Cox (1972). Mantel and Haenszel propose a hypergeometric variance estimate, Cochran and Cox suggest a slightly different variance, and Peto and Peto refer to a permutation distribution. The limiting weight function is $K_\theta^{(0)}(t) = \rho(t)\rho_\theta(t)/(p_1\rho(t) + p_2\rho_\theta(t))$, and therefore

$$\mu_0(\theta) = \int \frac{\rho\rho_\theta(\lambda - \lambda_\theta)}{p_1\rho + p_2\rho_\theta}.$$

If the censoring distributions H_θ and H are equal,

$$\mu_0(\theta) = \int \frac{\bar{F}\bar{F}_\theta\bar{H}}{p_1\bar{F} + p_2\bar{F}_\theta}(\lambda - \lambda_\theta)$$

and the null hypothesis variance is

$$\sigma_0^2 = \int \rho\lambda = \int \bar{H}f = P\{Y \leq C\}.$$

The null hypothesis variance is controlled by intensity of the censoring as measured by the probability of events being uncensored. These formula are equivalent to those deduced from Crowley (1974), who discussed the case in which group membership may change once in time.

Since $\kappa_\theta^{(0)}(t) = \bar{F}(t)\bar{F}_\theta(t)/(p_1\bar{F}(t) + p_2\bar{F}_\theta(t))$, a test whose asymptotic variance is distribution-free can be based on the weight function

$$L_n^{(0)}(t) = \frac{\hat{\bar{F}}_1(t)\hat{\bar{F}}_2(t)}{p_1\hat{\bar{F}}_1(t) + p_2\hat{\bar{F}}_2(t)} \left(\frac{\hat{\bar{H}}_1(t)\hat{\bar{H}}_2(t)}{p_1\hat{\bar{H}}_1(t) + p_2\hat{\bar{H}}_2(t)}\right)^{1/2}.$$

If all observations are complete, $L_n^{(0)}$ and $K_n^{(0)}$ are equal and this test is Savage's test. The mean function reduces to

$$\tilde{\mu}_0(\theta) = \int \frac{\bar{F}\bar{F}_\theta}{p_1\bar{F} + p_2\bar{F}_\theta} \bar{H}^{1/2}(\lambda - \lambda_\theta).$$

This expression differs from $\mu_0(\theta)$ only in the power of $\bar{H}$. Since Corollary 4.3 applies with $J(u) = 1$, $\tilde{\sigma}_0^2$ is one. The location family efficacies are inversely proportional to the reciprocals of the limiting variance, because $K_n^{(0)}(0) = L_n^{(0)}(0) = 1$. Table 5.1 gives the formulas for the two families and equal censoring.

Jensen's inequality implies that the standard logrank test is more efficacious for the proportional hazards, or exponential scale family, than the Savage test

TABLE 5.1
Savage Tests.

	Logrank Test	**Asymptotically Distribution-Free Test**
K_θ	$\dfrac{\bar{F}\bar{F}_\theta\bar{H}}{p_1\bar{F} + p_2\bar{F}_\theta}$	$\dfrac{\bar{F}\bar{F}_\theta\bar{H}^{1/2}}{p_1\bar{F} + p_2\bar{F}_\theta}$
$\sigma^2(0) = \int K_0^2 \frac{\lambda}{\rho}$	$\int \bar{H}f$	$\int f = 1$
Scale Family		
$\mu'(0)$	$\int_0^\infty \bar{H}(t)e^{-t}\,dt$	$\int_0^\infty \bar{H}^{1/2}(t)e^{-t}\,dt$
$(p_1p_2)^{-1/2}$ Efficacy	$\left(\int_0^\infty \bar{H}(t)e^{-t}\,dt\right)^{1/2}$	$\int_0^\infty \bar{H}^{1/2}(t)e^{-t}\,dt$
Location Family		
$\mu'(0)$	1	1
$(p_1p_2)^{-1/2}$ Efficacy	$\left(\int_0^\infty \bar{H}(t)e^{-t}\,dt\right)^{-1/2}$	1

with variance free of censoring. This comparison is expected, since Peto (1972), Prentice (1978) and Gill (1980) have argued that the logrank test is the most powerful rank test for this family. For the exponential location family, the efficacy comparison is similar.

EXAMPLE 5.2 *Generalized Wilcoxon tests.* Four extensions of the Wilcoxon test are described. The asymptotic variance of the fourth extension is distribution-free. Each test is discussed in turn, and the formulas for equal censoring are collected in Table 5.2.

1. The first extension of the Wilcoxon test to allow censored observations was that proposed by Gehan (1965) and Gilbert (1962). Tarone and Ware (1975) show that Gehan's test can be thought of as a weighted logrank test, with the weight at each event time being proportional to the number at risk. In the present notation, the weight function is

$$K_n^{(1)} = \frac{R_+}{n} K_n^{(0)} = \frac{R_1}{n_1} \frac{R_2}{n_2}.$$

This test reduces to the Wilcoxon test if none of the observations are censored. These formulas are equivalent to those of Gehan (1965) for the exponential scale family and those of Breslow (1970) reduced from k samples to 2 samples.

2. Peto and Peto (1972) derived another extension of the Wilcoxon test. This latter extension was also obtained by Prentice (1978). (As with generalized Savage tests, there are several variants, depending on how the variability of the test is assessed.) Prentice and Marek (1979) show that this statistic is also a weighted logrank test, with the weight at each time proportional to the value of the pooled product-limit estimator at that time:

$$K_n^{(2)} = K_n^{(0)} \hat{\bar{F}}_p.$$

Heuristically, this weight function is the multiple of the logrank weight function and the estimated proportion of the sample that would be under observation in the absence of censoring, rather than the actual proportion under observation, the factor used in $K_n^{(1)}$. Since the pooled product limit estimator is equal to R_+/n if no times are censored, this statistic reduces to the Wilcoxon statistic if no observations are censored. When the censoring distributions differ and $\theta \neq 0$, the pooled product limit estimator does not converge to $p_1\bar{F} + p_2\bar{F}$.

3. A third extension of the Wilcoxon test was proposed by Efron (1967). The weight function for Efron's test replaces R_j in Gehan's weight function with the corresponding product limit estimator $\bar{F}_j$:

$$K_n^{(3)} = \hat{\bar{F}}_{1,n(1)} \hat{\bar{F}}_{2,n(2)} J_{1,n(1)} J_{2,n(2)}.$$

The unstable behavior of the test under heavy censoring, as pointed out by Efron, is apparent in the presence of $\bar{H}$ in the denominator of the variance expression.

TABLE 5.2
Wilcoxon Tests.

	Gehan's Test	Peto and Peto's Test	Efron's Test	Asymptotically Distribution-Free Test
K_θ	$\bar{H}^2\bar{F}\bar{F}_\theta$	$\bar{H}\bar{F}\bar{F}_\theta$	$\bar{F}\bar{F}_\theta$	$\bar{F}\bar{F}_\theta\bar{H}^{1/2}$
$\sigma^2(0) = \int K_0^2 \frac{\lambda}{\rho}$	$\int \bar{H}^3\bar{F}^2 f$	$\int \bar{H}\bar{F}^2 f$	$\int \bar{H}^{-1}\bar{F}^2 f$	$\int -\bar{F}^2 f = \frac{1}{3}$
Scale Family				
$\mu'(0)$	$\int \bar{H}^2(t)e^{-2t}\,dt$	$\int \bar{H}(t)e^{-2t}\,dt$	$\frac{1}{2}$	$\int \bar{H}^{1/2}(t)e^{-2t}\,dt$
$(p_1p_2)^{-1/2}$ Efficacy	$\frac{\int \bar{H}^2(t)e^{-2t}\,dt}{(\int \bar{H}^3(t)e^{-3t}\,dt)^{1/2}}$	$\frac{\int \bar{H}(t)e^{-2t}\,dt}{(\int \bar{H}(t)e^{-3t}\,dt)^{1/2}}$	$\left(4\int \bar{H}^{-1}(t)e^{-3t}\,dt\right)^{-1/2}$	$3^{1/2}\int \bar{H}^{1/2}(t)e^{-2t}\,dt$
Location Family				
$\mu'(0)$	1	1	1	1
$(p_1p_2)^{-1/2}$ Efficacy	$\left(\int \bar{H}^3(t)e^{-3t}\,dt\right)^{-1/2}$	$\left(\int \bar{H}e^{-3t}\,dt\right)^{-1/2}$	$\left(\int \bar{H}^{-1}(t)e^{-3t}\,dt\right)^{-1/2}$	$3^{1/2}$

4. Asymptotically distribution-free extensions of the Wilcoxon test are possible. One such test uses

$$L_n = \hat{F}_1\hat{F}_2\left(\frac{\hat{H}_1\hat{H}_2}{p_1\hat{H}_1 + p_2\hat{H}_2}\right)^{1/2}.$$

The tables show that the order of the efficacies depends on the alternative. For the location family, if H is not identically 1, Gehan's test is most efficacious, followed in order by Peto and Peto's test, the logrank test and the asymptotically distribution-free logrank test. Efron's test is always less efficacious than the asymptotically distribution-free Wilcoxon, which is in turn always less efficacious than Peto and Peto's test. For the scale family, the logrank test is always most efficacious and Peto and Peto's test is always more efficacious than Gehan's test. This reverses the ordering of these three tests under location alternatives. Other comparisons depend on the censoring distribution.

In the absence of contiguity, tests of the form (1.1) may not have the best possible asymptotic behavior. For the exponential location family, an extension of Rosenbaum's (1954) test is more efficient. Such a test is based on the rank in the pooled deaths of the first death in the second sample. The asymptotic distribution of this test statistic is geometric, and the power of the test converges to 1 if $n\theta(n)$ diverges.

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DEPARTMENT OF STATISTICS
UNIVERSITY OF WISCONSIN, MADISON
1210 WEST DAYTON ST.
MADISON, WISCONSIN 53706