The Annals of Statistics
1984, Vol. 12, No. 1, 358–365

COMPARISON OF LINEAR EXPERIMENTS WITH KNOWN COVARIANCES[1]

BY CZESŁAW STĘPNIAK, SONG-GUI WANG[2] AND C. F. JEFF WU[3]

Agricultural University, Lublin; Chinese University of Science and Technology, Hefei; and University of Wisconsin, Madison

For two linear experiments $d_1 = L(X_1\beta, V_1)$ and $d_2 = L(X_2\beta, V_2)$ where the covariances V_1 and V_2 are known and can be singular or nonsingular, we characterize the following relations: d_1 at least as good as d_2, d_1 better than d_2, and d_1 equivalent to d_2. Sometimes only a subset of parameters is of interest to the experimenter. We extend the above relations between d_1 and d_2 to estimation of a common subset of parameters and give analogous characterizations. Three examples are given.

1. Introduction. A linear experiment with known covariances, denoted by $L(X\beta, V)$, is represented by

$$y = X\beta + \varepsilon, \quad E(\varepsilon) = 0, \quad \mathrm{Var}(\varepsilon) = V,$$

where y is an $n \times 1$ random vector of observations, X is an $n \times p$ (design) matrix, β is a $p \times 1$ vector of parameters of interest, and ε is an $n \times 1$ random vector of errors with mean 0 and covariance matrix V(singular or nonsingular). Ehrenfeld (1955) defined that $d_1 = L(X_1\beta, V_1)$ is at least as good as $d_2 = L(X_2\beta, V_2)$, denoted by $d_1 \geq d_2$, iff for any $c'\beta$ estimable in d_2 it is also estimable in d_1 and $\mathrm{Var}(c'\hat{\beta}_1) \leq \mathrm{Var}(c'\hat{\beta}_2)$ for all such c, where $\hat{\beta}_i$ is the best linear unbiased estimator (BLUE) of β under d_i. She proved that $d_1 \geq d_2$ if $X_1'V_1^{-1}X_1 - X_2'V_2^{-1}X_2$ is nonnegative definite when V_1 and V_2 are nonsingular. Subsequent results were given by Kiefer (1959). Comparison of linear experiments was also considered by Hansen and Torgersen (1974) and Stepniak and Torgersen (1981), using more general concepts like risk function, statistical decision rule, etc. For known covariance, this more general comparison of linear experiments is equivalent to the previous one in terms of performance of linear estimation. For comparison of general statistical experiments, see Goel and DeGroot (1979) and the review paper of Torgersen (1976).

In Theorem 1 of Section 2 we extend Ehrenfeld's result to linear experiments where V_1 and V_2 can be singular or nonsingular. We then consider in what ways experiment d_1 is *strictly* better than experiment d_2 and characterize such an ordering relation in Theorem 2. A definition of equivalence of experiments is considered and a characterization is given in Theorem 3. An alternative characterization of $d_1 \geq d_2$ is given in Theorem 4.

Received December 1981; revised September 1983.

[1] This joint work was stimulated by the earlier work of the first author, and is based on the papers Stepniak (1982) and Wang and Wu (1981) cited in the References.

[2] Work done while visiting at the Dept. of Statistics, Univ. of Wisconsin, Madison.

[3] Research supported by the National Science Foundation Grant No. MCS-7901846.

AMS 1980 *subject classifications.* Primary 62J05; secondary 62B15, 62J10.

Key words and phrases. Comparison of experiments, linear estimation, g-inverse, block design.

Two experiments can be compared in terms of their performances in estimating a subset of parameters. For example, treatment effects in a block design are of more interest to the experimenter than block effects. Block design d_1 is said to be at least as good as block design d_2 if any (estimable) treatment contrast can be estimated at least as precisely under d_1 as under d_2. A precursor is Kiefer (1959). In Section 3 we extend all the results of Section 2 to the situation where estimation of a subset of parameters is of interest. In Section 4 we give three examples to illustrate the main results of the paper.

2. Comparison of linear experiments with known covariances. We first state two lemmas. For a matrix A denote its column space by $\mathscr{M}(A)$ and any g-inverse of A by A^-, i.e. $AA^-A = A$.

LEMMA 1. *For any symmetric nonnegative definite* (n.n.d.) *matrix* A,

$$\sup\left\{\frac{(z'y)^2}{z'A^-z}: z \neq 0,\ z \in \mathscr{M}(A)\right\} = y'Ay,$$

and equality attains when $z = kAy$, $k \neq 0$.

PROOF. From the Cauchy-Schwarz inequality, we have $(x'Ay)^2 \leq (x'Ax)(y'Ay)$. By taking $z = Ax$ and the definition of g-inverse, the result is proved. □

Note that $z'A^-z$ is independent of the choice of A^- since $z \in \mathscr{M}(A)$. In the case of nonsingular A, Lemma 1 was used fruitfully in another context (Wu, 1980a).

For any two n.n.d. matrices A and B, $A \geq B$ means $A - B$ is n.n.d. The following lemma provides the key tool of the paper.

LEMMA 2. *For any two* n.n.d. $k \times k$ *matrices* Q_1 *and* Q_2, $Q_1 \geq Q_2$ iff

(i) $\mathscr{M}(Q_2) \subset \mathscr{M}(Q_1)$,

(ii) $v'Q_1^- v \leq v'Q_2^- v$ *for any* $v \in \mathscr{M}(Q_2)$, *where* Q_i^- *is a* g-*inverse of* Q_i.

PROOF. "*Necessity*". (i) is obvious. To prove (ii), note that the expressions in (ii) are independent of the choice of g-inverse Q_i^-. Since $Q_1 \geq Q_2$, from Theorem 5 of Wu (1980b), there exist a pair of g-inverses Q_1^- and Q_2^- such that $Q_1^- \leq Q_2^-$, thus proving (ii). "*Sufficiency*". We want to prove $y'Q_1y \geq y'Q_2y$ for any $k \times 1$ vector y. From Lemma 1, $y'Q_iy = \sup\{(z'y)^2/z'Q_i^-z: z \neq 0, z \in \mathscr{M}(Q_i)\}$. For each $z \in \mathscr{M}(Q_2) \subset \mathscr{M}(Q_1)$, we have $z'Q_1^-z \leq z'Q_2^-z$, which together with (i) proves the result. □

We are now ready to prove the main result of the paper.

THEOREM 1. *For two linear experiments* $d_1 = L(X_1\beta, V_1)$ *and* $d_2 = L(X_2\beta, V_2)$, $d_1 \geq d_2$ iff

(1) $X_1'(V_1 + kX_1X_1')^-X_1 \geq X_2'(V_2 + kX_2X_2')^-X_2$ *for any scalar* $k > 0$.

PROOF. According to Rao (1973, page 300), a BLUE of an estimable function $c'\beta$ under d_i is given by $c'\hat{\beta}_i$ with $\hat{\beta}_i = (X_i' T_i^- X_i)^- X_i' T_i^- y_i$, where y_i is a realization of d_i, $T_i = V_i + kX_iX_i'$, and k is any positive scalar. The variance of $c'\hat{\beta}_i$ under d_i is

$$\mathrm{Var}(c'\hat{\beta}_i) = c'(X_i' T_i^- X_i)^- c - kc'c. \tag{2}$$

From the definition and (2), $d_1 \geq d_2$ iff

$$\mathscr{M}(X_2') \subset \mathscr{M}(X_1') \tag{3}$$

and

$$c'(X_1' T_1^- X_1)^- c \leq c'(X_2' T_2^- X_2)^- c \textit{ for any } c \in \mathscr{M}(X_2'). \tag{4}$$

Since $\mathscr{M}(X_i) \subset \mathscr{M}(T_i)$ from the definition of T_i, $X_i' T_i^- X_i$ is independent of the g-inverse T_i^- and is therefore n.n.d.

It remains to prove (1) is equivalent to (3) and (4). From Lemma 2, (1) is equivalent to

$$\mathscr{M}(X_2' T_2^- X_2) \subset \mathscr{M}(X_1' T_1^- X_1) \tag{3$'$}$$

and

$$c'(X_1' T_1^- X_1)^- c \leq c'(X_2' T_2^- X_2)^- c \text{ for any } c \in \mathscr{M}(X_2' T_2^- X_2). \tag{4$'$}$$

To prove (3) and (4) are equivalent to (3)$'$ and (4)$'$, it remains to prove $\mathscr{M}(X_i' T_i^- X_i) = \mathscr{M}(X_i')$, $i = 1, 2$, which follows from Rao (1973, page 300). □

REMARK. Condition (1) can be replaced by a more general one

$$X_1'(V_1 + X_1 U X_1')^- X_1 \geq X_2'(V_2 + X_2 U X_2')^- X_2, \tag{1$'$}$$

where U is any symmetric matrix satisfying, for $i = 1, 2$,

$$\mathrm{rank}(V_i + X_i U X_i') = \mathrm{rank}(V_i : X_i) \tag{5a}$$

and

$$V_i + X_i U X_i' \text{ are n.n.d.} \tag{5b}$$

This more general version of Theorem 1 was originally given in Wang and Wu (1981). It can be proved in exactly the same way except that Example 30 of Rao (1971, page 77) is used instead. Condition (1)$'$ was independently conjectured by D. A. Harville.

It is easy to see that $U = 0$ satisfies (5) when $\mathscr{M}(X_i) \subset \mathscr{M}(V_i)$ or V_i is nonsingular, $i = 1, 2$. With this remark the following result follows as a special case of Theorem 1.

COROLLARY 1. *Let* $d_i = L(X_i\beta, V_i)$, $i = 1, 2$.
(*a*) *If* $\mathscr{M}(X_i) \subset \mathscr{M}(V_i)$ *for* $i = 1, 2$, $d_1 \geq d_2$ iff $X_1' V_1^- X_1 \geq X_2' V_2^- X_2$.
(b) *If* V_1 *and* V_2 *are nonsingular*, $d_1 \geq d_2$ iff $X_1' V_1^{-1} X_1 \geq V_2' V_2^{-1} X_2$.

We should point out that, if V_1 is nonsingular and V_2 is singular, $U = 0$ does not necessarily satisfy (5) and a condition like $X_1' V_1^{-1} X_1 \geq X_2' T_2^- X_2$ does not characterize $d_1 \geq d_2$. In this case we should use Theorem 1 or its more general version in the above remark.

The "if" part of Corollary 1(b) was proved in Ehrenfeld (1955) and Kiefer (1959). Corollary 1(a) was noted in Remark 2 of Stepniak and Torgersen (1981).

We next investigate in what sense $d_1 = L(X_1\beta, V_1)$ can be strictly better than $d_2 = L(X_2\beta, V_2)$. We say $d_1 > d_2$ iff $d_1 \geq d_2$ and either one of the following holds true:

(a) $\mathscr{M}(X_2') \subsetneq \mathscr{M}(X_1')$, (b) $\mathrm{Var}(c'\hat{\beta}_1) < \mathrm{Var}(c'\hat{\beta}_2)$ for some $c \in \mathscr{M}(X_2')$.

THEOREM 2. $d_1 > d_2$ iff

$$M_1(k) \geq M_2(k) \quad \text{and} \quad M_1(k) \neq M_2(k)$$

for any $k > 0$, *where* $M_i(k) = X_i'(V_i + k X_i X_i')^- X_i$.

Theorem 2 can be proved in the same way as Theorem 1 except that the following variant of Lemma 2 replaces the role of Lemma 2 in the proof:

"$Q_1 \geq Q_2$ and $Q_1 \neq Q_2$ iff conditions (i) and (ii) of Lemma 2 and either one of the following holds true: (iii) $\mathscr{M}(Q_2) \subsetneq \mathscr{M}(Q_1)$, (iv) $v'Q_1^- v < v'Q_2^- v$ for some $v \in \mathscr{M}(Q_2)$."

A natural definition of equivalence of experiments is: d_1 is equivalent to d_2, denoted by $d_1 \simeq d_2$, iff $d_1 \geq d_2$ and $d_2 \geq d_1$.

THEOREM 3. $d_1 \simeq d_2$ iff $M_1(k) = M_2(k)$ *for any* $k > 0$.

This follows trivially from Theorem 1.

As in Corollary 1, when $\mathscr{M}(X_i) \subset \mathscr{M}(V_i)$ or V_i is nonsingular, $i = 1, 2$, we can take $k = 0$ in $M_i(k)$ in Theorems 2 and 3 to simplify conditions.

An alternative characterization of $d_1 \geq d_2$ can be provided via the following lemma.

LEMMA 3. *Let* P *be the orthogonal projection matrix onto* $\mathscr{M}(X_1' V_1^\perp)$, *where* $V_1^\perp$ *is any matrix of maximum rank* s.t. $V_1 V_1^\perp = 0$. *Define the "new" experiments* $\tilde{d}_i = L(X_i(I - P)\beta, V_i)$. *Then* $d_1 \geq d_2$ iff $\tilde{d}_1 \geq \tilde{d}_2$.

PROOF. Let n, m be the numbers of rows of X_1 and X_2. The conditions $d_1 \geq d_2$ and $\tilde{d}_1 \geq \tilde{d}_2$ can be written, respectively, as 1) and 2),

1) for any vector $b \in R^m$, there is a vector $g_b \in R^n$ s.t.

$$g_b' X_1 = b' X_2 \text{ and } g_b' V_1 g_b \leq b' V_2 b, \tag{6}$$

2) for any vector $b \in R^m$, there is a vector $h_b \in R^n$ s.t.

$$h_b' X_1(I - P) = b' X_2(I - P) \text{ and } h_b' V_1 h_b \leq b' V_2 b.$$

By taking $h_b = g_b$, 1) implies 2) trivially. Suppose 2) holds. Then $c = X_1' h_b - X_2' b \in \mathscr{M}(X_1' V_1^\perp)$ from the definition of P. Therefore there exists a vector $\alpha \in R^n$ s.t. $c = X_1' \alpha$ and $V_1 \alpha = 0$. By taking $g_b = h_b - \alpha$ in (6), we obtain 1). □

THEOREM 4. *Let P and $V_i^{\perp}$ be defined in Lemma* 3. *Then* $d_1 \geq d_2$ iff

$$\mathcal{M}(X_2' V_2^{\perp}) \subset \mathcal{M}(X_1' V_1^{\perp}) \tag{7}$$

and

$$(I - P)(X_1' V_1^{-} X_1 - X_2' V_2^{-} X_2)(I - P) \geq 0. \tag{8}$$

PROOF. Writing $\tilde{X}_i = X_i(I - P)$, observe that, as a consequence of the definition of P, $\mathcal{M}(\tilde{X}_1' V_1^{\perp}) = 0$ which implies

$$\mathcal{M}(\tilde{X}_1) \subset \mathcal{M}(V_1). \tag{9}$$

From the definition of P, (7) is equivalent to $(I - P)X_2' V_2^{\perp} = 0$, and hence to

$$\mathcal{M}(\tilde{X}_2) \subset \mathcal{M}(V_2). \tag{10}$$

Suppose $d_1 \geq d_2$. Then (7) holds by noting that $\mathcal{M}(X_i' V_i^{\perp})$ consists of all vectors c such that $c'\beta$ can be estimated unbiasedly with zero variance in d_i. Consider the experiments $\tilde{d}_i = L(\tilde{X}_i\beta, V_i)$. From Lemma 3, $d_1 \geq d_2$ implies $\tilde{d}_1 \geq \tilde{d}_2$. Under (7), (10) holds. Therefore recalling (9) and applying Corollary 1(a) to $\tilde{d}_1$ and $\tilde{d}_2$, we obtain (8). Conversely, (7) and (8) imply (8)–(10), which imply $d_1 \geq d_2$ via Lemma 3 and Corollary 1(a). □

Conditions (7) and (8) are usually not as easy to verify as condition (1) of Theorem 1.

By using a standard method (Rao, 1973, page 544) for re-expressing a multivariate linear model as a univariate linear model, the results of this section can be extended to the comparison of multivariate linear experiments in a straightforward manner.

3. Comparison of linear experiments for estimating a subset of parameters.

A linear experiment ofen involves two kinds of parameters, those of interest to the experimenter and the remaining ones, which are nuisance parameters. In block designs, treatment effects are the parameters of interest and block effects are the nuisance parameters; in factorial designs we may only be interested in the main effects and treat the higher order interactions as nuisance parameters. If two linear experiments involve a common subset of parameters of interest, their comparison should be made in terms of the performance of the BLUE for this subset of parameters. Formally, let $d_1 = L(X_1\beta + Z_1\gamma, V_1)$ and $d_2 = L(X_2\beta + Z_2\delta, V_2)$, where γ and δ may be different sets of parameters. (For example, for comparison of two block designs with different numbers of blocks, the two vectors of block effects are different.) We say that d_1 is at least as good as d_2 for estimating β, denoted $d_1 \geq d_2$ w.r.t. β, iff for any $c'\beta$ estimable in d_2 it is estimable in d_1 and $\mathrm{Var}(c'\hat{\beta}_1) \leq \mathrm{Var}(c'\hat{\beta}_2)$ for all such c, where $\hat{\beta}_i$ is the BLUE of β under d_i (Kiefer, 1959).

THEOREM 5. *For two linear experiments $d_1 = L(X_1\beta + Z_1\gamma, V_1)$ and $d_2 = L(X_2\beta + Z_2\delta, V_2)$* $d_1 \geq d_2$ w.r.t. β iff

$$C_1(k) \geq C_2(k) \tag{11}$$

for any $k > 0$, *where*

$$C_i(k) = X_i' T_i^- X_i - X_i' T_i^- Z_i (Z_i' T_i^- Z_i)^- Z_i' T_i^- X_i$$

and

$$T_i = V_i + k(X_i X_i' + Z_i Z_i'), \quad i = 1, 2.$$

PROOF. If $c'\beta$ is estimable in d_i, there exists a vector α s.t. $c = X_i'\alpha$ and $0 = Z_i'\alpha$, i.e., $c \in \mathscr{M}(X_i' Z_i^\perp)$ where $Z_i^\perp$ is any matrix of maximum rank s.t. $Z_i' Z_i^\perp = 0$. Define $Y_i = [X_i \vdots Z_i]$. Then $\text{Var}(c'\hat{\beta}_i) = (c', 0)[(Y_i' T_i^- Y_i)^- - kI]\binom{c}{0} = c' C_i(k)^- c - kc'c$ by the formula for a g-inverse of a partitioned matrix (Pringle and Rayner, 1971, page 46). From the definition, $d_1 \geq d_2$ w.r.t. β iff $\mathscr{M}(X_2' Z_2^\perp) \subset \mathscr{M}(X_1' Z_1^\perp)$ and $c', C_1(k)^- c \leq c' C_2(k)^- c$ for any $c \in \mathscr{M}(X_2' Z_2^\perp)$. In view of Lemma 2, to show that these two conditions are equivalent to (11), it remains to show $\mathscr{M}(X_i' Z_i^\perp) = \mathscr{M}(C_i(k))$, $i = 1, 2$. For the remaining proof, we drop the subscript i. Since the choice of g-inverse T^- is irrelevant, we choose a symmetric nonsingular T^- and decompose $T^- = B'B$ with B nonsingular. Writing $F = BX$ and $G = BZ$, we have $C(k) = F'F - F'G(G'G)^- G'F = F' P_{\mathscr{M}(G)^\perp} F$, where $P_{\mathscr{M}(G)^\perp}$ is the projection matrix onto the orthogonal complement of $\mathscr{M}(G)$. Therefore $\mathscr{M}(C(k)) = \mathscr{M}(F' P_{\mathscr{M}(G)^\perp}) = \{c\colon c = F'\alpha, G'\alpha = 0 \text{ for some } \alpha\} = \{c\colon c = X'\delta, Z'\delta = 0 \text{ for some } \delta = B'\alpha\} = \mathscr{M}(X' Z^\perp)$, thus completing the proof. □

As in Corollary 1, we can have $k = 0$ in $C_i(k)$ in special cases.

COROLLARY 2. *For* d_1 *and* d_2 *in Theorem* 5,
(a) *if* $\mathscr{M}(X_i \vdots Z_i) \subset \mathscr{M}(V_i)$, $i = 1, 2$, $d_1 \geq d_2$ w.r.t. β iff $C_1(0) \geq C_2(0)$, *where* $C_i(0) = X_i' V_i^- X_i - X_i' V_i^- Z_i (Z_i' V_i^- Z_i)^- Z_i' V_i^- X_i$, $i = 1, 2$;
(b) *if* V_1 *and* V_2 *are nonsingular,* $d_1 \geq d_2$ iff $C_1 \geq C_2$, *where* $C_i = X_i' V_i^{-1} X_i - X_i' V_i^{-1} Z_i (Z_i' V_i^{-1} Z_i)^- Z_i' V_i^{-1} X_i$, $i = 1, 2$.

Definitions of $d_1 > d_2$ w.r.t. β and $d_1 \simeq d_2$ w.r.t. β are obvious. Their characterizations can be readily obtained from Theorems 2 and 3 by replacing $M_i(k)$ by $C_i(k)$. The "if" part of Corollary 2(b) was noted in Kiefer (1959).

4. Examples.

A. *Experiment with an additional constraint on parameters.* Let experiment $d_1 = L(X_1\beta, V_1)$ and experiment $d_2 = d_1$ together with an additional constraint $h'\beta = b$, X_1 is an $n \times p$ matrix. We can rewrite d_2 as $L(X_2\beta, V_2)$ with

$$X_2 = \begin{pmatrix} X_1 \\ h' \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} V_1 & \mathbf{0} \\ \mathbf{0}' & 0 \end{pmatrix}$$

where $\mathbf{0}$ is the $n \times 1$ vector of zeros, and h is a $p \times 1$ vector. Using formula (3.6.8) of Rao and Mitra (1971) for a g-inverse of a partitioned matrix, we can show that $M_2 = M_1 + (I - M_1)h\, h'(I - M_1)/(h'h - h'M_1 h)$ if $M_1 h \neq h$, and $M_2 = M_1$ if $M_1 h = h$, where M_i is $M_i(k)$ with $k = 1$ in Theorem 2. As applications of Theorems

1 to 3, we conclude:
(i) $d_2 \geq d_1$,
(ii) $d_2 > d_1$ iff $M_1 h \neq h$,
(iii) $d_2 \simeq d_1$ iff $M_1 h = h$.

Extension of the previous results to several parameter constraints is straightforward.

We now give a statistical interpretation of result (iii). Let the rank of V_1 be $t \leq n$ and let P be the $n \times (n - t)$ matrix with its column vectors as the eigenvectors of V_1 with zero eigenvalues. Let y be a realization of d_1, i.e. $y = X_1\beta + \varepsilon$ with $E(\varepsilon) = 0$ and $\text{Var}(\varepsilon) = V_1$. Since $P'\varepsilon$ has mean 0 and variance-covariance 0, $P'\varepsilon = 0$ with probability one, which implies that $P'X_1\beta$ is estimated unbiasedly by $P'y$ with zero variance. We want to show that, when $M_1 h = h$, $h'\beta$ can be estimated unbiasedly with zero variance and hence the constraint $h'\beta = b$ is *redundant.* This explains why d_1 is equivalent to d_2. Let $\mathbf{a}' = (\mathbf{0}', h')$, $V_1 = BB'$ and $A = (B \,\vdots\, X_1)$. Then $M_1 h = h$ implies $\mathbf{a}'A'(AA')^{-}A\,\mathbf{a} = \mathbf{a}'\mathbf{a}$. Since $A'(AA')^{-}A$ is a projection matrix, $\mathbf{a} \in \mathcal{M}(A')$, i.e., there exists a vector α s.t. $B'\alpha = 0$ and $X_1'\alpha = h$, or $V_1\alpha = 0$ and $X_1'\alpha = h$, which implies $h \in \mathcal{M}(X_1'P)$, thus completing the proof.

B. *Augmentation of an experiment with an additional run.* Let experiment $d_1 = L(X_1\beta, V_1)$ be defined as in A and $d_2 = d_1$ together with the $(n + 1)$th run $y_{n+1} = x_{n+1}'\beta + \varepsilon_{n+1}$, where x_{n+1} is a $p \times 1$ vector, $E\,\varepsilon_{n+1} = 0$, $\text{Var}(\varepsilon_{n+1}) = \sigma^2 > 0$, and y_{n+1} is independent of the other y's. We can rewrite d_2 as $L(X_2\beta, V_2)$ with

$$X_2 = \begin{pmatrix} X_1 \\ x_{n+1}' \end{pmatrix}, \quad V_2 = \begin{pmatrix} V_1 & \mathbf{0} \\ \mathbf{0}' & \sigma^2 \end{pmatrix}.$$

In computing M_2 (same as $M_2(k)$ with $k = 1$ in Theorem 2), we need to compute a g-inverse of

$$V_2 + X_2X_2' = \begin{pmatrix} V_1 + X_1X_1' & X_1x_{n+1} \\ x_{n+1}'X_1' & x_{n+1}'x_{n+1} \end{pmatrix} = \begin{pmatrix} AA' & A\mathbf{b} \\ \mathbf{b}'A' & \mathbf{b}'\mathbf{b} \end{pmatrix},$$

where $A = (B, 0, X_1)$, $\mathbf{b}' = (0, \sigma, x_{n+1}')$, $V_1 = BB'$. Again, using (3.6.8) of Rao and Mitra (1971) and after some simplifications, we have $M_2 = M_1 + (I - M_1)x_{n+1}x_{n+1}'(I - M_1)/d$, where $d = \sigma^2 + x_{n+1}'x_{n+1} - x_{n+1}'M_1x_{n+1}$ is always positive. As applications of Theorems 1 to 3, we conclude:
(i) $d_2 \geq d_1$,
(ii) $d_2 > d_1$ iff $M_1x_{n+1} \neq x_{n+1}$,
(iii) $d_2 \simeq d_1$ iff $M_1x_{n+1} = x_{n+1}$.

We should point out that the previous results can be readily extended to simultaneous addition of several runs.

C. *Comparison of block designs for estimating treatment contrasts.* A block design consists of b blocks each of size k with v treatments assigned to the bk plots. The usual additivity model specifies that the expectation of an observation on treatment i in block j equals constant + ith treatment effect + jth block effect, and that the bk observations are uncorrelated with common variance σ^2. The C-matrix in Corollary 2(b) (with $V = \sigma^2 I$) is

$$\sigma^{-2}[\mathrm{diag}(r_1, \cdots, r_v) - k^{-1}NN'], \tag{12}$$

where r_i = number of replications of treatment i, $N = [n_{ij}]_{v \times b}$ is the incidence matrix with n_{ij} = number of appearances of treatment i in block j. The C-matrix (12) is called the "reduced information matrix for estimating treatment contrasts". According to Corollary 2(b), block design d_1 (with common variance σ_1^2) is at least as good as block design d_2 (with common variance σ_2^2) for estimating treatment contrasts iff $\sigma_1^{-2}[C\text{-matrix of } d_1] \geq \sigma_2^{-2}[C\text{-matrix of } d_2]$.

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CZESŁAW STEPNICK
INSTITUTE OF APPLIED MATH.
AGRICULTURAL UNIVERSITY
LUBLIN AKADEMICKA 13
20-934 LUBLIN,
POLAND

SONG-GUI WANG
DEPARTMENT OF MATHEMATICS
CHINESE UNIV. OF SCIENCE AND TECHNOLOGY
HEFEI, ANHUI
PEOPLE'S REPUBLIC OF CHINA

C. F. JEFF WU
DEPARTMENT OF STATISTICS
UNIVERSITY OF WISCONSIN, MADISON
1210 WEST DAYTON ST.
MADISON, WISCONSIN 53706