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LOCATION ESTIMATORS AND SPREAD

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In the location estimation problem, translation equivariant estimators are considered. It is shown that under a mild regularity condition the distribution of such estimators is more spread out than a particular distribution which is defined in terms of the sample size and the density of the i.i.d. observations. Some consequences of this so-called spread-inequality are discussed, namely the Cramér-Rao inequality, an asymptotic minimax inequality and the efficiency of the maximum likelihood estimator in some nonregular cases.

1. Introduction and main result. We shall consider one of the classical problems in statistical inference, to wit the estimation of a location parameter. Let $X_1, \cdots, X_n$ be independent and identically distributed random variables with common density $f(\cdot - \theta)$, $\theta \in \mathbb{R}$, with respect to Lebesgue measure on $(\mathbb{R}, \mathscr{B})$. The location parameter θ is estimated by an estimator T_n which is a measurable function t_n: $\mathbb{R}^n \to \mathbb{R}$ of the random variables $X_1, \cdots, X_n$, i.e. $T_n = t_n(X_1, \cdots, X_n)$. We are interested in the distribution of T_n under $f(\cdot - \theta)$.

Our estimation problem is invariant under translation. Hence it is natural to estimate the parameter θ with a translation equivariant estimator whenever we want to be impartial with respect to the possible values which the parameter can adopt. Therefore, we assume that T_n is translation equivariant, i.e. for all real a and Lebesgue almost all $x_1, \cdots, x_n$

$$t_n(x_1 + a, \cdots, x_n + a) = t_n(x_1, \cdots, x_n) + a. \tag{1.1}$$

Because of the translation equivariance of T_n we have

$$P_{f(\cdot - \theta)}(T_n \le x) = P_f(T_n \le x - \theta), \quad x \in \mathbb{R}, \quad \theta \in \mathbb{R}, \tag{1.2}$$

and we see that it suffices to study the distribution of T_n under f, i.e. with $\theta = 0$.

Let a_n be positive. We denote the distribution function of $a_n T_n$ under f by G_n,

$$G_n(x) = P_f(a_n T_n \le x), \quad x \in \mathbb{R}. \tag{1.3}$$

Furthermore, we assume that the density f is absolutely continuous with an integrable Radon-Nikodym derivative f' and we define the distribution function K_n for some $w \in (0, 1)$ by

$$K_n^{-1}(u) = \int_w^u \frac{1}{\int_s^1 H_n^{-1}(t)\, dt}\, ds, \quad 0 < u \le 1, \tag{1.4}$$

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where H_n^{-1} is the inverse distribution function $H_n^{-1}(t) = \inf\{x \mid H_n(x) \geq t\}$ of H_n defined by

$$(1.5) \qquad H_n(x) = P_f(a_n^{-1} \textstyle\sum_{i=1}^n [-(f'/f)(X_i)] \leq x), \quad x \in \mathbb{R}.$$

Note that the distribution function of the score function of K_n equals H_n.

The distribution functions G_n and K_n are related by the fact that any two quantiles of G_n are further apart than the corresponding quantiles of K_n; more precisely

THEOREM 1.1. *If the density f is absolutely continuous with respect to Lebesgue measure with Radon-Nikodym derivative f′ satisfying*

$$(1.6) \qquad \int |f'| < \infty$$

and if T_n *is translation equivariant* (*cf.* (1.1)), *then* G_n *and* K_n *are differentiable with derivatives* g_n *respectively* k_n *satisfying* (*cf.* (1.3), (1.4) *and* (1.5))

$$(1.7) \qquad g_n(G_n^{-1}(s)) \leq k_n(K_n^{-1}(s)) = \int_s^1 H_n^{-1}(t)\, dt, \quad 0 < s < 1.$$

This implies

$$(1.8) \qquad G_n^{-1}(v) - G_n^{-1}(u) \geq K_n^{-1}(v) - K_n^{-1}(u), \quad 0 \leq u \leq v \leq 1.$$

We say that G_n is more spread out than K_n. This concept of spread has been introduced by Bickel and Lehmann (1979). Note that the inequalities (1.7) and (1.8) are insensitive to translations and that hence the choice of $w \in (0, 1)$ is immaterial. The important point in the spread-inequality (1.8) is that K_n is defined in terms of the sample size n and the density f of the observations. Hence K_n does not depend on T_n and consequently Theorem 1.1 gives a uniform upper bound to the accuracy of translation equivariant estimators T_n. Well-known upper bounds to the accuracy of estimators are provided by the Cramér-Rao inequality and by the asymptotic minimax theory of Hájek and Le Cam (see Hájek, 1972 and Le Cam, 1979). Restricted to the location estimation problem we are considering, the Cramér-Rao inequality and the asymptotic inequality of Hájek (1972) are implied by Theorem 1.1. Our spread-inequality also implies that the maximum likelihood estimator is asymptotically efficient in the nonregular location estimation problem of Woodroofe (1972), which has been shown in Weiss and Wolfowitz (1973) for a somewhat different efficiency concept, and that its rate of convergence to its limit distribution is of the right order in the nonregular location estimation problem of Woodroofe (1974), which is suggested by the results of Polfeldt (1970).

These consequences of Theorem 1.1 will be discussed in the next section. Section 3 consists of the proofs.

2. Some consequences of the spread-inequality. From the spread-inequality (1.8), nontrivial lower bounds may be obtained for the risk of translation equivariant location estimators, both for finite sample sizes and asymptotically. Such bounds are presented in the following theorem.

THEOREM 2.1. *Let* $\ell: \mathbb{R} \to \mathbb{R}$ *be a measurable function, which is nonincreasing on* $(-\infty, 0]$ *and nondecreasing on* $[0, \infty)$. *Under the conditions of Theorem* 1.1 *we have*

$$\text{(2.1)} \qquad \inf_{a\in\mathbb{R}} E_f \ell(a_n T_n - a) \geq \inf_{a\in\mathbb{R}} \int_0^1 \ell(K_n^{-1}(u) - a)\, du.$$

Furthermore, if for some distribution function K the sequence $\{K_n\}$ *converges weakly to K as n tends to infinity and if at least one of the following conditions holds*:

$$\text{(2.2)} \qquad \ell \text{ is lower semicontinuous,}$$

$$\text{(2.3)} \qquad \int_A dK = 0 \quad \text{for each countable set} \quad A \subset \mathbb{R},$$

then

$$\text{(2.4)} \qquad \liminf_{n\to\infty} \inf_{a\in\mathbb{R}} E_f \ell(a_n T_n - a) \geq \inf_{a\in\mathbb{R}} \int_0^1 \ell(K^{-1}(u) - a)\, du.$$

Finally, if $G_n \to_w G$ *and* $K_n \to_w K$ *as* $n \to \infty$ *for some distribution functions G and K, then*

$$\text{(2.5)} \qquad G^{-1}(v) - G^{-1}(u) \geq K^{-1}(v) - K^{-1}(u), \quad 0 \leq u \leq v \leq 1.$$

For quadratic loss functions, inequality (2.1) of Theorem 2.1 implies an extension of the Cramér-Rao inequality.

COROLLARY 2.1. (Cramér-Rao inequality). *Under the conditions of Theorem* 1.1.

$$\text{(2.6)} \qquad \operatorname{var}_f a_n T_n \geq \operatorname{var}_{k_n} X.$$

If f has finite Fisher information $I(f) = \int (f'/f)^2 f$, *this implies*

$$\text{(2.7)} \qquad \operatorname{var}_f T_n \geq (nI(f))^{-1}.$$

In the remainder of this section we will discuss three special cases of inequalities (2.4) and (2.5) of Theorem 2.1. The first one is closely related to the result of Hájek (1972) and arises if f has finite Fisher information.

COROLLARY 2.2. *If f has finite Fisher information* $I(f)$ *then for* $a_n = (nI(f))^{1/2}$ *the sequence* $\{K_n\}$ *with* $w = 1/2$ *converges weakly to the standard normal distribution function* Φ *as n tends to infinity and hence* (2.4) *and* (2.5) *hold for* T_n *translation equivariant,* $a_n = (nI(f))^{1/2}$ *and* $K = \Phi$.

Furthermore, we'll consider densities f of the following very special type. Let

$c \in (0, \infty)$. If f satisfies (1.6), f vanishes on $(-\infty, 0]$, $\lim_{x\downarrow 0} f'(x) = c$ and if $\int_\varepsilon^\infty (f'(x)/f(x))^2 f(x)\,dx < \infty$ for all $\varepsilon > 0$, then f will be said to belong to the class $D(c)$. The gamma and Weibull distributions with shape parameter 2 are of this type. We note that $I(f) = \infty$ for all $f \in D(c)$. Nevertheless the following analogue of Corollary 2.2 holds.

COROLLARY 2.3. *If $f \in D(c)$, then for $a_n = (\frac{1}{2} cn \log n)^{1/2}$ the sequence $\{K_n\}$ with $w = \frac{1}{2}$ converges weakly to Φ as n tends to infinity and hence, if T_n is translation equivariant, we have* (2.4) *and* (2.5) *with $a_n = (\frac{1}{2} cn \log n)^{1/2}$ and $K = \Phi$.*

Woodroofe (1972) has shown, under some regularity conditions, that the asymptotic distribution of the maximum likelihood estimator for this case is standard normal if it is normed by $(\frac{1}{2} cn \log n)^{1/2}$. Consequently Corollary 2.3 implies that both the maximum likelihood estimator and the spread-inequality (2.5) are asymptotically efficient in this nonregular case (cf. Weiss and Wolfowitz, 1973).

Finally, we'll consider densities f which behave like $x^{\alpha-1}$, $1 < \alpha < 2$, near the origin.

COROLLARY 2.4. *Let $\alpha \in (1, 2)$, f vanish on $(-\infty, 0]$, f satisfy* (1.6) *and let $f'(x) \sim \alpha(\alpha-1)x^{\alpha-2}L(x)$ as $x \downarrow 0$, where $L(x)$ varies slowly as $x \downarrow 0$. Furthermore let $\int_\varepsilon^\infty (f'(x)/f(x))^2 f(x)\,dx < \infty$ for all $\varepsilon > 0$. If $\{a_n\}$ is such that*

$$\lim_{n\to\infty} n a_n^{-\alpha} L(a_n^{-1}) = 1, \tag{2.8}$$

then $H_n \to_w H$ as $n \to \infty$, where H is a stable distribution function with exponent α and cumulant generating function

$$\psi_H(t) = -d\,|t|^\alpha(1 + i \operatorname{sgn} t(\tan \tfrac{1}{2}\alpha\pi)) \tag{2.9}$$

with

$$d = (\alpha-1)^{\alpha-1}\Gamma(2-\alpha)[-\cos \tfrac{1}{2}\alpha\pi]. \tag{2.10}$$

Furthermore, $K_n \to_w K$ as $n \to \infty$, where K is defined by

$$K^{-1}(u) = \int_w^u \frac{1}{\int_s^1 H^{-1}(t)\,dt}\,ds, \quad 0 < u \le 1, \tag{2.11}$$

and hence, if T_n is translation equivariant, (2.4) *and* (2.5) *hold with a_n and K as in* (2.8) *respectively* (2.11).

In Woodroofe (1974) the asymptotic distribution $\hat{G}$ of the maximum likelihood estimator for this case has been derived under some regularity conditions. This has been done with the norming constants a_n as in (2.8). We infer that the rates of convergence of both the maximum likelihood estimator and the spread lower bound K_n are of the right order, i.e. that the a_n defined by (2.8) are suitable norming constants for the maximum likelihood estimator to attain a limit distribution without mass at infinity and for K_n to have a nondegenerate limit distribution. However, $\hat{G}$ and K are different as can be seen by studying their tails. This is not surprising since in the limit experiment for the situation of Corollary 2.4 the optimal estimators for different loss functions need not be the same (see Theorem VI.6.2 of Ibragimov and Has'minskiĭ (1981) and note that $\tilde{t}_n$ may be chosen translation equivariant there). Consequently no estimator sequence can be asymptotically optimal for all loss functions simultaneously or can attain equality in (2.5).

As a curiosity we mention the following immediate consequence of (1.7)

$$g_n(x) \le \frac{1}{2} \int_0^1 | H_n^{-1}(t) | \, dt, \quad x \in \mathbb{R}. \tag{2.12}$$

With $n = 1$, $a_n = 1$ and $T_1 = X_1$ this reduces to the simple inequality

$$f(x) \le \frac{1}{2} \int | f' |, \quad x \in \mathbb{R}, \tag{2.13}$$

which can easily be proved directly.

In the above we have discussed some consequences of the spread-inequality (1.8). Other consequences of it can be found in Klaassen (1981), which restricts attention to the case of symmetric densities with finite Fisher information. Finite sample results on the tail behavior of the distributions of location estimators have been obtained by Jurečková (1981a, 1981b).

3. Proofs.

PROOF OF THEOREM 1.1. By classical analysis (see the proof of Lemma 3.1 in Klaassen, 1979) we obtain

$$\begin{aligned} \lim_{\theta \to 0} \int \cdots \int_{\mathbb{R}^n} \Big| \theta^{-1}\big(\textstyle\prod_{i=1}^n f(x_i + \theta) - \prod_{i=1}^n f(x_i)\big) \\ - \Big(\textstyle\sum_{j=1}^n \frac{f'}{f}(x_j)\Big) \prod_{i=1}^n f(x_i) \Big| \, dx_1 \cdots dx_n = 0. \end{aligned} \tag{3.1}$$

By the translation equivariance of T_n

$$\begin{aligned} &\theta^{-1}(G_n(y + \theta) - G_n(y)) \\ &= \int \cdots \int_{a_n t_n(x_1, \cdots, x_n) > y} - \theta^{-1}\{\textstyle\prod_{i=1}^n f(x_i + a_n^{-1}\theta) - \prod_{i=1}^n f(x_i)\} \, dx_1 \cdots dx_n \end{aligned} \tag{3.2}$$

holds and it follows from (3.1) that G_n is differentiable with derivative g_n given by

$$g_n(y) = \int \cdots \int_{a_n t_n(x_1, \cdots, x_n) > y} a_n^{-1} \sum_{j=1}^n \Big[-\frac{f'}{f}(x_j) \Big] \prod_{i=1}^n f(x_i) \, dx_1 \cdots dx_n. \tag{3.3}$$

With

$$S_n = a_n^{-1} \textstyle\sum_{j=1}^{n} \left[-\frac{f'}{f} (X_j) \right] \tag{3.4}$$

and $y = G_n^{-1}(s)$, formula (3.3) may be rewritten as (cf. (1.5))

$$g_n(G_n^{-1}(s)) = \int_0^1 H_n^{-1}(t) E_f(1_{(G_n^{-1}(s),\infty)}(a_n T_n) \mid S_n = H_n^{-1}(t))\, dt. \tag{3.5}$$

Furthermore it is easy to verify that

$$1 - s = \int_0^1 E_f(1_{(G_n^{-1}(s),\infty)}(a_n T_n) \mid S_n = H_n^{-1}(t))\, dt. \tag{3.6}$$

Since the integrand in (3.6) takes on values in [0, 1] and since H_n^{-1} is nondecreasing the Neyman-Pearson lemma (cf. Theorem 5(ii) with $m = 1$ of Chapter 3 of Lehmann, 1959) applied to (3.5) and (3.6) yields

$$g_n(G_n^{-1}(s)) \leq \int_s^1 H_n^{-1}(t)\, dt, \quad 0 < s < 1. \tag{3.7}$$

Because H_n is nondegenerate with mean 0, we have for all $s \in (0, 1)$

$$0 < \int_s^1 H_n^{-1}(t)\, dt \leq \frac{1}{2} \int_0^1 | H_n^{-1}(t) |\, dt \leq \frac{1}{2} a_n^{-1} n \int | f' | < \infty.$$

Furthermore, $\int_s^1 H_n^{-1}(t)\, dt$ is concave and hence,

$$\inf_{u \leq s \leq v} \int_s^1 H_n^{-1}(t)\, dt = \min \left\{ \int_u^1 H_n^{-1}(t)\, dt, \int_v^1 H_n^{-1}(t)\, dt \right\}, \quad 0 < u \leq v < 1.$$

Consequently K_n^{-1} is well defined by (1.4) and is differentiable with a positive and finite derivative on (0, 1). Hence K_n is differentiable with a positive and finite derivative k_n on $(K_n^{-1}(0+), K_n^{-1}(1))$ and K_n satisfies

$$x = \int_w^{K_n(x)} \frac{1}{\int_s^1 H_n^{-1}(t)\, dt}\, ds, \quad x \in (K_n^{-1}(0+),\ K_n^{-1}(1)). \tag{3.8}$$

Differentiating (3.8) and combining the result with (3.7), we see that (1.7) holds.

Combining (1.4) and (1.7) we obtain

$$
\begin{aligned}
G_n^{-1}(v) - G_n^{-1}(u) &= \int_{-\infty}^{\infty} 1_{[G_n^{-1}(u), G_n^{-1}(v)]}(x)\, dx \\
&\geq \int_{-\infty}^{\infty} 1_{[G_n^{-1}(u), G_n^{-1}(v)]}(x) \frac{1}{g_n(x)}\, dG_n(x) \\
&= \int_0^1 1_{[G_n^{-1}(u), G_n^{-1}(v)]}(G_n^{-1}(s)) \frac{1}{g_n(G_n^{-1}(s))}\, ds \qquad (3.9)\\
&= \int_u^v \frac{1}{g_n(G_n^{-1}(s))}\, ds \geq \int_u^v \frac{1}{\int_s^1 H_n^{-1}(t)\, dt}\, ds \\
&= K_n^{-1}(v) - K_n^{-1}(u), \quad 0 < u \leq v \leq 1.
\end{aligned}
$$

Hereby (1.8) and the theorem have been proved. □

PROOF OF THEOREM 2.1. If $G_n^{-1}(0+) > -\infty$ and $a < G_n^{-1}(0+)$ hold, we have $G_n^{-1}(v) - a \geq K_n^{-1}(v) - K_n^{-1}(0+) + G_n^{-1}(0+) - a$, $v \in (0, 1)$, in view of (1.8). If $G_n^{-1}(1) < \infty$ and $G_n^{-1}(1) < a$ hold, we have $G_n^{-1}(v) - a \leq K_n^{-1}(v) - K_n^{-1}(1) + G_n^{-1}(1) - a$, $v \in (0, 1)$. For every $a \in [G_n^{-1}(0+), G_n^{-1}(1)]$ there exist $\alpha, u \in [0, 1]$ such that $G_n^{-1}(v) - a = \alpha[G_n^{-1}(v) - G_n^{-1}(u)] + (1 - \alpha)[G_n^{-1}(v) - G_n^{-1}(u+)]$, which implies $G_n^{-1}(v) - a \leq K_n^{-1}(v) - K_n^{-1}(u)$, $v \in (0, u)$, and $G_n^{-1}(v) - a \geq K_n^{-1}(v) - K_n^{-1}(u)$, $v \in (u, 1)$. Using these inequalities for $G_n^{-1}(v) - a$ and the properties of ℓ, we obtain (2.1).

Let b_n, $n = 1, 2, \cdots$, be such that

$$
\begin{aligned}
&\inf_{a\in\mathbb{R}} \int_0^1 \ell(K_n^{-1}(u) - a)\, du \qquad (3.10)\\
&\qquad \geq \int_0^1 \ell(K_n^{-1}(u) - b_n)\, du - \frac{1}{n}, \quad n = 1, 2, \cdots.
\end{aligned}
$$

Now we obtain from (2.1)

$$\liminf_{n\to\infty} \inf_{a\in\mathbb{R}} E_f \ell(a_n T_n - a) \geq \liminf_{n\to\infty} \int_0^1 \ell(K_n^{-1}(u) - b_n)\, du. \tag{3.11}$$

Let $\{n_i\}$ be a sequence of positive integers and let b_0 be in $[-\infty, \infty]$ such that

$$\liminf_{n\to\infty} \int_0^1 \ell(K_n^{-1}(u) - b_n)\, du = \lim_{i\to\infty} \int_0^1 \ell(K_{n_i}^{-1}(u) - b_{n_i})\, du \tag{3.12}$$

and

$$\lim_{i\to\infty} b_{n_i} = b_0. \tag{3.13}$$

From Satz 2.11 of Witting and Nölle (1970) it follows that

$$\lim_{i\to\infty} K_{n_i}^{-1}(u) = K^{-1}(u) \tag{3.14}$$

for Lebesgue almost all $u \in (0, 1)$. Under either of the conditions (2.2) and (2.3) we arrive by (3.13) and (3.14) at

$$\begin{aligned} &\lim_{i\to\infty} \int_0^1 \ell(K_{n_i}^{-1}(u) - b_{n_i})\, du \\ &\geq \int_0^1 \liminf_{i\to\infty} \ell(K_{n_i}^{-1}(u) - b_{n_i})\, du \geq \int_0^1 \ell(K^{-1}(u) - b_0)\, du \\ &\geq \inf_{a\in\mathbb{R}} \int_0^1 \ell(K^{-1}(u) - a)\, du. \end{aligned} \tag{3.15}$$

Combining (3.11), (3.12) and (3.15), we obtain (2.4). Since (3.14) holds also for G_n^{-1} and K_n^{-1}, the left continuity of G^{-1} and K^{-1} yields (2.5). □

PROOF OF COROLLARY 2.1. With $\ell(x) = x^2$, inequality (2.6) is a special case of (2.1). Let $u_0 \in (0, 1)$ be such that H_n^{-1} is nonpositive on $(0, u_0)$ and nonnegative on $(u_0, 1)$. By Fubini's theorem we have

$$\begin{aligned} &\int_0^1 K_n^{-1}(u) H_n^{-1}(u)\, du \\ &= \int_0^1 \int_{u_0}^u \frac{H_n^{-1}(u)}{\int_s^1 H_n^{-1}(t)\, dt}\, ds\, du \\ &= \int_0^{u_0} \frac{-\int_0^s H_n^{-1}(u)\, du}{\int_s^1 H_n^{-1}(t)\, dt}\, ds + \int_{u_0}^1 \frac{\int_s^1 H_n^{-1}(u)\, du}{\int_s^1 H_n^{-1}(t)\, dt}\, ds = 1. \end{aligned} \tag{3.16}$$

Consequently, if $I(f) < \infty$, the Cauchy-Schwarz inequality yields

$$\operatorname{var}_{k_n} X \geq \left(\int_0^1 [H_n^{-1}(u)]^2\, du\right)^{-1} = a_n^2 (nI(f))^{-1}$$

and hence (2.7). □

For the proofs of Corollaries 2.2 through 2.4, the following lemma is useful.

LEMMA 3.1. *Let H be a nondegenerate distribution function such that $H_n \to_w H$ as $n \to \infty$. If*

$$\lim_{n\to\infty} E\,|S_n| = \int_0^1 |H^{-1}(u)|\, du \tag{3.17}$$

then

$$K_n \to_w K \quad as \quad n \to \infty, \tag{3.18}$$

where the absolutely continuous distribution function K is defined by (*cf.* (1.4))

$$K^{-1}(u) = \int_w^u \frac{1}{\int_s^1 H^{-1}(t)\,dt}\,ds, \quad 0 < u \le 1. \tag{3.19}$$

Furthermore, if H is stable with exponent $\alpha \in (1, 2]$, *then* (3.17) *and consequently* (3.18) *hold.*

PROOF. Since $H_n^{-1}(t) \to H^{-1}(t)$ for Lebesgue almost all $t \in (0, 1)$, we obtain from (3.17) and Vitali's theorem

$$\lim_{n\to\infty} \int_s^1 H_n^{-1}(t)\,dt = \int_s^1 H^{-1}(t)\,dt \tag{3.20}$$

and hence by dominated convergence (3.18).

Using Theorem 4.1 and Remark 4.1 of Kruglov (1979) with $\psi(x) = 1 + |x|$ and $F = H$ stable with exponent $\alpha \in (1, 2]$, we see that in order to prove (3.17) it suffices to show

$$\lim_{R\to\infty} \sup_n n a_n^{-1} E\{|Y_1| 1_{(R,\infty)}(a_n^{-1}|Y_1|)\} = 0, \tag{3.21}$$

where Y_1 has the same distribution as $-f'(X_1)/f(X_1)$ under f. Denoting $E\{Y_1^2 1_{[0,y)}(|Y_1|)\}$ by $\mu(y)$ we obtain from formulas (5.16), (5.17) and (5.24) of Chapter XVII of Feller (1971)

$$y^2 P(|Y_1| > y)/\mu(y) \le 1 \quad \text{for } y \text{ large}, \tag{3.22}$$

$$\mu(y) \text{ is regularly varying with exponent } 2-\alpha \quad \text{as } y \to \infty, \tag{3.23}$$

$$n\mu(a_n R) a_n^{-2} \to CR^{2-\alpha} \quad \text{as } n \to \infty. \tag{3.24}$$

Applying these results and Karamata's theorem (cf. Theorem VIII.9.1 of Feller, 1971) we arrive at

$$\begin{aligned}
&\lim_{R\to\infty} \limsup_{n\to\infty} n a_n^{-1} E\{|Y_1| 1_{(a_n R,\infty)}(|Y_1|)\} \\
&= \lim_{R\to\infty} \limsup_{n\to\infty} n a_n^{-1} \left\{ \int_{a_n R}^{\infty} P(|Y_1| > y)\,dy + a_n R P(|Y_1| > a_n R) \right\} \\
&\le \lim_{R\to\infty} \limsup_{n\to\infty} n a_n^{-1} \left\{ \int_{a_n R}^{\infty} y^{-2}\mu(y)\,dy + (a_n R)^{-1}\mu(a_n R) \right\} \\
&= \lim_{R\to\infty} \frac{\alpha}{\alpha-1} CR^{1-\alpha} = 0,
\end{aligned} \tag{3.25}$$

which, in view of the monotonicity in R of the expectation in the left hand side of (3.21), implies (3.21). □

PROOF OF COROLLARY 2.2. Let $a_n = (nI(f))^{1/2}$. By the central limit theorem $H_n \to_w \Phi$ and hence Lemma 3.1 with $\alpha = 2$ and $w = ½$ yields $K_n \to_w \Phi$. □

PROOF OF COROLLARY 2.3. Let $a_n = (\frac{1}{2} cn \log n)^{1/2}$. From Lemma 3.2 of Woodroofe (1972) it follows that $H_n \to_w \Phi$ and hence Lemma 3.1 with $\alpha = 2$ and $w = \frac{1}{2}$ yields $K_n \to_w \Phi$. □

PROOF OF COROLLARY 2.4. We define

$$\mu(x) = \int_{|f'(y)/f(y)| \le x} \left(\frac{f'}{f}(y)\right)^2 f(y)\, dy, \quad x > 0. \tag{3.26}$$

Let $\varepsilon \in (0, 1)$. There exists a $\delta > 0$ such that (cf. Lemma 4.1 of Woodroofe, 1974)

$$\left|\frac{f'}{f}(y) = \frac{\alpha - 1}{y}\right| \le \frac{(\alpha - 1)\varepsilon}{y}, \quad 0 < y \le \delta. \tag{3.27}$$

In view of the properties of f and Karamata's theorem, this yields for $x \to \infty$

$$\begin{aligned}\mu(x) &\le \int_\delta^\infty \left(\frac{f'}{f}(y)\right)^2 f(y)\, dy \\ &\quad + \int_{(\alpha-1)(1-\varepsilon)x^{-1}}^{\delta} \alpha(\alpha - 1)^2(1 + \varepsilon)^2 y^{\alpha-3} L(y)\, dy \\ &\sim \alpha(\alpha - 1)^\alpha (2 - \alpha)^{-1}(1 + \varepsilon)^2(1 - \varepsilon)^{\alpha-2} x^{2-\alpha} L(x^{-1})\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}\mu(x) &\ge \int_{(\alpha-1)(1+\varepsilon)x^{-1}}^{\delta} \alpha(\alpha - 1)^2(1 - \varepsilon)^2 y^{\alpha-3} L(y)\, dy \\ &\sim \alpha(\alpha - 1)^\alpha (2 - \alpha)^{-1}(1 - \varepsilon)^2(1 + \varepsilon)^{\alpha-2} x^{2-\alpha} L(x^{-1}).\end{aligned} \tag{3.29}$$

Consequently $\mu(x)$ is regularly varying with exponent $2 - \alpha$ and

$$\lim_{n\to\infty} n a_n^{-2} \mu(a_n) = \alpha(\alpha - 1)^\alpha (2 - \alpha)^{-1}. \tag{3.30}$$

Since for ε and δ as above and $x \to \infty$

$$P_f\left(-\frac{f'}{f}(X_1) > x\right) \le x^{-2} E_f\left\{\left(\frac{f'}{f}(X_1)\right)^2 1_{(\delta,\infty)}(X_1)\right\} = O(x^{-2}) \tag{3.31}$$

and

$$\begin{aligned}P_f\left(-\frac{f'}{f}(X_1) < -x\right) &\ge P_f(0 < X_1 \le \delta,\ (\alpha - 1)(1 - \varepsilon) > xX_1) \\ &= (\alpha - 1)^\alpha (1 - \varepsilon)^\alpha x^{-\alpha} L(x^{-1}),\end{aligned} \tag{3.32}$$

we also have

$$\lim_{x\to\infty} P_f\left(-\frac{f'}{f}(X_1) > x\right)\left[P_f\left(\left|\frac{f'}{f}(X_1)\right| > x\right)\right]^{-1} = 0. \tag{3.33}$$

From (3.30), (3.33), the regular variation of μ and $\int f' = 0$ we obtain the weak convergence of H_n to H by Theorem XVII.5.3 of Feller (1971).

Here we note that the + and − sign in (3.18) of section XVII.4 of Feller (1971) should be interchanged. In view of this misprint the + sign in formula (2.4) of Woodroofe (1974) should be replaced by a − sign and consequently the remark at the beginning of Section 3 of Woodroofe (1974) should be (in our notation): $\hat{G}(0) = 1 - \alpha^{-1}$, which for $\alpha \downarrow 1$ tends to 0.

Again the proof is completed by Lemma 3.1. □

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