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TESTING FOR HOMOGENEITY OF NOISY SIGNALS EVOKED BY REPEATED STIMULI[1]

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Two statistics are proposed for testing the homogeneity of signals, when these are noisy and repeatedly recorded. The distribution of the statistics is well approximated by the χ^2 and the normal distribution, respectively, as shown by theoretical and simulation results. Power and insensitivity to other types of variation are discussed. The approach is developed in the context of analysing recordings of electric brain potentials being evoked by some stimulation, as traditionally investigated in neurophysiology and psychophysiology. In particular, applications to visual potentials evoked by light flashes of a group of 41 children are given.

1. Introduction and outline. In this paper, we propose two test statistics for detecting inhomogeneity of signals which are noisy and repeatedly recorded. This approach was developed in the context of analysing electroencephalographic (EEG) recordings. Nevertheless, the methods are rather general and may as well apply to other fields.

The measurements of so-called event-related potentials (ERP) has become common in neurophysiology and psychophysiology (cf. Callaway et al., 1978, Thomson and Patterson, 1973). These are electric potentials of the brain evoked as a response to external stimuli, e.g. flash lights or sounds. As long as human beings are investigated, the data is usually recorded by electrodes placed (with respect to some reference—see Section 6) at various topographical locations of the scalp. (We will, however, consider only one derivation.) The ERP is low in amplitude compared to the spontaneous, not event-related activity of the brain, which is the main contribution to the noise. Therefore, the stimulation is repeated and further signal extraction methods are applied. The recorded data can be written as

$$x_i(t), \quad i = 1, \cdots, n, \quad t = 1, \cdots, T$$

where i is the index of stimulus repetition and t is the index of post-stimulus time, sampled at some interval Δt. Following EEG-jargon, we will refer to x_i as the ith sweep.

The most common and simplest signal extraction method, i.e. ERP-estimation procedure, consists of averaging all the sweeps over stimulus repetitions

$$\bar{x}(t) = (1/n) \textstyle\sum_{i=1}^{n} x_i(t). \tag{1}$$

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This provides an unbiased estimate of the brain response as long as the single responses are identical (assuming also noise with zero mean). The underlying model can be written as

$$x_i(t) = s(t) + \varepsilon_i(t) \tag{2}$$

where $s(t)$ denotes the fixed deterministic brain response, i.e. the signal, and $\varepsilon_i(t)$ represents the background noise not related to the stimulus. The noise $\varepsilon_i(t)$, ($t = 1, \cdots, T$) is viewed as a realization of a zero mean stationary ergodic Gaussian process (see Glaser and Ruchkin, 1976, Gasser et al., 1982). A further assumption is the independence of the noise across stimuli (its validity depends on the choice of interstimulus intervals).

There is, however, general agreement in ERP research that the invariance of the signal in (2) is an oversimplification and that responses may vary from one stimulus to the other. This hypothesis is substantiated e.g. by the fact that average evoked potentials change intraindividually, if the psychological state of the subjects is systematically altered from one experiment to the other (cf. Garcia-Austt et al., 1964, Callaway, 1975). The amplitudes are affected by attention and alertness of the subjects (we are focusing here on cortical potentials, whereas so-called brain-stem potentials may have different properties). Furthermore, potentials recorded directly at the cortex are known to vary from one stimulus to the other (see for example John, 1973, for results with cats, where subdural derivations lead to a better signal-to-noise ratio).

Model (2), therefore, assumes that the psychological state of the subject should be roughly constant during the stimulation session. This might be true for one subject, but grossly wrong for another, and may differ between groups, in particular in control design studies, where the experimental group has some psychiatric or behavioral disorder. Such background variables are difficult to control or to quantify, and these problems are, therefore, crucial for ERP measurement. Since there are no valid models for describing response variation, it is no longer clear what we should estimate. Simply replacing the expression "ERP" by "average ERP", as is usually done, seems to us no escape from the difficulties. If serious response variation due to psychological variation is present, the interpretation of $\bar{x}$ in physiological terms will be difficult. (Of course, this does not imply that looking for the average ERP is useless). On the other hand, quantification of response variability provides additional, perhaps important, information about the experiment. Callaway (1975) and John et al. (1978) give careful discussion of the problems arising from response variation. Approaches to the estimation problem are given in John et al. (1978) and Gasser et al. (1983).

In the present paper, we propose two statistics for testing for violations of model (2) with respect to the presence of response variation. Roughly speaking, the first test is sensitive to variations of the signal amplitude while the second one is likely to detect every kind of variation for which there are on the average "small" variations from one stimulation to the next, and "large" changes over the entire experiment.

The assumptions in (2) concerning the noise are doubtful as well. While the assumption of Gaussianity at least univariately seems to be fairly well fulfilled

(Gasser et al., 1982), stationarity within the sweeps and across repetitions appears shaky. The likely occurrence of inhomogeneity of the noise across the sweeps sets a higher demand on a test, since it should be sensitive to variations in the responses but not in noise. The tests proposed fulfill this requirement in some sense as dicussed below.

The statistics are described in Section 2 and their approximate distributions are derived in Section 3, with proofs being relegated to the appendix. In Section 4, some simulation results, supporting the validity of the approximate distribution, are given. They are followed by some power and robustness considerations. Applications based on real data are given in Section 6 including some details about data recording.

2. Models and test statistics. The data recorded consists of n independent time series $x_i(t)$ assumed to obey to the model

$$x_i(t) = s_i(t) + \varepsilon_i(t), \quad i = 1, \cdots, n \tag{3}$$

where $s_i(t)$ is the (deterministic) signal and $\varepsilon_i(t)$ an additive Gaussian noise. To simplify the analysis, we shall assume for the moment that the ε_i have the same power $\Pi(\varepsilon)$ (i.e. variance—the term power is used because we look at the series mostly along the time axis). We are interested in testing the hypothesis that $s_i(t) = s(t)$ for all i ("standard model"). If nothing else is specified, there is a large class of possible alernatives, and we could not expect a test to be powerful for all of them. Thus, two alternative models are introduced for which the tests should be sensitive:

H_A: The amplitude variation model: $s_i(t) = a_i s(t)$, the a_i being unknown constants.

H_B: The slowly changing signal model: $s_i(t)$ is close to $s_{i+1}(t)$ and is quite different from $s_j(t)$ for $|j - i|$ large.

Alternative H_A is not far-fetched when comparing the results about amplitude variations of the average ERP reported in the introduction. Alternative H_B is interesting in the light of the well known effect of subjects becoming habituated to the stimuli in the course of the experiment.

To construct a test for alternative H_A, we observe that

$$E\ \mathrm{cro}(x_i, s) = a_i \cdot P(s)$$

where

$$\mathrm{cro}(x_i, s) = (1/T) \textstyle\sum_{t=1}^{T} x_i(t) \cdot s(t)$$

is the cross-product of x_i and s and $P(s) = \mathrm{cro}(s, s)$ the power of s. This suggests considering the sample variance of $\mathrm{cro}(x_i, s)$, which tends to be large if the a_i are widely dispersed and small if they are equal. However, s is unknown, so one might try to replace it by the estimate $\bar{x}$ given by (1). But then $\mathrm{cro}(x_i, \bar{x})$ is a biased estimate of $a_i \bar{a} P(s)$, $\bar{a} = \Sigma a_i/n$, the bias being $\Pi(\varepsilon_i)/n$ where $\Pi(\varepsilon_i)$ is the noise power of the ith sweep. This causes no harm if the noise power is invariant as assumed in the standard model. However, noise inhomogeneity may well be present and we would like our test to be insensitive to this type of departure. Therefore, we use the one-leave-out technique, that is, instead of $\bar{x}$, we use

$$\bar{x}_i = (n-1)^{-1} \textstyle\sum_{j\neq i} x_j$$

$$E \operatorname{cro}(x_i, \bar{x}_i) = a_i \cdot \bar{a}_i P(s), \quad \bar{a}_i = (n-1)^{-1} \textstyle\sum_{j\neq i} a_j.$$

The expectation of the cross-products is then approximately proportional to a_i (if $\bar{a}_i$ is treated as a constant). Let $c_i = \operatorname{cro}(x_i, \bar{x}_i)$ and let $v(c_i)$ denote the sample variance of the c_i, i.e.

$$v(c_i) = (n-1)^{-1} \textstyle\sum_{i=1}^{n} (c_i - \bar{c})^2, \quad \bar{c} = (1/n) \sum_{i=1}^{n} c_i. \tag{4}$$

Then we would reject the standard model if $v(c_i)$ were much larger than its expected value.

THEOREM 1. *Assume the standard model and white noise. Then*

$$Ev(c_i) = \frac{1}{T}\Pi(\varepsilon)\left(\frac{n-2}{n-1}\right)\left[\frac{n-2}{n-1}P(s) + \frac{1}{n-1}\Pi(\varepsilon)\right].$$

When replacing the unknown quantities $\Pi(\varepsilon)$ and $P(s)$ in the above theorem by the following unbiased estimates

$$\hat{\Pi}(\varepsilon) = (n-1)^{-1} \textstyle\sum_{i=1}^{n} P(x_i - \bar{x}) = (n-1)^{-1} \sum_{i=1}^{n} [P(x_i) - nP(\bar{x})] \tag{5}$$

$$\hat{P}(s) = P(\bar{x}) - (1/n)\,\hat{\Pi}(\varepsilon), \tag{6}$$

we obtain an estimate of $Ev(c_i)$

$$\tilde{v} = \frac{1}{T}\hat{\Pi}(\varepsilon)\left(\frac{n-2}{n-1}\right)\left[\frac{n-2}{n-1}\hat{P}(s) + \frac{1}{n-1}\hat{\Pi}(\varepsilon)\right].$$

Intuitively, $\tilde{v}$ should have about the same expected value as $v(c_i)$ under the standard model. Under the alternative H_A the expected value of $v(c_i)$ should be larger than that of $\tilde{v}$ (compare Section 5, Theorem 1′). Thus, we introduce the statistic:

$$A = v(c_i)/\tilde{v}$$

and reject the hypothesis if A is too large. The distributional properties of A will be studied in Section 3.

Turning to the construction of a test for alternative H_B, let us consider the successive differences $x_i - x_{i+1}$, $i = 1, \cdots, n-1$. Under the standard model, the power of the above difference can be used to estimate the noise power since the signals cancel out. Estimators for the noise power based on the differences have in fact been proposed (see Callaway, 1975). The $P(x_i - x_{i+1})$ can still provide a good estimate of the noise power even under alternative H_B, since the signal changes slowly. More precisely

$$EP(x_i - x_{i+1}) = 2\Pi(\varepsilon) + P(s_i - s_{i+1})$$

where the last term should be small by assumption. Therefore, the statistic

$$\tilde{\Pi}(\varepsilon) = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} P(x_i - x_{i+1})$$

estimates $\Pi(\varepsilon)$ under alternative H_B with only a small positive bias. The standard estimate $\hat{\Pi}(\varepsilon)$ given by (5) will have a high positive bias since $EP(x_i - \bar{x}) = \Pi(\varepsilon_i - \bar{\varepsilon}) + P(s_i - \bar{s})$ and since s_i would be quite different from $\bar{s}$. Thus, we introduce the test statistic

$$B = \hat{\Pi}(\varepsilon)/\tilde{\Pi}(\varepsilon) \tag{8}$$

and reject the standard model if B is too large. (As to distributional properties compare Section 3).

Note that white noise was assumed in Theorem 1 for the derivation of statistic A, in contrast to B. However, this assumption will be needed for the derivation of the approximate distribution for both statistics. Since the noise in ERP data is definitely not white, further considerations are needed, to be supplied in Section 6.

3. Distributional properties of the test statistics. To obtain an approximate distribution for statistic A, we first show

LEMMA 1. *Under the standard model assuming white noise, $v(c_i)$ is distributed like $v(U_i - P_i/(n-1))$ where $U_i = (n-2)(n-1)^{-1}$ cro$(u_i, \bar{x})$, $P_i = P(u_i - \bar{u})$, the $u_i(t)$, $t = 1, \cdots, T$ being independent Gaussian white noise, independent of $\bar{x}(t)$, with variance $\Pi(\varepsilon)$. Consequently, the P_i are uncorrelated with U_i and have variance $2T^{-1}(n-1)^2 n^{-2}\Pi(\varepsilon)^2$ and covariance* $\mathrm{cov}(P_i, P_j) = 2T^{-1}n^{-2}\Pi(\varepsilon)^2$.

From the above results we see that the conditional distribution of the $U_i - P_i/(n-1)$ given $\bar{x}$ is close to a Gaussian distribution if n is large since the U_i are Gaussian and independent of $\bar{x}$, or if T is large, because of the central limit theorem. Thus if we treat the above distribution as Gaussian, it can be seen from Lemma 1 that $v(c_i)$ is distributed like

$$\left[\frac{1}{T}\left(\frac{n-2}{n-1}\right)^2 \Pi(\varepsilon)\left\{P(\bar{x}) + \frac{2}{n(n-2)}\Pi(\varepsilon)\right\}\right]\frac{1}{n-1}\chi^2_{n-1}$$

where χ^2_{n-1} denotes a χ^2 variate with $n-1$ degrees of freedom independent of $\bar{x}$. Since $\hat{\Pi}(\varepsilon)/\Pi(\varepsilon) \to 1$ as $n \cdot T \to \infty$, the above bracket [ ] is quite close to $\tilde{v}$, and hence $(n-1)A$ is approximately distributed like χ^2_{n-1}.

The above argument is only a heuristic one. However, for $T \to \infty$ and n fixed, one can prove the result rigorously as follows: Without loss of generality, one can assume a constant signal by using the argument given at the beginning of Section 4. The $U_i - P_i/(n-1)$ are then the time average of i.i.d. random variables

$$\frac{n-2}{n-1} \cdot u_i(t) \cdot \bar{x}(t) - \frac{[u_i(t) - \bar{u}(t)]^2}{n-1}, \quad t = 1, \cdots, T.$$

Thus by the central limit theorem and Lemma 1, as $T \to \infty$ and n fixed,

$\sqrt{T}[U_i - P_i/(n-1) + \Pi(\varepsilon)/n]$, $i = 1, \cdots, n$ converges in distribution to jointly normal variates $G_1, \cdots, G_n$ with zero mean and

$$\operatorname{var}(G_i) = \left(\frac{n-2}{n-1}\right)^2 \Pi(\varepsilon) \cdot EP(\bar{x}) + \frac{2}{n^2}\Pi(\varepsilon)^2$$

$$\operatorname{cov}(G_i, G_j) = \frac{2}{n^2(n-1)^2}\Pi(\varepsilon)^2, \quad i \neq j.$$

It can be verified that the G_i are distributed like $Z_i - \sum Z_i/n$, where the Z_i are i.i.d. normal variates with variance

$$\operatorname{var}(G_i) - \operatorname{cov}(G_i, G_j) = \left(\frac{n-2}{n-1}\right)^2 \Pi(\varepsilon)\left[EP(\bar{x}) + \frac{2}{n(n-2)}\Pi(\varepsilon)\right].$$

On the other hand, the above right hand side is the limit in probability of $T\tilde{v}$ as $T \to \infty$. Thus, we have proved

THEOREM 2. *Under the standard model and assuming white noise, $(n-1)A$ converges in distribution to a χ^2 variate with $n-1$ degrees of freedom as $T \to \infty$ and for n fixed.*

Turning to statistic B, we can write it in the form:

$$B = 1 + \frac{\hat{\Pi}(\varepsilon) - \tilde{\Pi}(\varepsilon)}{\tilde{\Pi}(\varepsilon)} = 1 + \frac{D}{\tilde{\Pi}(\varepsilon)}.$$

THEOREM 3. *Under the standard model*

$$D = \frac{1}{n-1}\left[\left(1 - \frac{2}{n}\right)\textstyle\sum_{i=1}^{n-1} \operatorname{cro}(\varepsilon_i, \varepsilon_{i+1}) + \frac{1}{2}P(\varepsilon_1) + \frac{1}{2}P(\varepsilon_n) - \frac{1}{n}\sum_{i=1}^{n} P(\varepsilon_i) - \frac{1}{n}\sum\sum_{|i-j|>1} \operatorname{cro}(\varepsilon_i, \varepsilon_j)\right]$$

and has zero mean, and (assuming white noise) variance $(n-2)(n-1)^{-2}T^{-1}\Pi(\varepsilon)^2$.

By the above theorem, D is the sum of a large number of random variables where each is independent of most of the other. For example $\operatorname{cro}(\varepsilon_i, \varepsilon_{i+1})$ is independent of $\operatorname{cro}(\varepsilon_j, \varepsilon_k)$ if $j \notin \{i, i+1\}$ and $k \notin \{i, i+1\}$. Therefore, the central limit theorem is expected to hold, implying that $(n-1)\sqrt{T/(n-2)} \cdot D/\tilde{\Pi}(\varepsilon)$ is approximately distributed as a standard normal variate.

THEOREM 4. *Under the standard model $(n-1)\sqrt{T/(n-2)} \cdot (B-1)$ converges in distribution to a standard normal variance as $nT \to \infty$.*

4. Simulation study of the distribution of the test statistics under the standard model. The above χ^2 and normal approximations for the distribution of statistics A and B are obtained under the assumption that n, T or both are large. In practice, the values of n and T are limited by experimental constraints and it is worthwhile to get an idea of how good the approximation is for moderate values of n and T, especially for those which appear typically in ERP data. This is the purpose of our simulation study. Notice that the distribution of the $\mathrm{cro}(x_i, x_j)$ is that of the elements of a noncentral Wishart matrix with the noncentrality parameter depending only on $P(s)$ (cf. Arnold, 1981). Therefore, the joint distribution of $v(c_i)$ and $\tilde{v}$ depends only on $P(s)$ and $\Pi(\varepsilon)$ and since the statistic A is scale free, its distribution depends only on the signal-to-noise ratio $P(s)/\Pi(\varepsilon)$. Thus, in our simulation study, we take as s a constant signal. Normal pseudo random numbers were generated from uniform ones by the polar method; the latter themselves were generated by a two-seed congruential scheme following Marsaglia (1972) and McLaren and Marsaglia (1975). Each row of Table 1 was obtained independently from the results of 1000 runs, in each of which n noise sweeps of length T were generated and statistics A and B were computed. For statistic A, the signal power was chosen to give two signal-to-noise ratios .05 and 1. To compute the summarizing statistics, we used the SAS-package, together with the Kolmogorov-Smirnov Test from the IMSL-subroutine package.

The fit to the approximate distribution is good, even in cases of low T and n. Note that just one of the Kolmogorov-Smirnov test-probabilities has a low value of 0.025 which, after inspection of the other p-values, might be attributed to chance.

5. Power considerations and the effect of noise inhomogeneity. To get an idea about the power of the test, we look at the expected value of $v(c_i)$, $\tilde{v}$, $\hat{\Pi}(\varepsilon)$, and $\tilde{\Pi}(\varepsilon)$ under the general model (3). In addition, we will allow the noise to be inhomogeneous in the sense that its power may vary from one sweep to the other.

For convenience, the over bar will be used to denote the average over the sweeps. Also, for any quantity y indexed by the sweep number i, $v(y_i)$ denotes the sample variance of $y_1, \cdots, y_n$, similar to (4). Finally, V_s denotes $\sum P(s_i - \bar{s})/(n-1) = n \cdot [\overline{P(s)} - P(\bar{s})]/(n-1)$, a measure of signal variation. Starting with test A, we have:

THEOREM 1′. *Assume white noise, then*

$$Ev(c_i) = v(\mathrm{cro}(s_i, \bar{s}_i)) + \frac{1}{T}\frac{n-2}{n-1}\overline{\Pi(\varepsilon)}\left[\frac{n-2}{n-1}P(\bar{s}) + \frac{1}{n-1}\overline{\Pi(\varepsilon)}\right]$$
$$+ \frac{1}{T}\frac{n^2-4}{n(n-1)^2}\overline{\Pi(\varepsilon)}V_s - \frac{1}{T}\frac{n-2}{n(n-1)^2}v(\Pi(\varepsilon_i)) + r_1$$

$$E\tilde{v} = \frac{1}{T}\left(\frac{n-2}{n-1}\right)\overline{\Pi(\varepsilon)}\left[\frac{n-2}{n-1}P(\bar{s}) + \frac{1}{n-1}\overline{\Pi(\varepsilon)} + \frac{1}{T}\frac{2}{n(n-1)^2}\overline{\Pi(\varepsilon)}\right]$$
$$+ \frac{1}{T}\left[\left\{\frac{n^2-4}{n(n-1)^2} + \frac{1}{T}\frac{n-2}{n(n-1)^3}\right\}\overline{\Pi(\varepsilon)} + \left(\frac{n-2}{n-1}\right)^2 P(\bar{s}) + \frac{2(n-2)}{n(n-1)^2}V_s\right]V_s$$
$$+ \frac{1}{T^2}\left(\frac{n-2}{n-1}\right)^2\frac{2}{n^2}v(\Pi(\varepsilon_i)) + r_2$$

TABLE 1

Test A and Test B simulation results for fit to the approximate distribution (each column N = 1000 runs) (Skewness and kurtosis according to k-statistics)

				Test Statistic					Quantile Fit to the Theoretical Distribution								
				Mean	**Variance**	**Skewness**	**Kurtosis**	**KS-test prob.***									
Test A	n = 30		Theory	1.0	. 069	.525	.414	—	1%	5%	10%	25%	50%	75%	90%	95%	99%
		T = 9	SNR = .05	1.007	.063	.494	.379	.516	1.42	6.12	11.50	27.00	52.04	75.63	90.08	94.28	98.18
			SNR = 1	.995	.069	.585	.601	.641	1.42	4.97	8.92	24.12	50.01	73.56	89.06	94.83	99.20
		T = 18	SNR = .05	1.006	.064	.415	.027	.422	1.50	5.93	11.26	25.72	52.14	76.26	90.11	94.64	98.61
			SNR = 1	.999	.063	.497	.136	.871	1.35	5.99	11.37	26.39	49.04	75.53	88.88	95.10	98.88
	n = 64		Theory	1.0	.032	.356	.190	—	1%	5%	10%	25%	50%	75%	90%	95%	99%
		T = 9	SNR = .05	.991	.027	.264	.144	.025	1.66	5.32	10.19	25.79	49.26	71.38	86.63	93.36	98.20
			SNR = 1	1.006	.028	.393	.132	.111	2.00	6.35	12.14	27.58	50.84	74.04	89.81	93.83	98.52
		T = 18	SNR = .05	.999	.027	.421	.088	.155	1.60	6.92	12.27	26.29	48.64	72.66	88.41	94.34	98.72
			SNR = 1	.995	.030	.175	−.096	.842	.89	4.68	9.65	24.52	49.20	75.15	89.06	94.22	98.31
Test B			Theory	1.0	1.0	0	0	—	1%	5%	10%	25%	50%	75%	90%	95%	99%
	n = 30	T = 9		1.004	1.034	.222	−.137	.252	1.61	6.62	11.94	25.57	50.46	76.94	92.50	96.72	99.19
		T = 18		1.003	1.021	.200	−.066	.220	2.19	6.12	11.28	26.75	51.98	77.10	91.53	96.69	99.46
	n = 64	T = 9		1.003	.988	.269	.064	.204	2.19	6.41	11.55	26.81	52.29	76.77	89.97	95.85	99.67
		T = 18		1.001	.928	.042	−.161	.796	2.02	6.40	10.81	25.02	50.49	74.68	89.52	94.09	98.65

*Kolmogorov-Smirnov Test Probability.

where $\bar{s}_i$ is the average of the s_j, $j \neq i$ and r_1, r_2 are terms of order T^{-1} and T^{-2} given in the appendix.

The terms r_1, r_2 in the above result can be viewed as interaction terms since they vanish as soon as either the signal or the noise is homogeneous. As to $Ev(c_i)$, the second term is the expected value of $v(c_i)$ under the standard model, the first and the third term describe the effect of signal inhomogeneity and the fourth that of noise inhomogeneity. Similarly, in the expression for $E\tilde{v}$, the first term is the expected value of $\tilde{v}$ under the standard model, the second and the third term describe the effect of signal and noise inhomogeneity respectively. The above result shows that: (i) under the standard model, $E\tilde{v}$ is almost equal to $Ev(c_i)$, (ii) for inhomogeneous signals, the main increase in $Ev(c_i)$ comes from the term $v(\text{cro}(s_i, \bar{s}_i))$ which is of higher order of magnitude than the increase of $E\tilde{v}$, because of the factor $1/T$, and (iii) the effect of noise inhomogeneity is quite small, of the order $1/(n^2T)$ and $1/(nT)^2$ respectively (assuming signal homogeneity) as far as $Ev(c_i)$ and $E\tilde{v}$ are concerned. This insensitivity to noise inhomogeneity is actually due to the one-leave-out technique. Neglecting terms of order $(nT)^{-1}$ and the interaction terms, $Ev(c_i)$ roughly amounts to $v(\text{cro}(s_i, \bar{s}_i)) + \overline{\Pi(\varepsilon)}P(\bar{s})/T$ and $E\tilde{v} \approx (\overline{\Pi(\varepsilon)} + V_s)P(\bar{s})/T$. Under H_A (treating $\bar{a}_i = \bar{a}$), the ratio of the above quantities is approximately $[T \cdot V_s + \overline{\Pi(\varepsilon)}]/[V_s + \overline{\Pi(\varepsilon)}]$, and thus the power of test A is determined by the ratio of $V_s = v(a_i) \cdot P(s)$ to the average noise power. However, the next term accounting for the increase of $Ev(c_i)$, i.e. $(n^2 - 4)n^{-1}(n - 1)^{-2}T^{-1}\overline{\Pi(\varepsilon)}V_s$, is also present in $E\tilde{v}$. Thus, the above analysis only applies if $\overline{\Pi(\varepsilon)} \cdot V_s/(nT)$ is negligible with resepct to $v(\text{cro}(s_i, \bar{s}_i))$. This implies in particular, that the power of test A deteriorates when the average signal approaches zero. Further note that

$$[v(\text{cro}(s_i, \bar{s}_i))] \approx v(\text{cro}(s_i, \bar{s})) = (n-1)^{-1} \textstyle\sum_{i=1}^{n} (\text{cro}(s_i - \bar{s}, \bar{s}))^2 \leq V_s \cdot P(\bar{s})$$

by the Schwarz-inequality, equality holding for H_A. This yields the mild condition that $P(\bar{s})/\overline{\Pi(\varepsilon)}$ should be much greater than $(nT)^{-1}$ for test A to be powerful under H_A. For more general alternatives, however, test A might fail even when $V_sP(\bar{s})$ is large: an example is "morphological" variation, where $s_i = s + d_i$ with $\bar{d} = 0$ and $\text{cro}(s, d_i) = 0$.

Turning to test B, we have under the general model (3):

$$E\hat{\Pi}(\varepsilon) = \overline{\Pi(\varepsilon)} + V_s$$

$$E\tilde{\Pi}(\varepsilon) = \frac{n}{n-1}\overline{\Pi(\varepsilon)} - \frac{1}{2(n-1)}[\Pi(\varepsilon_1) + \Pi(\varepsilon_n)] + \frac{1}{2(n-1)} \textstyle\sum_{i=1}^{n-1} P(s_i - s_{i+1}).$$

These expressions show that the effect of noise inhomogeneity is small, i.e. $[(\Pi(\varepsilon_1) + \Pi(\varepsilon_n))/2 - \overline{\Pi(\varepsilon)}]/(n-1)$ for $E\hat{\Pi}(\varepsilon) - E\tilde{\Pi}(\varepsilon)$. To get an idea about the power, observe that

$$V_s = [n \cdot (n-1)]^{-1} \sum_{i<j} P(s_i - s_j).$$

Indeed the right hand side equals

$$[2n(n-1)]^{-1} \sum_i \sum_j P(s_i - \bar{s} + \bar{s} - s_j) = \sum_i P(s_i - \bar{s})/(n-1).$$

Therefore, assuming homogeneous noise, $E\hat{\Pi}(\varepsilon) - E\tilde{\Pi}(\varepsilon)$ is given by the difference of half the average power of all $s_i - s_j (i < j)$ with half the average power of the $s_i - s_{i+1}$. It is clear when the signal is slowly changing that this quantity would be large and the power of the test is determined by its ratio to the noise power. As an example, consider $s_i = s$ for $i = 1, \cdots, m$ and $s_i = s'$ for $i = m + 1, \cdots, n$. Then

$$E\hat{\Pi}(\varepsilon) - E\tilde{\Pi}(\varepsilon) = \frac{1}{n-1}\left[\frac{m(n-m)}{n} - \frac{1}{2}\right]P(s - s').$$

Unlike test A, test B could still detect signal variability, even if the signals average zero. However, test B relies on alternative H_B to be powerful and may not detect other types of signal variation. This differential performance of the tests is desirable since the joint application to real data might give hints about modes of response variation (Möcks et al., 1983). Intuitively, the tests are expected to be sensitive to the signal-to-noise ratio, like any visual judgment on signal variation would be. To look at this, since there is no unique definition of the signal-to-noise ratio within the general model, a scale parameter f is introduced for the signals by replacing s_i by $f \cdot s_i$. It is easily seen that the ratios used before for the power considerations increase with f^2.

6. Application to brain potentials evoked by light flashes. In this application, we used the data of a group of $n = 41$ normal children (age range 7–15 years), who volunteered as a control group for a study on neurophysiological aspects of mental retardation. The recordings took place in an electrically shielded room, with the subjects comfortably seated in a reclining chair with eyes closed. A photic stimulator placed 2m away from the subjects at the height of their head, produced $n = 64$ flashes with random interstimulus intervals (Min = 1063 msec; Max = 2938 msec). For about 2 min brain activity was first recorded on analog tape (at 8 locations on the scalp relative to linked earlobes; the results here refer to one derivation at the back of the head, occipital region O_1) and then digitized off-line at a rate of 408.5 Hz (= step of 2.448 msec). Sweeps with 512 points, 256 pre- and 256 poststimulus, entered further analysis. However, 128 points ($\approx$313 msec) poststimulus (signal domain) is the span where the main features of the responses are expected to occur. To give an impression of what the signals look like, Figure 1 shows three examples of averaged evoked potentials, slightly smoothed. The tests were applied to response data and real noise data using EEG data stretches from prestimulus activity of the same subjects.

We have already pointed out that the assumption of white noise is not fulfilled for ERP data. In order to achieve an approximately flat noise spectrum, one may apply prewhitening, i.e. zero-phase digital filtering of the sweeps with gain $\hat{f}(\nu)^{-1/2}$ where $\hat{f}(\nu)$ is some estimate of the noise spectrum at frequency ν. In our application, $\hat{f}(\nu)$ was computed as the average of the periodograms of the residual

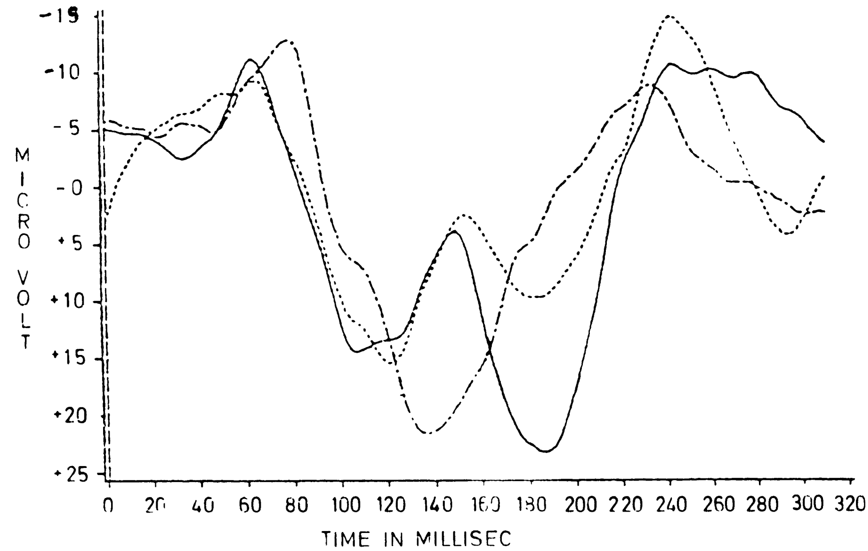

FIG. 1. *Average flash-evoked potentials (slightly smoothed) of three children. Electrode in the occipital region (O_1).*

sweeps $x_i - \bar{x}$. We should mention here that estimates of the noise spectrum from nonstimulus EEG data are not valid for this purpose due to the impact of the stimulus on the background activity. More specifically, the following procedure was applied: "Long" sweeps with 64 pre- and 192 poststimulus points (to give a total length of 256 points, the signal domain placed in the middle) were tapered (cosine) using 32 points on each side of the signal domain, the remaining part was set zero. The gain of the prewhitening filter was estimated and filtering was performed by Fast Fourier Transform using these "long" sweeps (i.e. Fourier transform forward, multiplication with gain function, Fourier transform backward). Further analysis was restricted to 128 poststimulus points. Since the relevant signal power is concentrated in some band, it is desirable to restrict the analysis to this band, in order to improve the signal-to-noise ratio. The results of the previous sections are then still applicable, provided T is replaced by T^*, the number of elementary frequencies falling in the band considered. This is seen rather straightforwardly by looking at the cross-products in the frequency domain using the finite Fourier transform. One can verify that crossproducts evaluated in some band can be viewed as crossproducts of two (real) sequences of length T^* to which all previous assumptions apply. In our application we restricted the analysis from above by 25 Hz dropping zero frequency as well. This led to $T^* = 16$ elementary frequencies falling in this band.

Table 2 summarizes some elementary statistics, in the first part the results for response data, in the second those for the real noise application are shown. The statistics had been transformed to have under the standard model mean 100 and standard deviation 10. The SNR was completed after prewhitening by $P(s)/\hat{\Pi}(\varepsilon)$, see equation (5), (6) in Section 2. The striking message of the noise-

TABLE 2
Application to flash evoked potentials of $N = 41$ subjects. Elementary statistics of test A, test B, and signal-to-noise ratio (SNR) after prewhitening. (Skewness and kurtosis according to k-statistics).

	Response Data			Noise Data		
	A	*B*	SNR	*A*	*B*	SNR
Mean	106.41	109.45	.504	101.65	100.07	$4 \cdot 10^{-5}$
Std Dev	14.55	13.96	.456	9.55	10.61	.004
Median	105.45	105.87	.333	100.30	98.55	$5 \cdot 10^{-4}$
Min	78.89	79.56	.050	79.15	81.72	−.009
Max	145.82	140.74	2.246	122.33	127.12	.006
Skewness	.7461	.2789	1.820	.2299	.7550	−.5025
Kurtosis	.8153	−.2659	4.192	−.0206	−.0951	−.8107

data results is that the distributional approximations are again rather good under realistic conditions, even for signal-to-noise ratio zero. We should note here that in this application serious deviations in the empirical distribution of A were found when using simply 128 pts poststimulus for all computations, instead of the "long" sweep procedure described above. These may be attributed to leakage effects. The response data results demonstrate that response variation is in fact present in our data. Using the 5% significance level, we get 7 rejections of the standard model in case of test A and 15 for test B, and there are 3 subjects where the tests were jointly significant. Further data-analytic results can be found in Möcks et al. (1983).

APPENDIX

Proofs of Results

We begin by proving Theorem 1′ and Theorem 1, which is a special case of the former. We first show

LEMMA A. *Let $\hat{\Pi}(\varepsilon)$ be given by* (5). *Under the general model and white noise assumption*

$$\operatorname{var} P(\bar{x}) = \frac{2}{T}\left[\frac{1}{n^2}\,\overline{\Pi(\varepsilon)}^2 + \frac{2}{n}\,\overline{\Pi(\varepsilon)}\,P(\bar{s})\right]$$

$$\operatorname{var} \hat{\Pi}(\varepsilon) = \frac{1}{T}\frac{2}{n-1}\left[\overline{\Pi(\varepsilon)}^2 + \frac{n-2}{n}\,v(\Pi(\varepsilon_i)) + \frac{2}{n-1}\sum_{i=1}^{n}\Pi(\varepsilon_i)P(s_i-\bar{s})\right]$$

$$\operatorname{cov}\{P(\bar{x}),\,\hat{\Pi}(\varepsilon)\} = \frac{1}{T}\frac{2}{n}\left[\frac{1}{n}\,v(\Pi(\varepsilon_i)) + \frac{2}{n-1}\sum_{i=1}^{n}\{\Pi(\varepsilon_i) - \overline{\Pi(\varepsilon)}\}\operatorname{cro}(s_i,\bar{s})\right].$$

PROOF. In this proof, we shall often use the fact that moments of third order between zero mean Gaussian variables vanish. Thus $P(\bar{x}) = P(\bar{s}) + 2\operatorname{cro}(\bar{\varepsilon}, s) + P(\bar{\varepsilon})$ with the two latter terms being uncorrelated. Since $P(\bar{\varepsilon})/\Pi(\bar{\varepsilon})$ is a χ^2 variate with T degrees of freedom, the first result of the lemma follows. To show the second result, we observe that $\hat{\Pi}(\varepsilon) = [\sum P(x_i) - nP(\bar{x})]/(n-1)$. Hence

$$\text{var}\,\hat{\Pi}(\varepsilon) = \frac{1}{(n-1)^2}\,[\textstyle\sum_{i=1}^n \{\text{var}\,P(x_i) - 2n\,\text{cov}(P(x_i), P(\bar{x}))\} + n^2\,\text{var}\,P(\bar{x})].$$

We need only to compute $\text{cov}(P(x_i), P(\bar{x}))$ since

$$\text{var}\,P(x_i) = (2/T)[\Pi(\varepsilon_i)^2 + 2\Pi(\varepsilon_i)\cdot P(s_i)]$$

by a similar argument as above. Write $P(x_i)$ in the form $P(s_i) + 2\,\text{cro}(\varepsilon_i, s_i) + P(\varepsilon_i)$ and observe that $\text{cro}(\varepsilon_i, s_i)$, $P(\bar{\varepsilon})$ and $\text{cro}(\bar{\varepsilon}, \bar{s})$, $P(\varepsilon_i)$ are uncorrelated, respectively, we get

$$\text{cov}\{P(x_i), P(\bar{x})\} = 4\,\text{cov}\{\text{cro}(\varepsilon_i, s_i), \text{cro}(\bar{\varepsilon}, \bar{s})\} + \text{cov}\{P(\varepsilon_i), P(\bar{\varepsilon})\}.$$

The first covariance of the above right hand side is

$$(1/n)\,\text{cov}\{\text{cro}(\varepsilon_i, s_i), \text{cro}(\varepsilon_i, \bar{s})\} = (1/T)(1/n)\Pi(\varepsilon_i)\text{cro}(s_i, \bar{s})$$

and the second covariance is

$$(1/n^2)\text{cov}[P(\varepsilon_i), \textstyle\sum_{j,k}\text{cro}(\varepsilon_j, \varepsilon_k)] = (1/n^2)\text{var}\,P(\varepsilon_i) = (1/T)(2/n^2)\Pi(\varepsilon_i)^2.$$

Therefore

$$\text{cov}\{P(x_i), P(\bar{x})\} = (1/T)(2/n)[(1/n)\,\Pi(\varepsilon_i)^2 + 2\,\Pi(\varepsilon_i)\text{cro}(s_i, \bar{s})].$$

Summing up, the variance of $\hat{\Pi}(\varepsilon)$ is seen to be

$$\frac{1}{T}\,\frac{2}{(n-1)^2}\,[(n-2)\,\overline{\Pi(\varepsilon)^2} + \overline{\Pi(\varepsilon)}^2 + 2\,\textstyle\sum_{i=1}^n \Pi(\varepsilon_i)\{P(s_i) - 2\,\text{cro}(s_i, \bar{s}) + P(\bar{s})\}]$$

and the second result of the theorem follows.

Finally

$$\text{cov}\{P(\bar{x}), \hat{\Pi}(\varepsilon)\} = (n-1)^{-1}[\textstyle\sum_{i=1}^n \text{cov}\{P(\bar{x}), P(x_i)\} - n\,\text{var}\,P(\bar{x})]$$

and we obtain the last result of the lemma after some algebraic arrangement.

PROOF OF THEOREM 1′. It is easily seen that

$$Ev(c_i) = v(Ec_i) + (n-1)^{-1}[\textstyle\sum_{i=1}^n \text{var}(c_i) - n\,\text{var}(\bar{c})].$$

To compute $\text{var}(c_i)$, we observe that

$$c_i = \text{cro}(s_i, \bar{s}) + \text{cro}(\varepsilon_i, \bar{s}_i) + \text{cro}(\bar{\varepsilon}_i, s_i) + \text{cro}(\varepsilon_i, \bar{\varepsilon}_i)$$

and that the last three terms of the above right-hand sides are uncorrelated. Therefore their variances sum to the variance of c_i. Clearly, the variances of $\text{cro}(\varepsilon_i, \bar{s}_i)$ and of $\text{cro}(\bar{\varepsilon}_i, s_i)$ are $\Pi(\varepsilon_i)P(\bar{s}_i)$ and $\Pi(\bar{\varepsilon}_i)P(s_i)$ respectively. As for $\text{cro}(\varepsilon_i, \bar{\varepsilon}_i)$, we have

$$E[\textstyle\sum_t \varepsilon_i(t)\bar{\varepsilon}_i(t)]^2 = \sum_t E[\varepsilon_i(t)\bar{\varepsilon}_i(t)]^2 = T\Pi(\varepsilon_i)\Pi(\bar{\varepsilon}_i)$$

which shows that $\text{var}[\text{cro}(\varepsilon_i, \bar{\varepsilon})] = \Pi(\varepsilon_i)\Pi(\bar{\varepsilon}_i)/T$. Thus

$$\text{var}\,c_i = (1/T)[\Pi(\varepsilon_i)P(\bar{s}_i) + \Pi(\bar{\varepsilon}_i)P(s_i) + \Pi(\varepsilon_i)\Pi(\bar{\varepsilon}_i)].$$

Writing $\bar{s}_i = \bar{s} - (s_i - \bar{s})/(n-1)$, $\Pi(\bar{\varepsilon}_i) = (n-1)^{-2}[n\overline{\Pi(\varepsilon)} - \Pi(\varepsilon_i)]$, we get

$$\sum_{i=1}^n \text{var}\, c_i = \frac{1}{T}\sum_{i=1}^n \Pi(\varepsilon_i)\left[P\left\{\bar{s} - \frac{1}{n-1}(s_i - \bar{s})\right\} - \frac{1}{(n-1)^2}P(\bar{s} + s_i - \bar{s})\right]$$

$$+ \frac{1}{T}\frac{n^2}{(n-1)^2}\overline{\Pi(\varepsilon)P(\bar{s})} + \frac{1}{T}\frac{n}{n-1}\overline{\Pi(\varepsilon)V_s} + \frac{1}{T}\sum_{i=1}^n \Pi(\varepsilon_i)\Pi(\bar{\varepsilon}_i)$$

$$= \frac{1}{T}\left[\frac{n^2(n+1)}{(n-1)^2}\overline{\Pi(\varepsilon)P(\bar{s})} - \frac{2n}{(n-1)^2}\sum \text{cro}(s_i, \bar{s})\Pi(\varepsilon_i)\right.$$

$$\left. + \frac{n}{n-1}\overline{\Pi(\varepsilon)V_s} + \frac{n}{n-1}\overline{\Pi(\varepsilon)}^2 - \frac{1}{n-1}v(\Pi(\varepsilon_i))\right].$$

Note that the last two terms of the above expression come from the relation

$$\sum_{i=1}^n \Pi(\varepsilon_i)\Pi(\bar{\varepsilon}_i) = (n-1)^{-2}[(n\overline{\Pi(\varepsilon)})^2 - n\overline{\Pi(\varepsilon)^2}]$$

$$= (n-1)^{-1}[n\overline{\Pi(\varepsilon)}^2 - v(\Pi(\varepsilon_i))].$$

Turning to var $\bar{c}$, we observe that $\bar{c} = P(\bar{x}) - \hat{\Pi}(\varepsilon)/n$. Hence, from Lemma A

$$\text{var}\,\bar{c} = \text{var}\,P(\bar{x}) - \frac{2\,\text{cov}\{P(\bar{x}), \hat{\Pi}(\varepsilon)\}}{n} + \frac{\text{var}\{\hat{\Pi}(\varepsilon)\}}{n^2}$$

$$= \frac{1}{T}\frac{2}{n}\left[\frac{1}{n-1}\overline{\Pi(\varepsilon)}^2 - \frac{1}{n(n-1)}v(\Pi(\varepsilon_i)) + 2\overline{\Pi(\varepsilon)P(\bar{s})}\right.$$

$$- \frac{4}{n(n-1)}\sum_{i=1}^n \{\Pi(\varepsilon_i) - \overline{\Pi(\varepsilon)}\}\text{cro}(s_i, \bar{s})$$

$$\left. + \frac{2}{n(n-1)^2}\sum_{i=1}^n \Pi(\varepsilon_i)P(s_i - \bar{s})\right].$$

Summing up, we obtain the first result of the theorem, with:

$$r_1 = -\frac{1}{T}\frac{1}{n-1}\sum_{i=1}^n [\Pi(\varepsilon_i) - \overline{\Pi(\varepsilon)}]$$

$$\cdot\left[\left(\frac{n-2}{n-1}\right)^2 \frac{2}{n}\text{cro}(s_i, \bar{s}) + \frac{4}{n(n-1)^2}P(s_i - \bar{s})\right].$$

Consider now $E\tilde{v}$. We have

$$E\tilde{v} = \frac{1}{T}\left(\frac{n-2}{n-1}\right)^2 E\left[\hat{\Pi}(\varepsilon)\left\{P(\bar{x}) + \frac{2}{n(n-2)}\hat{\Pi}(\varepsilon)\right\}\right]$$

$$= \frac{1}{T}\left(\frac{n-2}{n-1}\right)^2 [E\hat{\Pi}(\varepsilon)]E\left[P(\bar{x}) + \frac{2}{n(n-2)}\hat{\Pi}(\varepsilon)\right]$$

$$+ \frac{1}{T}\left(\frac{n-2}{n-1}\right)^2\left[\text{cov}\{\hat{\Pi}(\varepsilon), P(\bar{x})\} + \frac{2}{n(n-2)}\text{var}\,\hat{\Pi}(\varepsilon)\right].$$

The first term of the right-hand side is equal to

$$\frac{1}{T}\left(\frac{n-2}{n-1}\right)^2 [\overline{\Pi(\varepsilon)} + V_s]\left[P(\bar{s}) + \frac{1}{n-2}\overline{\Pi(\varepsilon)} + \frac{2}{n(n-2)} V_s\right]$$

$$= \frac{1}{T}\frac{n-2}{n-1}\overline{\Pi(\varepsilon)}\left[\frac{n-2}{n-1}P(\bar{s}) + \frac{1}{n-1}\overline{\Pi(\varepsilon)}\right]$$

$$+ \frac{1}{T}\left[\frac{n^2-4}{n(n-1)^2}\overline{\Pi}(\varepsilon) + \left(\frac{n-2}{n-1}\right)^2 P(\bar{s}) + \frac{2(n-2)}{n(n-1)^2} V_s\right]V_s$$

and the second term, by Lemma A, is equal to

$$\frac{2}{T^2}\left[\frac{n-2}{n(n-1)^3}\overline{\Pi(\varepsilon)}^2 + \left(\frac{n-2}{n-1}\right)^2 \frac{2}{n^2} v(\Pi(\varepsilon_i)) + \frac{n-2}{n(n-1)^3}\overline{\Pi(\varepsilon)}V_s\right.$$

$$\left. + \frac{1}{n-1}\sum_{i=1}^n \{\Pi(\varepsilon_i) - \overline{\Pi(\varepsilon)}\}\left\{\left(\frac{n-2}{n-1}\right)^2 \frac{2}{n}\operatorname{cro}(\bar{s}, s_i) - \frac{2(n-2)}{n(n-1)^3} P(s_i - \bar{s})\right\}\right].$$

We then deduce the last result of the theorem, with

$$r_2 = \frac{1}{T^2}\frac{1}{n-1}\sum_{i=1}^n [\Pi(\varepsilon_i) - \overline{\Pi(\varepsilon)}]$$

$$\cdot \left[\left(\frac{n-2}{n-1}\right)^2 \frac{2}{n}\operatorname{cro}(s_i, \bar{s}) - \frac{2(n-2)}{n(n-1)^3} P(s_i - \bar{s})\right].$$

PROOF OF LEMMA 1. Since $\bar{x}_i = \bar{x} - (x_i - \bar{x})/(n-1)$, we have

$$c_i = \operatorname{cro}(x_i, \bar{x}) - \frac{1}{n-1}\operatorname{cro}(x_i, x_i - \bar{x})$$

$$= \frac{n-2}{n-1}\operatorname{cro}(x_i - \bar{x}, \bar{x}) - \frac{1}{n-1} P(x_i - \bar{x}) + P(\bar{x}).$$

Note that, under the standard model, the $x_i - \bar{x} = \varepsilon_i - \bar{\varepsilon}$ are distributed jointly independently to $\bar{x}$. Therefore, and since $P(\bar{x})$ does not depend on i, the distribution of $v(c_i)$ is the same as the distribution of

$$v\left\{\frac{n-2}{n-1}\operatorname{cro}(u_i - \bar{u}, \bar{x}) - \frac{1}{n-1} P(u_i - \bar{u})\right\}$$

where $u_i(t)$, $t = 1, \cdots, T$ are independent Gaussian white noise, with variance $\Pi(\varepsilon)$. Putting $U_i = (n-2)(n-1)^{-1}\operatorname{cro}(u_i, \bar{x})$, $P_i = P(u_i - \bar{u})$, we get the first result of the lemma. Since $u_i(t) - \bar{u}(t)$ has variance $[(n-1)/n]\Pi(\varepsilon)$, P_i is distributed as $(n-1)/n$ times a χ^2 variate with T degrees of freedom and hence has variance $T^{-1}(n-1)^2 n^{-2}\Pi(\varepsilon)^2$. On the other hand $EU_iP_i = 0$ since moments of third order between zero mean Gaussian variables vanish. Finally, $\operatorname{cov}(P_i, P_j)$

equals

$$\frac{1}{T^2} \sum_t \operatorname{cov}[\{u_i(t) - \bar{u}(t)\}^2, \{u_j(t) - \bar{u}(t)\}^2] = \frac{2}{T} \frac{1}{n^2} \Pi(\varepsilon)^2$$

since the $u_i(t) - \bar{u}(t)$ and $u_j(t) - \bar{u}(t)$ are jointly normal with zero mean and covariance $\Pi(\varepsilon)/n$.

PROOF OF THEOREM 3. We have

$$\hat{\Pi}(\varepsilon) = \frac{1}{n} \sum_{i=1}^{n} P(\varepsilon_i) - \frac{2}{n(n-1)} \sum\sum_{i<j} \operatorname{cro}(\varepsilon_i, \varepsilon_j)$$

and

$$\tilde{\Pi}(\varepsilon) = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} [P(\varepsilon_i) + P(\varepsilon_{i+1}) - 2 \operatorname{cro}(\varepsilon_i, \varepsilon_{i+1})].$$

Thus the difference $\hat{\Pi}(\varepsilon) - \tilde{\Pi}(\varepsilon)$ is

$$D = \frac{1}{n-1} \left[\frac{1}{2} P(\varepsilon_1) + \frac{1}{2} P(\varepsilon_n) - \frac{1}{n} \sum_{i=1}^{n} P(\varepsilon_i) + \left(1 - \frac{2}{n}\right) \sum_{i=1}^{n-1} \operatorname{cro}(\varepsilon_i, \varepsilon_{i+1}) - \frac{2}{n} \sum_{i=1}^{n-1} \sum_{j>i+1} \operatorname{cro}(\varepsilon_i, \varepsilon_j) \right].$$

Using the fact that the ε_i are independent zero mean Gaussian white noise, it can be checked that the terms in the above bracket are uncorrelated. Moreover

$$\operatorname{var} P(\varepsilon_i) = (2/T)\Pi(\varepsilon)^2$$

$$\operatorname{var} \operatorname{cro}(\varepsilon_i, \varepsilon_j) = (1/T^2) \sum_t \operatorname{var}[\varepsilon_i(t)\varepsilon_j(t)] = (1/T)\Pi(\varepsilon)^2.$$

Therefore

$$\operatorname{var} D = \frac{1}{T} \frac{1}{(n-1)^2} \cdot \left[4\left(\frac{1}{2} - \frac{1}{n}\right)^2 + 2\frac{n-2}{n^2} + \left(1 - \frac{2}{n}\right)^2 (n-1) + \frac{4}{n^2} \frac{(n-1)(n-2)}{2} \right] \Pi(\varepsilon)^2$$

$$= \frac{1}{T} \frac{n-2}{(n-1)^2} \Pi(\varepsilon)^2.$$

PROOF OF THEOREM 4. We first prove that $(n-1)\sqrt{T/(n-2)} \cdot D$ is asymptotically normal with mean 0 and variance $\Pi(\varepsilon)^2$. Since $nT \to \infty$, if n does not go to infinity, then T must go to infinity. In this case, from the central limit theorem, $\sqrt{T}P(\varepsilon_i)$, $\sqrt{T} \operatorname{cro}(\varepsilon_i, \varepsilon_j)$ converge jointly in distribution to independent normal variates with variances $2\Pi(\varepsilon)^2$, $\Pi(\varepsilon)^2$ respectively. The asymptotic normality of D follows. Thus, we need only consider the case $n \to \infty$. Now, from the proof of Theorem 3

$$(n-1)D = (1 - 2/n) \sum_{i=1}^{n-1} \operatorname{cro}(\varepsilon_i, \varepsilon_{i+1}) + r$$

where the variance of the residual term r can be bounded by C/T, C being a constant. Thus

$$\frac{(n-1)\sqrt{T}}{\sqrt{n-2}} D = \frac{\sqrt{(n-2)(n-1)}}{n} \sqrt{\frac{T}{n-1}} \sum_{i=1}^{n-1} \operatorname{cro}(\varepsilon_i, \varepsilon_{i+1}) + o(1)$$

where $o(1) \to 0$ in probability as $nT \to \infty$. By the central limit theorem, the first term of the above right-hand side converges in distribution as $nT \to \infty$ to a normal variate with zero mean and the first assertion is proved.

Now, it is easily seen that $\hat{\Pi}(\varepsilon)$ converges in probability to $\Pi(\varepsilon)$ as $nT \to \infty$. By Theorem 3 the same holds for $\tilde{\Pi}(\varepsilon)$, and hence $(n-1)\sqrt{T/(n-2)} \cdot D/\tilde{\Pi}(\varepsilon)$ converges in distribution to a standard normal variate. This closes the present proof.

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