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MONOTONICITY IN THE NONCENTRALITY PARAMETER OF THE RATIO OF TWO NONCENTRAL t-DENSITIES[1]

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Let $p_\nu(t, \delta)$ be the density at t of a noncentral t-variable with ν degrees of freedom and noncentrality parameter δ. It is proved that for any $d > 0$ and fixed t, $p_\nu(t, \delta + d)/p_\nu(t, \delta)$ is a strictly decreasing function of δ.

1. Statement and motivation. Let T have a noncentral t-distribution with ν degrees of freedom and noncentrality parameter δ, i.e., the distribution of $(Z + \delta)/(\nu^{-1}W)^{1/2}$, where Z and W are independent, $Z \sim N(0, 1)$, and $W \sim \chi^2_\nu$. Let $p_\nu(t, \delta)$ be the density of T at t. Let $d > 0$ be fixed and consider the density ratio $r(t, \delta) = p_\nu(t, \delta + d)/p_\nu(t, \delta)$. It will be shown that $r(t, \delta)$ is decreasing in δ for fixed t. This monotonicity property should not be confused with the well-known monotone likelihood ratio property, first proved by Kruskal (1954), which under the present circumstances implies that $r(t, \delta)$ is increasing in t for fixed δ. We restate our result in the following.

LEMMA. *Let $p_\nu(t, \delta)$ be the density at t of a noncentral t-variable with degrees of freedom $\nu \geq 1$ and noncentrality parameter δ, $-\infty < \delta < \infty$. Then for any $d > 0$, $p_\nu(t, \delta + d)/p_\nu(t, \delta)$ is a strictly decreasing function of δ.*

The proof will be given in Section 2. It will be convenient there to replace T by $U = T/(\nu + T^2)^{1/2}$. Obviously, $|U| < 1$. The ratio of densities is of course invariant under this transformation. If T arises from a sample $X_1, \cdots, X_n$ from $N(\mu, \sigma^2)$, then $\nu = n - 1$, $\delta = \sqrt{n}\,\mu/\sigma$, $T = \sqrt{n}\,\bar{X}_n/s_n$ where $s_n^2 = (n-1)^{-1}\sum (X_i - \bar{X}_n)^2$, and $U = \bar{X}_n/(n^{-1}\sum X_i^2)^{1/2}$.

As an application of the Lemma consider the construction of a sequential confidence interval for $\gamma = \mu/\sigma$ in a $N(\mu, \sigma^2)$ population from a family of sequential t-tests. Specifically, let $X_1, X_2, \cdots$ be iid $N(\mu, \sigma^2)$ and test γ versus $\gamma + d$, for each γ, at level α and power $1 - \beta$, where α, β (both small) and $d > 0$ are chosen in advance. The Wald SPRT based on the sequence of probability ratios $R_2(\gamma), R_3(\gamma), \cdots$ of t-statistics chooses stopping bounds A, B depending on α, β, and stops at $N =$ smallest integer $n \geq 2$ such that $R_n(\gamma) \leq B$ in which case γ is accepted, or $\geq A$ in which case γ is rejected. The corresponding confidence set includes (excludes) those γ's that are accepted (rejected) at sampling stage n among the γ's that were not decided yet at any earlier stage. The monotonicity of $R_n(\gamma)$ in γ implies that the inequalities $R_n(\gamma) \leq B$ and $R_n(\gamma) \geq A$ define disjoint half-infinite intervals for γ, and therefore guarantees that the resulting confidence set is an interval of the form $[\gamma_0, \infty)$. The possibility of constructing a sequential confidence set from a family of sequential tests was discussed in Wijsman (1981).

2. Proof of the Lemma. As remarked before, $p_\nu(t, \delta + d)/p_\nu(t, \delta)$ equals the corresponding ratio of densities of $U = T/(\nu + T^2)^{1/2}$ at $u = t/(\nu + t^2)^{1/2}$. Writing the latter ratio as $r_\nu(u, \delta)$, it suffices to show that $\log r_\nu(u, \delta)$ is strictly decreasing in δ. In order to write down an expression for this quantity it is convenient to introduce the function

$$f_\nu(x) = \int_0^\infty \exp[-\tfrac{1}{2}z^2 + xz]z^\nu\,dz, \quad -\infty < x < \infty, \quad \nu > -1. \tag{1}$$

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It is defined for all real $\nu > -1$, even though for statistical applications it is only needed for integer $\nu \geq 1$. This function is closely related to the function Hh_ν defined in Airey, Irwin, and Fisher (1931): $f_\nu(x) = \Gamma(\nu + 1)\exp(\frac{1}{2}x^2)Hh_\nu(-x)$. (The name "Hermitian probability functions" for the functions Hh seems to have been adopted since the 1946 second edition.) In terms of (1) one has

$$\log r_\nu(u, \delta) = -\tfrac{1}{2}(\delta + d)^2 + \tfrac{1}{2}\delta^2 + \log f_\nu((\delta + d)u) - \log f_\nu(\delta u). \tag{2}$$

This may be obtained for instance from Kruskal (1954), equation (4.2), or from Johnson and Kotz (1970), Chapter 31, equation (6) (the latter equation has a few small misprints). We shall show that $\log r_\nu(u, \delta)$ has a negative derivative with respect to δ. Using (2), this derivative equals

$$-d + uf_\nu'((\delta + d)u)/f_\nu((\delta + d)u) - uf_\nu'(\delta u)/f_\nu(\delta u). \tag{3}$$

Since $|u| < 1$, the claim will follow if it is shown that

$$|f_\nu'((\delta + d)u)/f_\nu((\delta + d)u) - f_\nu'(\delta u)/f_\nu(\delta u)| < d. \tag{4}$$

This, in turn, will be true if the function $f_\nu'(x)/f_\nu(x)$ has a derivative bounded in absolute value by 1. Indeed, it will be shown that

$$|f_\nu f_\nu'' - (f_\nu')^2| < f_\nu^2, \quad \nu \geq 0. \tag{5}$$

Note that from (1) it follows that $f_\nu' = f_{\nu+1}$. It will be shown now that

$$f_\nu f_{\nu+2} - f_{\nu+1}^2 > 0, \quad \nu > -1, \tag{6}$$

and

$$f_\nu f_{\nu+2} < f_\nu^2 + f_{\nu+1}^2, \quad \nu \geq 0. \tag{7}$$

Then (6) and (7) together imply (5).

Inequality (6), which is the same as $f_\nu f_\nu'' - (f_\nu')^2 > 0$, is equivalent to the statement that $\log f_\nu$ is strictly convex. That this is indeed true is easily seen by observing that $f_\nu(x)$ is of the form $\int \exp(xz)\mu(dz) \equiv \exp[b(x)]$, say, and considering $\exp[xz - b(x)]\mu(dz)$ a one-parameter exponential family indexed by the parameter x. It is well-known that in such a family the function $b(\cdot)$ is strictly convex, provided the measure μ is not supported on one point. (Note also that (1) defines the moment generating function of a χ-distribution with $\nu + 1$ degrees of freedom, except for a multiplicative factor.) Thus, (6) has been verified.

After replacing ν by $\nu + 2$ in (1) and a partial integration one obtains the recurrence relation

$$f_{\nu+2}(x) = (\nu + 1)f_\nu(x) + xf_{\nu+1}(x), \quad \nu > -1. \tag{8}$$

Substituting (8) into (7), the latter is equivalent to

$$[-x + f_{\nu+1}(x)/f_\nu(x)]f_{\nu+1}(x)/f_\nu(x) > \nu, \quad \nu \geq 0. \tag{9}$$

In order to prove (9), consider first the special case $\nu = 0$. By direct computation, $f_1(x) - xf_0(x) = 1$, so that (9) holds for $\nu = 0$. Now let $\nu > 0$. Then (8) can be used once more with ν replaced by $\nu - 1$:

$$-x + f_{\nu+1}(x)/f_\nu(x) = \nu f_{\nu-1}(x)/f_\nu(x), \quad \nu > 0. \tag{10}$$

After substitution of (10) into (9) the latter is equivalent to $f_{\nu+1}f_{\nu-1} > f_\nu^2$, $\nu > 0$. But this is (6), which has been shown to hold. □

REMARK. It is not known to this writer whether (7) (or the equivalent (9)) holds for $-1 < \nu < 0$. Of course, for statistical applications, only integer values of $\nu \geq 1$ matter.

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