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SIMULTANEOUS INTERVAL ESTIMATION IN THE GENERAL MULTIVARIATE ANALYSIS OF VARIANCE MODEL

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Let $M \in M(m, p)$ be the matrix of means of interest in the GMANOVA problem. Our main results characterize all confidence sets for M in a given class (invariant plus a weak additional restriction) that are exact for the families of parametric functions $a'Mb$ for all $a \in \mathbb{R}^m$, $b \in \mathbb{R}^p$ and tr $N'M$ for all $N \in M(m, p)$. The corresponding families of smallest exact simultaneous confidence intervals are also given. Similar results are obtained for the MANOVA problem under triangular group reduction.

1. Introduction. Wijsman (1979, 1980) developed a general method for constructing smallest simultaneous confidence sets for parametric functions and applied this method to the MANOVA problem, as well as to several other problems in multivariate analysis. The notation, definitions, and results of Wijsman (1980) are assumed known. Our main purpose is to extend Wijsman's results to the general multivariate analysis of variance (GMANOVA) model of Potthoff and Roy (1964). Along with others, the confidence set determined by Roy's maximum root criterion is found to be exact for the families of parametric functions considered. A class of functions used in our characterizations is described in Section 2. Some new results on the MANOVA problem are given in Section 4.

2. Upper and lower self-reproducing functions. In this section we define a class of pairs of indicator functions used in characterizing exact confidence sets and corresponding smallest exact simultaneous confidence intervals. Let $\mathbb{R}^n \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^n$.

DEFINITION 2.1. If ℓ and u are functions mapping $\mathbb{R}^n \cup \{\infty\}$ into $\{0, 1\}$ then ℓ is *lower self-reproducing*, u is *upper self-reproducing*, and ℓ and u are *related*, provided

(2.1a) $$\ell(x) = \max\{u(y) : x'y \geq 1\} \quad \text{and}$$

(2.1b) $$u(x) = \min\{\ell(y) : x'y \geq 1\}$$

for all $x \in \mathbb{R}^n \cup \{\infty\}$.

In the above definition we adopt the convention $r \cdot \infty = r/0 = r \pm \infty = \infty$ for all $r \in \mathbb{R}$, so that $\ell(0) = u(\infty)$ and $u(0) = \ell(\infty)$. Definition 2.1 is extended to include functions of matrices $A \in M(m, n)$ by treating these as functions of the (mn) column vector vec A. Note that (vec A)$'$ vec B = tr $A'B$.

We observe immediately from (2.1) that ℓ (u, respectively) is increasing (decreasing) along rays emanating from the origin. Note also that if (ℓ, u) satisfies (2.1) then so does (ℓ_1, u_1), where $\ell_1 = 1 - u$ and $u_1 = 1 - \ell$. Thus ℓ is lower self-reproducing if and only if $1 - \ell$ is upper self-reproducing.

LEMMA 2.1. *The following are equivalent*: (i) *u is upper self-reproducing*; (ii) *for some* $g: \mathbb{R}^n \cup \{\infty\} \to \{0, 1\}$, $u(x) = \min\{g(y): x'y \geq 1\}$ *for all* $x \in \mathbb{R}^n \cup \{\infty\}$; (iii) $u(x) = \min_{x'y \geq 1} \max_{y'z \geq 1} u(z)$ *for all* $x \in \mathbb{R}^n \cup \{\infty\}$.

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PROOF. That (i) $\Rightarrow$ (ii) is immediate. That (ii) $\Rightarrow$ (iii) follows from the fact that we have $u \le \min \max u$ for arbitrary u and $g \ge \max \min g$ for arbitrary g. To see that (iii) $\Rightarrow$ (i), define ℓ by (2.1a) and observe that (2.1b) holds. □

REMARK 2.1. For $x \notin \{0, \infty\}$, let $\mathscr{H}(x)$ denote the set of closed half-spaces not containing 0 determined by a hyperplane passing through x. Lemma 2.1 shows that u is upper self-reproducing if and only if $u(0) = \max\{u(z) : z \in \mathbb{R}^n \cup \{\infty\}\}$ and

$$u(x) = \min_{H \in \mathscr{H}(x)} \max_{z \in H} u(z) \quad \text{for all} \quad x \notin \{0, \infty\}. \tag{2.2}$$

THEOREM 2.1. *Let u be the indicator function of C, where $0 \in C \subseteq \mathbb{R}^n$. If u is upper self-reproducing then C is convex. If C is convex and either open, closed, or strictly convex then u is upper self-reproducing.*

PROOF. Suppose (ℓ, u) satisfies (2.1) and $u(x) = u(z) = 1$. Then we have $\ell(y) = 0$ implies $\max\{x'y, z'y\} < 1$, which implies $(\alpha x + (1 - \alpha)z)'y < 1$ for $0 \le \alpha \le 1$. So we have $u(\alpha x + (1 - \alpha)z) = 1$.

Conversely, suppose C is convex and either open, closed, or strictly convex. We show that u satisfies (2.2). Since we have $u \le \min \max u$ for u arbitrary, it suffices to establish (2.2) when $u(x) = 0$; i.e., for $x \notin C$, $x \ne \infty$, we must show that there exists $H \in \mathscr{H}(x)$ with $H \cap C$ empty. This follows, with a small argument, from the fact that $0 \in C$ and from the Separating Hyperplane Theorem; see, e.g., Ferguson (1967). □

EXAMPLE 2.1. Put $\mathbb{B}^p = \{y \in \mathbb{R}^p : \|y\| \le 1\}$ and let $A \in M(n, p)$. If u is the indicator function of the n-dimensional ellipsoid $A\mathbb{B}^p \equiv \{Ay : y \in \mathbb{B}^p\}$ then u is upper self-reproducing with related ℓ the indicator function of $\{x \in \mathbb{R}^n : x'AA'x \ge 1\}$. We note that if AA' is nonsingular then $A\mathbb{B}^p = \{x \in \mathbb{R}^n : x'(AA')^{-1}x \le 1\}$.

EXAMPLE 2.2. Partition $x \in \mathbb{R}^n$ as $x' = (x_1', \cdots, x_k')$, where $x_i \in \mathbb{R}^{n_i}$ and $\sum n_i = n$. If u is the indicator function of $\{x \in \mathbb{R}^n : \|x_i\| \le c_i,\ i = 1, \cdots, k\}$ then u is upper self-reproducing with related ℓ the indicator function of $\{x \in \mathbb{R}^n : \sum c_i \|x_i\| \ge 1\}$.

3. GMANOVA. Following Marden (1980) and Hooper (1982), we consider the GMANOVA model in a partially reduced canonical form:

$$[X_1 : X_2] \sim N_{m \times (p+q)}([M : 0], I_m \otimes \Sigma), \quad S \sim W_{p+q}(\nu, \Sigma)$$

with $[X_1 : X_2]$ and S independent; i.e., the m rows of $[X_1 : X_2]$ are independent multivariate normal with common covariance matrix Σ and S has a Wishart distribution with mean $\nu\Sigma$. We assume that Σ is positive definite and that we have $\nu \ge p + q$. Put $X = (X_1, X_2, S)$.

Consider the group $G = M(m, p) \times \mathscr{A} \times O(m)$, where $\mathscr{A} \subseteq GL(p + q)$ consists of all lower block triangular matrices $A = (A_{ij})$ with $A_{11} \in GL(p)$, $A_{22} \in GL(q)$, and $A_{12} = 0$. The group actions are $[X_1 : X_2] \to \Gamma'[X_1 : X_2]A + [F : 0]$, $S \to A'SA$, $M \to \Gamma'MA_{11} + F$, $\Sigma \to A'\Sigma A$ for $(F, A, \Gamma) \in G$. Partition $S = (S_{ij})$ with S_{11} $p \times p$ and S_{22} $q \times q$. A maximal invariant function of (X, M) under $M(m, p) \times \mathscr{A}$ is

$$(T_1, T_2) = (X_{1\cdot 2} S_{11\cdot 2}^{-1} X_{1\cdot 2}', X_2 S_{22}^{-1} X_2')$$

where

$$X_{1\cdot 2} = (I_m + T_2)^{-1/2}(X_1 - M - X_2 S_{22}^{-1} S_{21}), \quad S_{11\cdot 2} = S_{11} - S_{12} S_{22}^{-1} S_{21}.$$

The action of $O(m)$ on (T_1, T_2) is $T_1 \to \Gamma' T_1 \Gamma$, $T_2 \to \Gamma' T_2 \Gamma$. A maximal invariant is $(\Gamma_2' T_1 \Gamma_2, \lambda(T_2))$, where $\lambda(T_2)$ is the vector of ordered eigenvalues of T_2 and the columns of $\Gamma_2 \in O(m)$ are the eigenvectors of T_2; i.e., $\Gamma_2' T_2 \Gamma_2 = \mathrm{diag}(\lambda(T_2))$. Put

$$W = \Gamma_2'(I_m + T_2)^{-1/2}(X_1 - M - X_2 S_{22}^{-1} S_{21}) S_{11\cdot 2}^{-1/2}. \tag{3.1}$$

Since the group G acts transitively on the parameter space, the maximal invariant $(WW', \lambda(T_2))$ is an invariant pivotal quantity; see Wijsman (1980, Lemma 3.1). From Lemmas 3.1 and 3.3 of Kariya (1978), we have $X_{1\cdot 2} \sim N_{m\times p}(0, I_m \otimes \Sigma_{11\cdot 2})$, $S_{11\cdot 2} \sim W_p(\nu - q, \Sigma_{11\cdot 2})$ and $X_{1\cdot 2}$, $S_{11\cdot 2}$, and T_2 are independent. Now Γ_2 is a random orthogonal matrix depending only on T_2, so $\Gamma_2' X_{1\cdot 2}$ has the same distribution as $X_{1\cdot 2}$ and is independent of T_2. Consequently WW' is independent of T_2 with the same distribution as T_1. The density of WW' for $m \le p$ is given in Theorem 4.2 of Olkin and Rubin (1964).

Our results will be given in terms of indicator functions $\phi = \phi(X, M)$ of confidence sets. We write $C_\phi(X, \cdot) = \{M : \phi(X, M) = 1\}$. A set estimator ϕ is invariant under G if and only if

$$\phi(X, M) = F(WW', \lambda(T_2)) \tag{3.2}$$

for some function F. We restrict attention to invariant set estimators. In addition we require that set estimators ϕ for M be *star-shaped*, meaning that the sets $C_\phi(X, \cdot)$ are star-shaped about $X_1 - X_2 S_{22}^{-1} S_{21}$, the usual point estimate for M; i.e., $M \in C_\phi(X, \cdot)$ implies $\alpha M + (1 - \alpha)(X_1 - X_2 S_{22}^{-1} S_{21}) \in C_\phi(X, \cdot)$ for all $0 \le \alpha \le 1$. If ϕ satisfies (3.2) then ϕ is star-shaped if and only if $F(\alpha WW', \lambda)$ is decreasing in $\alpha \ge 0$ for each fixed value of (WW', λ). The star-shaped restriction allows a much simpler characterization of exact set estimators. A characterization without this restriction is derived in Lemma 4.4.1 of Wijsman (1980) for the MANOVA problem under triangular group reduction. A corresponding result for the GMANOVA problem is given in Hooper (1981). A complete class theorem of Marden (1980) implies that invariant set estimators which are not star-shaped are inadmissible within the class of invariant set estimators.

Wijsman (1979, 1980) considered the following families of parametric functions: $\{a'M : a \in \mathbb{R}^m\}$, $\{Mb : b \in \mathbb{R}^p\}$, and $\{\operatorname{tr} N'M : N \in M^r(m, p)\}$, where $M^r(m, p)$ is the set of $m \times p$ matrices of rank at most r. Writing $N = ab'$ shows that $\{\operatorname{tr} N'M : N \in M^1(m, p)\} = \{a'Mb : a \in \mathbb{R}^m, b \in \mathbb{R}^p\}$. Simultaneous confidence regions for the vector-valued functions $a'M$ and Mb seem to be of interest primarily for their application in related simultaneous testing problems. Our results presented here concern the families of real-valued functions $\operatorname{tr} N'M$, which are more directly useful for estimation. Results for $\{a'M\}$ and $\{Mb\}$ are given in Hooper (1981), where the following implications are established for invariant star-shaped set estimators: ϕ is exact for $\{a'Mb\}$ if and only if ϕ is exact for $\{a'M\}$; if ϕ is exact for $\{a'Mb\}$ then ϕ is exact for $\{Mb\}$; if ϕ is exact for $\{a'Mb\}$ then ϕ is exact for $\{\operatorname{tr} N'M : N \in M^r(m, p)\}$ for all $r \ge 1$. The last statement is a consequence of the following general result.

LEMMA 3.1. *Let Y have distribution P_θ, let $\gamma = \gamma(\theta)$ be a parameter of interest, and let $\{\psi_i : i \in I_2\}$ be a family of functions of γ. If we have $I_1 \subseteq I_2$ and $\phi = \phi(Y, \gamma)$ is exact for $\{\psi_i : i \in I_1\}$ then ϕ is exact for $\{\psi_i : i \in I_2\}$.*

PROOF. For an arbitrary index set I, let $\zeta = \zeta(Y, \psi, i)$ be a simultaneous set estimator for $\{\psi_i : i \in I\}$; i.e., for each $i \in I$, $\{\psi : \zeta(Y, \psi, i) = 1\}$ is a confidence set for $\psi_i(\gamma)$. The T and T^{-1} operations relative to $\{\psi_i : i \in I\}$ are defined as follows:

$$\begin{aligned} \zeta = T\phi \quad &\text{if} \quad \zeta(Y, \psi, i) = \max\{\phi(Y, \gamma) : \psi_i(\gamma) = \psi\}, \\ \phi = T^{-1}\zeta \quad &\text{if} \quad \phi(Y, \gamma) = \min\{\zeta(Y, \psi_i(\gamma), i) : i \in I\}. \end{aligned} \tag{3.3}$$

Theorem 2.3 of Wijsman (1980) shows that ϕ is exact for $\{\psi_i : i \in I\}$ if and only if ϕ is self-reproducing relative to T and T^{-1}; i.e., $\phi = T^{-1}T\phi$. Let T_j and T_j^{-1} denote, respectively, the T and T^{-1} operations relative to $\{\psi_i : i \in I_j\}$, $j = 1, 2$. We must show that $\phi = T_2^{-1}T_2\phi$. By assumption $\phi = T_1^{-1}T_1\phi$, so

$$\begin{aligned} \phi(Y, \gamma_0) &= \min_{i \in I_1} \max\{\phi(Y, \gamma) : \psi_i(\gamma) = \psi_i(\gamma_0)\} \\ &\ge \min_{i \in I_2} \max\{\phi(Y, \gamma) : \psi_i(\gamma) = \psi_i(\gamma_0)\}, \end{aligned}$$

which gives $\phi \ge T_2^{-1}T_2\phi$. But $\phi \le T_2^{-1}T_2\phi$ holds for ϕ arbitrary. □

The following lemma concerns the usual partial ordering of nonnegative definite symmetric matrices: $\Sigma_1 \le \Sigma_2$ if $\Sigma_2 - \Sigma_1$ is nonnegative definite. The proof is not difficult and is omitted.

LEMMA 3.2. *Given $A \in M(m, r)$ and $B \in M(m, p)$ with $r \le p$, the following are equivalent*: (i) $AA' \le BB'$; (ii) $A = B\Omega$ *for some* $\Omega \in M(p, r)$ *with* $\Omega'\Omega \le I_r$; (iii) A *lies in the convex hull of* $\{B\Omega : \Omega \in M(p, r) : \Omega'\Omega = I_r\}$; (iv) $A\mathbb{B}^r \subseteq B\mathbb{B}^p$.

THEOREM 3.1. *Fix $1 \le r \le \min(m, p)$. If ϕ is an invariant star-shaped set estimator for M then ϕ is exact for $\{\operatorname{tr} N'M : N \in M^r(m, p)\}$ if and only if*

$$\phi(X, M) = \min\{u(B, \lambda(T_2)) : B \in M(m, r),\ BB' \le WW'\} \tag{3.4}$$

where, for each λ, the function $u(\cdot, \lambda)$ defined on $M(m, r)$ is upper self-reproducing and right invariant under $O(r)$. The simultaneous set estimator ζ that is smallest exact with respect to ϕ is given by:

$$\zeta(X, \operatorname{tr} N'M, N) = \ell(\{\operatorname{tr} N'(X_1 - M - X_2 S_{22}^{-1} S_{21})\}^{-1}A, \lambda(T_2)) \tag{3.5}$$

for any $A \in M(m, r)$ satisfying $AA' = \Gamma_2'(I_m + T_2)^{1/2} N S_{11\cdot 2} N'(I_m + T_2)^{1/2}\Gamma_2$, where $\ell(\cdot, \lambda)$ is the lower self-reproducing function related to $u(\cdot, \lambda)$.

PROOF. The result is obtained by applying Theorem 2.3 of Wijsman (1980). Referring to (3.1), we observe that, given the data X, a confidence statement about M is equivalent to one about W. More precisely we define the correspondence: $\tilde{\phi}(X, W) = \phi(X, M)$, $\tilde{\zeta}(X, \operatorname{tr} \tilde{N}'W, \tilde{N}) = \zeta(X, \operatorname{tr} N'M, N)$, $\tilde{N} = \Gamma_2'(I_m + T_2)^{1/2} N S_{11\cdot 2}^{1/2}$. Observe that $\tilde{\zeta} = T\tilde{\phi}$ if and only if $\zeta = T\phi$, $\tilde{\phi} = T^{-1}\tilde{\zeta}$ if and only if $\phi = T^{-1}\zeta$, and so $\tilde{\phi}$ is self-reproducing if and only if ϕ is self-reproducing. It is convenient to work with $\tilde{\phi}$ and $\tilde{\zeta}$ rather than with ϕ and ζ. Note from (3.2) and (3.3) that the T and T^{-1} operations involve only the first argument of F. For notational convenience we suppress the second argument, $\lambda(T_2)$, as well as the tildes. Finally it is useful to work with functions of W instead of WW'. So we have $\phi(X, W) = F(W)$ where F is right invariant under $O(p)$ and, by the star-shaped condition, F is decreasing along rays (where this is understood to mean along rays emanating from the origin).

We begin by assuming that $\phi = T^{-1}T\phi$ and derive necessary conditions on the form of ϕ and $\zeta = T\phi$. Applying the T operation (3.3) to $\phi(X, W) = F(W)$ yields

$$\zeta(X, \psi, N) = \max\{F(W) : \operatorname{tr} N'W = \psi\} = \max\{F(W) : \psi^{-1}\operatorname{tr} N'W \ge 1\}$$

since F is decreasing along rays. Observe that $\zeta(X, \pm\psi, N\Gamma) = \zeta(X, \psi, N)$ for all $\Gamma \in O(p)$ since $\operatorname{tr}(N\Gamma)'W = \operatorname{tr} N'W\Gamma'$ and F is right invariant under $O(p)$. For each $N \in M^r(m, p)$ there exists $A \in M(m, r)$ such that $NN' = AA'$, or equivalently, $N = [A : 0]\Gamma$ for some $\Gamma \in O(p)$. Thus

$$\zeta(X, \operatorname{tr} N'W, N) = \ell(\{\operatorname{tr} N'W\}^{-1}A) \tag{3.6}$$

for any $A \in M(m, r)$ satisfying $AA' = NN'$, where ℓ is defined by

$$\ell(A) = \max\{F(W) : \operatorname{tr}[A : 0]'W \ge 1\}, \quad A \ne 0. \tag{3.7}$$

If $N = 0$ then $\{\operatorname{tr} N'W\}^{-1}A = \infty$ so 0 never appears as the argument of ℓ in (3.6). We define $\ell(0) = 0$. Note that ℓ is right invariant under $O(r)$ and increasing along rays.

Since ϕ was assumed to be self-reproducing, applying the T^{-1} operation (3.3) to (3.6) produces

$$\begin{aligned} F(W) &= \min\{\ell(\{\operatorname{tr} \Gamma'[A : 0]'W\}^{-1}A) : A \in M(m, r), \Gamma \in O(p)\} \\ &= \min\{\ell(\{\operatorname{tr} A'W\Omega\}^{-1}A) : A \in M(m, r), \Omega'\Omega = I_r\} \\ &= \min\{\ell(A) : \operatorname{tr} A'W\Omega \ge 1, \Omega'\Omega = I_r\}. \end{aligned}$$

The second equality follows by writing $\Gamma' = [\Omega : \Omega_2]$ and the third from ℓ increasing along rays. Thus $F(W) = \min\{u(W\Omega) : \Omega'\Omega = I_r\}$ where u is defined on $M(m, r)$ by

$$u(B) = \min\{\ell(A) : \operatorname{tr} A'B \geq 1\}. \tag{3.8}$$

Note that u is right invariant under $O(r)$ (since ℓ is) and, by Lemma 2.1, u is upper self-reproducing. Theorem 2.1 shows that $\{B : u(B) = 1\}$ is convex. An application of Lemma 3.2 then shows that

$$F(W) = \min\{u(B) : B \in M(m, r), BB' \leq WW'\} \tag{3.9}$$

and that $AA' \leq BB'$ implies $u(A) \geq u(B)$. Note that $F([B:0]) = u(B)$ for $B \in M(m, r)$. To show that ℓ is related to u, we refer to (3.7):

$$\begin{aligned} \ell(A) &= \max\{F([B : W_1]) : \operatorname{tr} A'B \geq 1,\ W_1 \in M(m, p-r)\} \\ &= \max\{F([B : 0]) : \operatorname{tr} A'B \geq 1\} = \max\{u(B) : \operatorname{tr} A'B \geq 1\}. \end{aligned} \tag{3.10}$$

The second equality follows from (3.9); i.e., we have $F([B : W_1]) \leq F([B:0])$. This establishes the necessity of the stated conditions.

Conversely, suppose $\phi(X, W) \equiv F(W)$ and ζ are defined by (3.9) and (3.6) where u is upper self-reproducing and right invariant under $O(r)$ and ℓ is the lower self-reproducing function related to u. Then $\zeta = T\phi$ follows from (3.10) and $\phi = T^{-1}\zeta$ from the fact that $T^{-1}\zeta$ was computed to be the right-hand side of (3.9) with u defined by (3.8). □

The following two corollaries are obtained by applying Lemma 3.2 and Theorem 3.1 with $r = 1$.

COROLLARY 3.1. *If ϕ is an invariant star-shaped set estimator for M then ϕ is exact for $\{a'Mb : a \in \mathbb{R}^m, b \in \mathbb{R}^p\}$ if and only if*

$$\phi(X, M) = \min\{u(x, \lambda(T_2)) : x \in W\mathbb{B}^p\}$$

where, for each λ, the function $u(\cdot, \lambda)$ defined on $\mathbb{R}^m$ is upper self-reproducing and symmetric under reflection through the origin. The simultaneous set estimator ζ that is smallest exact with respect to ϕ is given by:

$$\zeta(X, a'Mb, (a, b)) = \ell(\{a'(X_1 - M - X_2S_{22}^{-1}S_{21})b\}^{-1}\{b'S_{11.2}b\}^{1/2}\Gamma_2'(I_m + T_2)^{1/2}a, \lambda(T_2))$$

where $\ell(\cdot, \lambda)$ is the lower self-reproducing function related to $u(\cdot, \lambda)$.

COROLLARY 3.2. *Let ϕ be a set estimator for M of the form $\phi(X, M) = F(WW', \lambda(T_2))$ and suppose that $F(WW', \lambda) \leq F(xx', \lambda)$ for all $x \in W\mathbb{B}^p$. Let $C(\lambda)$ be the convex hull of the closure of $\{x \in \mathbb{R}^m : F(xx', \lambda) = 1\}$. The smallest closed confidence set containing $C_\phi(X, \cdot)$ that is exact for $\{a'Mb : a \in \mathbb{R}^m, b \in \mathbb{R}^p\}$ is given by $\{M : W\mathbb{B}^p \subseteq C(\lambda)\}$.*

The following corollary describes the class of exact confidence sets based on the eigenvalues of T_1 and T_2. The proof is given in Hooper (1981). An equivalent characterization, using symmetric gauge functions, follows easily from Wijsman (1979, Theorem 4.2) by observing the similarity between $\lambda(T_1)$ and the pivotal quantity arising in the MANOVA problem under full invariance reduction. Let $\bar{\mathbb{R}}^s_{0+}$ denote the closed ordered positive cone: $\bar{\mathbb{R}}^s_{0+} = \{x \in \mathbb{R}^s : x_1 \geq \cdots \geq x_s \geq 0\}$. Let $d(W) \in \bar{\mathbb{R}}^s_{0+}$, $s = \min(m, p)$, denote the vector of ordered singular values of W; i.e., $d_i(W) = \{\lambda_i(WW')\}^{1/2} = \{\lambda_i(T_1)\}^{1/2}$. For $x = (x_1, \cdots, x_s)' \in \mathbb{R}^s$ and $r \leq s$ define $x^r = (x_1, \cdots, x_r)'$. Let G_s be the group of sign changes and permutations of the coordinates x_i of $x \in \mathbb{R}^r$.

COROLLARY 3.3. *Fix $1 \leq r \leq s$. If ϕ is a star-shaped set estimator for M of the form $\phi(X, M) = F(\lambda(T_1), \lambda(T_2))$ then ϕ is exact for $\{\operatorname{tr} N'M : N \in M^r(m, p)\}$ if and only if*

$$\phi(X, M) = u(d(W)^r, \lambda(T_2))$$

where, for each λ, *the function* $u(\cdot, \lambda)$ *defined on* $\mathbb{R}^r$ *is upper self-reproducing and invariant under* G_s. *The simultaneous set estimator* ζ *that is smallest exact with respect to* ϕ *is given by*

$$\zeta(X, \operatorname{tr} N'M, N) = \ell(\{\operatorname{tr} N'(X_1 - M - X_2S_{22}^{-1}S_{21})\}^{-1} d((I_m + T_2)^{1/2}NS_{11\cdot 2}^{1/2})^r, \lambda(T_2))$$

where $\ell(\cdot, \lambda)$ *is the lower self-reproducing function related to* $u(\cdot, \lambda)$.

REMARK 3.1. If in the above corollary $C_\phi(X, \cdot)$ is closed then ϕ is exact if and only if

$$C_\phi(X, \cdot) = \{M : d(W)^r \in C(\lambda(T_2))\} \tag{3.11}$$

where, for each λ, the set $C(\lambda) \subseteq \bar{\mathbb{R}}_{0+}^r$ is closed, convex, and monotone with respect to weak submajorization; see Marshall and Olkin (1979) for several equivalent definitions of weak submajorization. The above statement is proved by observing that the symmetric extension of $C \subseteq \bar{\mathbb{R}}_{0+}^r$ to $\mathbb{R}^r$ is convex if and only if C is convex and monotone with respect to weak submajorization.

When $r = 1$ the set (3.11) must take the form

$$\{M : \lambda_1(T_1) \leq c(\lambda(T_2))\}. \tag{3.12}$$

The following family of simultaneous confidence intervals is smallest exact with respect to (3.12):

$$\{a'Mb : (a'(X_1 - M - X_2S_{22}^{-1}S_{21})b)^2 \leq a'(I_m + T_2)ab'S_{11\cdot 2}bc(\lambda(T_2))\} \tag{3.13}$$

for all $a \in \mathbb{R}^m$, $b \in \mathbb{R}^p$. Roy's maximum root criterion is (3.12) with $c(\lambda)$ constant. The simultaneous confidence intervals (3.13) with $c(\lambda)$ constant were derived by Khatri (1966).

EXAMPLE 3.1. Define $T = (X_1 - M, X_2)S^{-1}(X_1 - M, X_2)'$ and consider the confidence set

$$\{M : \lambda_1(T) \leq c\}. \tag{3.14}$$

Marden (1980) proved that (3.14) is admissible within the class of invariant confidence sets. Some algebraic manipulation shows that $T = (I_m + T_2)^{1/2}T_1(I_m + T_2)^{1/2} + T_2$ and that (3.14) can be rewritten as

$$\{M : WW' \leq (I_m + D_2)^{-1/2}(cI_m - D_2)(I_m + D_2)^{-1/2}\} \tag{3.15}$$

where $D_2 = \operatorname{diag}(\lambda_1(T_2), \cdots, \lambda_m(T_2))$. Corollary 3.2 shows that (3.15) is exact for $\{a'Mb\}$. We observe that (3.15) is the empty set when $\lambda_1(T_2) > c$.

EXAMPLE 3.2. The confidence set determined by the locally most powerful invariant test of Kariya (1978) may be expressed in the form

$$\{M : \operatorname{tr}(I_m + D_2)^{-1}[c_0 WW'(I_m + WW')^{-1} - I_m] \leq c\}, \tag{3.16}$$

where $c_0 = (m + \nu - q)/p$. Note that we have $c_0 > 1$. Corollary 3.2 is applied to construct the smallest confidence set containing (3.16) that is exact for $\{a'Mb\}$. Using the formula $(I_m + xx')^{-1} = I_m - (1 + x'x)^{-1}xx'$, $x \in \mathbb{R}^m$, one may easily verify that (3.16) satisfies the conditions on $C_\phi(X, \cdot)$ in Corollary 3.2 and that

$$\begin{aligned}\{x \in \mathbb{R}^m &: \operatorname{tr}(I_m + D_2)^{-1}[c_0xx'(I_m + xx')^{-1} - I_m] \leq c\} \\ &= \{x \in \mathbb{R}^m : \textstyle\sum_{i=1}^m [c_0\eta_i - c - \sum_{j=1}^m \eta_j]x_i^2 \leq c + \sum_{j=1}^m \eta_j\},\end{aligned} \tag{3.17}$$

where $\eta_i = (1 + \lambda_i(T_2))^{-1}$, $i = 1, \cdots, m$. We observe that, depending on $\lambda(T_2)$, (3.17) is either an ellipsoid, the empty set, or an unbounded set. The convex hull $C(\lambda(T_2))$ of (3.17) equals (3.17) when $c_0\eta_i - c - \sum \eta_j \geq 0$ for all i and otherwise equals $\mathbb{R}^m$.

EXAMPLE 3.3. The following confidence set corresponds to a generalized Bayes test derived by Marden (1980):

$$\{M : |I_m + T_2|\,|I_m + T_1|^{c_0} \leq c\}. \tag{3.18}$$

where again $c_0 = (m + \nu - q)/p$. Hooper (1982) proved that, for appropriate $c = c_\alpha$, (3.18) has smallest expected volume within the class of invariant level $1 - \alpha$ confidence sets. Corollary 3.3 shows that (3.18) is not exact for $\{\operatorname{tr} N'M : N \in M(m, p)\}$.

REMARK 3.2. We note that, for each of the confidence sets described in the above three examples, the conditional probability of coverage given T_2 varies substantially with T_2. In fact, (3.14) and (3.18) produce the empty set with positive probability. The conditionality principle recommends that the conditional confidence level given the ancillary T_2 is more relevant than the unconditional level. Remark 3.1 shows that Roy's maximum root criterion gives the only star-shaped confidence set based on the eigenvalues of T_1 and T_2 that is exact for $\{a'Mb\}$ and has the same conditional coverage probability for all values of T_2.

4. MANOVA under triangular group. Consider the MANOVA model (GMANOVA with $q = 0$) and the subgroup $G_1 = M(m, p) \times \mathscr{A}_1 \times O(p)$, where $\mathscr{A}_1 = \{A : A' \in LT(p)\}$. Following Wijsman (1980, Section 4.4) we write $S = LL'$ for $L \in LT(p)$ and put $Z = (X_1 - M)L'^{-1}$. Then $Z'Z$ is a maximal invariant function of (X, M) under G_1. Note the similarity between the pivotal quantities $Z'Z$ and WW' defined at (3.1). As with W, given the data X, there is a one-to-one correspondence between M and Z. By making use of this correspondence, one can apply the proof of Theorem 3.1, with Z' taking the place of W, and obtain corresponding results.

EXAMPLE 4.1. Subbaiah and Mudholkar (1982) extended the step-down procedure of J. Roy (1958) to include situations where the variables are arranged in blocks, with the blocks ranked in order of importance, but with the variables in each block ranked equally. Partition $Z = [z_1 : z_2 : \cdots : z_k]$ with $z_i \in M(m, p_i)$ and $\sum p_i = p$. Set $Z_i = [z_1 : \cdots : z_i]$, $Z_0 = 0$. The generalized step-down confidence set is

$$\{M : \lambda_1(z_i'(I_m + Z_{i-1}Z_{i-1}')^{-1}z_i) \leq c_i,\ i = 1, \cdots, k\}. \tag{4.1}$$

For $x \in \mathbb{R}^p$ we write $x' = (x_1', \cdots, x_k')$, $x_i \in \mathbb{R}^{p_i}$. Put $c_1^* = c_1$, $c_i^* = c_i(1 + c_1^* + \cdots + c_{i-1}^*)$ for $i = 2, \cdots, k$, and $d_i^* = \{c_i^*\}^{1/2}$. Subbaiah and Mudholkar (1982) derived the following family of simultaneous confidence intervals, effectively by applying the T operation to (4.1):

$$\{a'Mb : |a'(X_1 - M)b| \leq \|a\| \textstyle\sum_{i=1}^{k} \|(L'b)_i\|\, d_i^*\} \tag{4.2}$$

for all $a \in \mathbb{R}^m$, $b \in \mathbb{R}^p$. We apply the obvious analogue of Corollary 3.2 to construct the smallest confidence set containing (4.1) that is exact for $\{a'Mb\}$. It will then follow that this confidence set is equivalent to the family (4.2). Observe that M lies in (4.1) if and only if

$$z_i z_i' \leq c_i(I_m + Z_{i-1}Z_{i-1}'), \quad i = 1, \cdots, k. \tag{4.3}$$

Now $Z'Z$ satisfies (4.3) if and only if xx' satisfies (4.3) for all $x \in Z'\mathbb{B}^m$. Defining

$$C_0 = \{x \in \mathbb{R}^p : \|x_i\|^2 \leq c_i(1 + \textstyle\sum_{j=1}^{i-1} \|x_j\|^2),\ i = 1, \cdots, k\},$$

we have that (4.1) equals $\{M : Z'\mathbb{B}^m \subseteq C_0\}$. The convex hull of C_0 is $C = d_1^*\mathbb{B}^{p_1} \times \cdots \times d_k^*\mathbb{B}^{p_k}$ and our desired confidence set is

$$\{M : Z'\mathbb{B}^m \subseteq C\} = \{M : \lambda_1(z_i'z_i) \leq c_i^*,\ i = 1, \cdots, k\}. \tag{4.4}$$

Example 2.2 and the analogue of Corollary 3.1 can be used to verify that the family (4.2) is smallest exact with respect to (4.4).

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