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INVARIANTLY SUFFICIENT EQUIVARIANT STATISTICS AND CHARACTERIZATIONS OF NORMALITY IN TRANSLATION CLASSES

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It is shown that an equivariant statistic S is invariantly sufficient iff the generated σ-algebra and the σ-algebra of the invariant Borel sets are independent, and that if S is invariantly sufficient and equivariant, then the Pitman estimator for location parameter γ is given by $S - E_0(S)$. For independent $X_1, \cdots, X_n$, the existence of an invariantly sufficient equivariant linear statistic is characterized by the normality of $X_1, \cdots, X_n$. Then, the independence of $X_1, \cdots, X_n$ is replaced by a linear framework in which there are established characterizations of the normality of $X = (X_1, \cdots, X_n)$ by properties (invariant sufficiency, admissibility, optimality) of the minimum variance unbiased linear estimator for γ.

1. Invariantly sufficient translation equivariant statistics. For fixed $n \in \mathbb{N}$ and all $\gamma \in \mathbb{R}$, let $T_\gamma: \mathbb{R}^n \to \mathbb{R}^n$ be the translation given by $T_\gamma(x_1, \cdots, x_n) = (x_1 + \gamma, \cdots, x_n + \gamma)$, $(x_1, \cdots, x_n) \in \mathbb{R}^n$. Further let $\mathscr{B}^n_{\mathscr{T}} = \{B \in \mathscr{B}^n \mid T_\gamma^{-1}(B) = B, \gamma \in \mathbb{R}\}$ denote the σ-algebra of the Borel subsets in $\mathbb{R}^n$ being invariant under all translations T_γ, $\gamma \in \mathbb{R}$. If P_0 is any probability measure on $\mathscr{B}^n$ and if $\mathscr{W} = \{P_\gamma = (P_0)^{T_\gamma} \mid \gamma \in \mathbb{R}\}$, where $(P_0)^{T_\gamma}$ is the image measure of P_0 under T_γ, is the corresponding n-dimensional translation class, we use the symbols E_γ and V_γ for the expectation and variance w.r.t. P_γ. $S: (\mathbb{R}^n, \mathscr{B}^n) \to (\mathbb{R}, \mathscr{B})$ being a statistic, we use the symbol E_γ^S for (a version of) the conditional expectation under S w.r.t. P_γ. A statistic $S: (\mathbb{R}^n, \mathscr{B}^n) \to (\mathbb{R}, \mathscr{B})$ is called (*translation*) *equivariant*, if $S \circ T_\gamma = S + \gamma$ holds, $\gamma \in \mathbb{R}$.

With these notations we can formulate a theorem which will be useful for characterizations of normality in translation classes by sufficiency. Actually, only partial sufficiency for invariant indicators is needed.

THEOREM 1.1. *If $\mathscr{W} = \{P_\gamma \mid \gamma \in \mathbb{R}\}$ is a translation class and $S: (\mathbb{R}^n, \mathscr{B}^n) \to (\mathbb{R}, \mathscr{B})$ is an equivariant statistic, then the following two statements are equivalent*:

(a) *For any $B \in \mathscr{B}^n_{\mathscr{T}}$ there exists a measurable function $g: (\mathbb{R}, \mathscr{B}) \to (\mathbb{R}, \mathscr{B})$ with $g \circ S = E_\gamma^S(1_B)$, $\gamma \in \mathbb{R}$*;

(b) *$S^{-1}(\mathscr{B})$ and $\mathscr{B}^n_{\mathscr{T}}$ are independent* (*under P_0*).

We will call a statistic S satisfying (a) (*translation*) *invariantly sufficient.*

PROOF. First let S be invariantly sufficient. Fix $B \in \mathscr{B}^n_{\mathscr{T}}$ and choose g according to (a). On account of the equivariance of S and of the invariance of B we get, for $t \in \bar{\mathbb{R}}$ and $\gamma \in \mathbb{R}$,

$$\text{(i)} \quad \begin{aligned} P_0(B \cap \{S < t\}) &= P_\gamma(B \cap \{S < t + \gamma\}) = \int_{\{S<t+\gamma\}} 1_B \, dP_\gamma \\ &= \int_{\{S<t+\gamma\}} g \circ S \, dP_\gamma = \int_{\{S<t\}} g(S + \gamma) \, dP_0. \end{aligned}$$

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Applying (i) for $\gamma = S(y), y \in \mathbb{R}^n$, and integrating both sides w.r.t. P_0, we obtain by Fubini's Theorem

$$P_0(B \cap \{S < t\}) = P_0(B) \cdot P_0\{S < t\}, \quad t \in \bar{\mathbb{R}}.$$

Since $B \in \mathscr{B}^n_{\mathscr{T}}$ was chosen arbitrarily, the independence of $S^{-1}(\mathscr{B})$ and $\mathscr{B}^n_{\mathscr{T}}$ is established.

Now suppose $S^{-1}(\mathscr{B})$ and $\mathscr{B}^n_{\mathscr{T}}$ are independent under P_0. Then it is readily verified that $S^{-1}(\mathscr{B})$ and $\mathscr{B}^n_{\mathscr{T}}$ are independent under P_γ, $\gamma \in \mathbb{R}$. Therefore, for any $B \in \mathscr{B}^n_{\mathscr{T}}$ it follows that

$$E^S_\gamma(1_B) = P_\gamma(B) = P_0(B), \quad \gamma \in \mathbb{R},$$

which proves the theorem. □

REMARK 1.2. The implication (a) ⇒ (b) of Theorem 1.1 is related to Theorem 2 in Basu (1955) where it is shown that $S^{-1}(\mathscr{B})$ and $\mathscr{B}^n_{\mathscr{T}}$ are independent if S is boundedly complete and sufficient. But, the above implication (a) ⇒ (b) is not covered by Basu's result. Firstly, in general an equivariant (invariantly) sufficient statistic need not be boundedly complete. E.g., the statistic $S = Id_{\mathbb{R}}$ in Example 3.7 in Lehmann and Scheffé (1950) is equivariant and sufficient, but not boundedly complete. Secondly, while evidently any sufficient statistic is invariantly sufficient, the converse does not hold. For, it is readily verified that for $n \geq 2$, any maximal invariant statistic is invariantly sufficient, but not sufficient.

The results stated below show that under the corresponding assumptions an invariantly sufficient linear statistic which is not invariant turns out to be complete and sufficient.

In the sequel, $X_j: \mathbb{R}^n \to \mathbb{R}$ denotes the jth projection, $1 \leq j \leq n$. Further, we define $Y = (X_2 - X_1, \cdots, X_n - X_1)$ for $n \geq 2$.

As a first consequence of Theorem 1.1 we get the following.

THEOREM 1.3. *Let* $\mathscr{W} = \{P_\gamma \mid \gamma \in \mathbb{R}\}$ *be a n-dimensional translation class with* $n \geq 2$.

(a) *If* $S \in \mathscr{L}^2(P_0)$ *is invariantly sufficient and equivariant, then the statistic* $\tilde{S} = S - E_0(S)$ *is the Pitman estimator for* γ, *i.e. the minimum mean squared error estimator for* γ *among all equivariant estimators.*

(b) *There exists essentially* (*i.e. up to* P_0*-equivalence and up to an additive constant*) *at most one invariantly sufficient and equivariant statistic in* $\mathscr{L}^2(P_0)$.

PROOF. Since $S \in \mathscr{L}^2(P_0)$ is equivariant, it is well known that the Pitman estimator for γ is given by $\tilde{S} = S - E_0^Y(S)$. Now, S is invariantly sufficient and $Y^{-1}(\mathscr{B}^{n-1}) = \mathscr{B}^n_{\mathscr{T}}$; hence Theorem 1.1 yields the independence of S and Y and therefore $E_0^Y(S) - E_0(S)$ which proves claim (a). The second statement is an immediate consequence of the uniqueness of the Pitman estimator. □

In this paper our interest will focus on linear statistics $\sum_{j=1}^n c_j X_j$. First we are concerned with necessary and sufficient conditions for (the existence of) sufficient linear statistics. For $X_1, \cdots, X_n$ being i.i.d. in Bartfai (1980) and Eberl (1983) respectively, see also Eberl and Moeschlin (1982), 2.3.14 and 2.3.15, it was shown that the sample mean $\bar{X}$ is sufficient iff $X_1, \cdots, X_n$ have a (possibly degenerate) normal distribution. Theorem 1.4 represents a generalization of this result.

THEOREM 1.4. *Let* $\mathscr{W} = \{P_\gamma \mid \gamma \in \mathbb{R}\}$ *be a translation class with* $n \geq 2$ *such that* $X_1, \cdots, X_n$ *are independent* (*under* P_0). *Then the following two statements are equivalent:*

(a) *There exists an invariantly sufficient statistic* $U = \sum_{j=1}^n c_j X_j$ *with* $c_j \neq 0$, $1 \leq j \leq n$, *and* $c = \sum_{j=1}^n c_j \neq 0$;

(b) *either the distributions of* X_j, $1 \leq j \leq n$, *are all point masses or they all are normal distributions.*

If (a) *and therefore* (b) *are valid, then the complete sufficient statistic* $\tilde{U} = (1/\tilde{c}) \sum_{j=1}^n \tilde{c}_j X_j$

with

$$\tilde{c}_j = \begin{cases} 1 & \text{for } V_0(X_j) = 0, \quad 1 \le j \le n, \\ 1/V_0(X_j) & \text{for } V_0(X_j) > 0, \quad 1 \le j \le n, \end{cases}$$

and $\tilde{c} = \sum_{j=1}^{n} \tilde{c}_j$ *is the essentially unique invariantly sufficient equivariant statistic in* $\mathscr{L}^2(P_0)$.

Proof. First, let U be a statistic according to (a). We assume without loss of generality that $c = 1$ and that the distribution of X_1 is degenerate if there is any X_j with a degenerate distribution. Due to Theorem 1.1, the statistics U and $V = (c_1 - 1)X_1 + \sum_{j=2}^{n} c_j X_j$ are independent (under P_0). Therefore, it follows by the Darmois-Skitovich Theorem that all distributions of X_j, $1 \le j \le n$, are normal, possibly degenerate. If the distribution of X_1 is degenerate, then the independence of U and V implies the same for the distribution of $\sum_{j=2}^{n} c_j X_j$, which implies that all distributions of X_j, $1 \le j \le n$, are degenerate. Considering for the converse first the degenerate case, and assuming $P_0\{X_j = \xi_j\} = 1$ with $\xi_j \in \mathbb{R}$, $1 \le j \le n$, we will establish the sufficiency of the statistic

$$V = \frac{1}{n} \sum_{j=1}^{n} (X_j - \xi_j)$$

which in turn implies the same for $\tilde{U} = \bar{X}$. Fix any $B \in \mathscr{B}^n$ and consider $g = 1_{(h \circ V)^{-1}(Z(B))}$ with $Z = (X_1 - \xi_1, \cdots, X_n - \xi_n)$ and $h: \mathbb{R} \to \mathbb{R}^n$ given by $h(y) = (y, \cdots, y)$, $y \in \mathbb{R}$. Then, writing $\xi + \gamma = (\xi_1 + \gamma, \cdots, \xi_n + \gamma)$, for $D \in \mathscr{B}$ and $\gamma \in \mathbb{R}$ it follows that

$$\text{(i)} \qquad \int_{V^{-1}(D)} 1_B \, dP_\gamma = 1_B(\xi + \gamma) \cdot 1_{V^{-1}(D)}(\xi + \gamma)$$

$$\text{(ii)} \qquad \int_{V^{-1}(D)} g \, dP_\gamma = g(\xi + \gamma) \cdot 1_{V^{-1}(D)}(\xi + \gamma).$$

It is readily verified that $\xi + \gamma \in B$ holds iff $\xi + \gamma \in (h \circ V)^{-1}(Z(B))$. Thus (i) and (ii) together with $(h \circ V)^{-1}(Z(B)) \in V^{-1}(\mathscr{B})$ imply $g = E_\gamma^V(1_B)$, $\gamma \in \mathbb{R}$. Since $B \in \mathscr{B}^n$ was chosen arbitrarily, this proves the sufficiency of V and of $\tilde{U} = \bar{X}$ for the case under consideration. The completeness of $\tilde{U}$ being easily shown, the degenerate case is accomplished.

The sufficiency and completeness of $\tilde{U}$ in the normal case being well known and the assertion concerning the uniqueness being implied by Theorem 1.3 (b), the theorem is proved. □

Remark 1.5. If in Theorem 1.4 the invariant sufficiency is replaced by sufficiency, then the condition $c \ne 0$ is superfluous. (But it is not otherwise; see Remark 1.2.)

Remark 1.6. In Theorem 1.4 the condition $c_j \ne 0$, $1 \le j \le n$, cannot be dropped, in general. In fact, if e.g. the distribution of X_1 under P_0 is degenerate, then one can show by arguments similar to the corresponding ones in the proof of Theorem 1.4 that X_1 is sufficient, irrespective of the distributions of $X_2, \cdots, X_n$ under P_0.

On the other hand, if the degenerate case is excluded in advance and the invariant sufficiency is replaced by sufficiency, then the condition $c_j \ne 0$, $1 \le j \le n$, is unnecessary. More precisely: If $S = \sum_{j=1}^{n} c_j X_j$ is sufficient and if there exists at least one j_0 with $c_{j_0} = 0$, then the distributions of all X_j with $c_j \ne 0$ are degenerate. Firstly, it follows along the same lines as in the proof of Theorem 1.4 that all these distributions are normal or all degenerate. Therefore the distributions of S are normal or degenerate. If they are normal, they obviously obey the overlapping property (i.e., for any $\gamma_1, \gamma_2 \in \mathbb{R}$ and any $B \in \mathscr{B}$ with $P_{\gamma_1}^S(B) = 1$ it follows that $P_{\gamma_2}^S(B) > 0$). Assuming without loss of generality that $c_1 = 0$, in this case by a theorem of Basu (see Basu, 1958, page 226) the distributions $P_\gamma^{X_1}$ turn out to be independent of γ. Thus, the distributions of S are degenerate, which proves the above claim.

REMARK 1.7. The only translation classes (with the assumptions encountered in Theorem 1.4) which at the same time are one-parameter exponential families in some linear statistic are those for which the marginal distributions are normal. This can be seen from Theorem 1.4, Remark 1.5 and Remark 1.6.

2. Characterizations of normality. Throughout this section let $\mathscr{W} = \{P_\gamma \mid \gamma \in \mathbb{R}\}$ be a n-dimensional translation class with $n \geq 2$, $A = (a_{jk})$ a regular $n \times n$ matrix and let $Z = (Z_1, \cdots, Z_n)$ be given by $Z = XA$ with $X = (X_1, \cdots, X_n)$ such that $Z_1, \cdots, Z_n$ are independent (under P_0), $E_0(Z_k) = 0$ and $0 < \sigma_k^2 = E_0(Z_k^2) < \infty$, $1 \leq k \leq n$. (The assumption $E_0(Z_k) = 0$ is made only for convenience.) In the given linear framework there will be derived characterizations of the normality of X by properties of the minimum variance unbiased linear estimator of γ. First, in an elementary, preliminary lemma this estimator is stated. We use the notations $A^{-1} = (a_{jk}^*)$ for the inverse of $A = (a_{jk})$, and $a_{.k} = \sum_{j=1}^n a_{jk}$, $1 \leq k \leq n$.

LEMMA 2.1. *The statistic*

$$\tilde{U} = c^{-1}U = c^{-1} \sum_{j=1}^n c_j X_j,$$

with

$$c_j = \sum_{k=1}^n \sigma_k^{-2} a_{jk} a_{.k} \text{ and } c = \sum_{j=1}^n c_j \neq 0,$$

is the minimum variance unbiased linear estimator for γ.

The proof can be accomplished by showing $\tilde{U}$ to be uncorrelated with any invariant linear statistic and is omitted.

The first characterization in the framework under consideration now represents an extension of Theorem 1.4.

THEOREM 2.2. *If the condition*

$$a_{.k} \neq 0, \quad 1 \leq k \leq n, \tag{2.3}$$

is fulfilled, then the following three statements are equivalent: (a) *There exists one* (*and only one, due to Theorem* 1.2) (*invariantly*) *sufficient unbiased linear statistic*; (b) *the statistic* $\tilde{U}$ *given in Lemma* 2.1 *is sufficient*; (c) $X = (X_1, \cdots, X_n)$ *is* (*n-variate*) *normally distributed.*

PROOF. Theorem 1.3 and Lemma 2.1 yield the validity of (a) $\Rightarrow$ (b); the implication (c) $\Rightarrow$ (b) is verified by elementary computations (linear transformation of the density of Z) which are omitted. To verify the remaining conclusion (b) $\Rightarrow$ (c) we first note that A^{-1} can have no row with all elements equal. (Actually, if we have $a_{jk}^* = a_j$, $1 \leq k \leq n$, for some fixed $j \in \{1, \cdots, n\}$, then necessarily $a_j \neq 0$ holds; due to (2.3) and to $\delta_{jk} = \sum_{m=1}^n a_{jm}^* a_{mk} = a_j a_{.k}$, $1 \leq k \leq n$, this leads to a contradiction.) Since $X = ZA^{-1}$, the statistic U given in Lemma 2.1 may be rewritten as

$$U = \sum_{k=1}^n Z_k a_{.k} / \sigma_k^2.$$

Now we fix $k_0 \in \{1, \cdots, n\}$, choose $i, j \in \{1, \cdots, n\}$ such that $a_{k_0 i}^* \neq a_{k_0 j}^*$ and consider the invariant statistic

$$U_0 = X_j - X_i = \sum_{k=1}^n Z_k (a_{kj}^* - a_{ki}^*).$$

Then U and U_0 are independent, by Theorem 1.1. Since the coefficients of Z_{k_0} in the linear statistics U and U_0 do not vanish, the Darmois-Skitovich Theorem yields the normality of Z_{k_0}. This proves the normality of $Z_1, \cdots, Z_n$ and of $X = (X_1, \cdots, X_n)$. □

The following lemma will be the main tool for the characterizations given below.

LEMMA 2.4. *Let $n \geq 3$. If* (2.3) *holds and if*

$$E_0^Y(U) = 0 \quad P_0\text{-a.e.}$$

with U given in Lemma 2.1, *then X is normally distributed.*

PROOF. Let f_k denote the characteristic function of Z_k, and let $g_k = f_k'/f_k = (d/dt)\log f_k$, $1 \leq k \leq n$. Further, let Ψ denote the characteristic function of P_0 and put $\psi = \log \Psi$. (All these functions are well defined in some neighbourhood of the origin.) From the linear connection between X and Z we infer that

$$\Psi(t_1, \cdots, t_n) = E_0\{\exp(i \textstyle\sum_{j=1}^n t_j X_j)\} = \prod_{k=1}^n f_k (\sum_{j=1}^n t_j a_{kj}^*).$$

Therefore

$$\partial\psi/\partial t_\nu = \partial \log \Psi/\partial t_\nu = \textstyle\sum_{k=1}^n a_{k\nu}^* g_k(\sum_{j=1}^n t_j a_{kj}^*), \quad 1 \leq \nu \leq n,$$

and

$$\sum_{\nu=1}^n c_\nu \frac{\partial\psi}{\partial t_\nu} = \sum_{\nu=1}^n \sum_{m=1}^n \sigma_m^{-2} a_{\nu m} a_{.m} \sum_{k=1}^n a_{k\nu}^* g_k (\textstyle\sum_{j=1}^n t_j a_{kj}^*)$$

$$= \textstyle\sum_{k=1}^n g_k(\sum_{j=1}^n t_j a_{kj}^*) a_{.k}/\sigma_k^2.$$

Thus Lemma 7.8.2 in Kagan, Linnik and Rao (1973) yields

(i) $$\sum_{k=1}^n \frac{a_{.k}}{\sigma_k^2} g_k(\textstyle\sum_{j=1}^n t_j a_{kj}^*) = 0 \quad \text{if} \quad \sum_{j=1}^n t_j = 0$$

for all $(t_1, \cdots, t_n)$ in some neighbourhood of $(0, \cdots, 0)$. Substituting $s_k = \sum_{j=1}^n t_j a_{kj}^*$, $1 \leq k \leq n$, we have $t_j = \sum_{k=1}^n s_k a_{jk}$, $1 \leq j \leq n$, and $\sum_{j=1}^n t_j = \sum_{j=1}^n \sum_{k=1}^n s_k a_{jk} = \sum_{k=1}^n s_k a_{.k}$; hence (i) implies

$$\sum_{k=1}^n \frac{a_{.k}}{\sigma_k^2} g_k(s_k) = 0 \quad \text{if} \quad \textstyle\sum_{k=1}^n s_k a_{.k} = 0$$

for all $(s_1, \cdots, s_n)$ in some neighbourhood of $(0, \cdots, 0)$. Putting $s_n = -(1/a_{.n})\sum_{k=1}^{n-1} s_k a_{.k}$ yields

(ii) $$\sum_{k=1}^{n-1} \frac{a_{.k}}{\sigma_k^2} g_k(s_k) = -\frac{a_{.n}}{\sigma_n^2} g_n\left(-\frac{1}{a_{.n}} \textstyle\sum_{k=1}^{n-1} s_k a_{.k}\right)$$

for all $(s_1, \cdots, s_{n-1})$ in some neighbourhood of $(0, \cdots, 0)$. Due to (2.3) and $n \geq 3$ differentiation of both sides of (ii) w.r.t. s_k leads to

$$g_k'(s) = \gamma_k \in \mathbb{R}, \quad 1 \leq k \leq n,$$

for all $s \in \mathbb{R}$ in some neighbourhood of 0. Because $g_k = f_k'/f_k$, this implies $f_k = \exp\{Q_k\}$, where Q_k is a polynomial of degree at most 2, $1 \leq k \leq n$. By Marcinkiewicz's Theorem (see Marcinkiewicz, 1938, or Kagan, Linnik and Rao, 1973, Lemma 1.4.2) this yields the normality of $Z_1, \cdots, Z_n$ and that of X. □

Next we give a characterization by admissibility.

THEOREM 2.5. *If $n \geq 3$ and if* (2.3) *holds, then the statistic $\tilde{U}$ given in Lemma* 2.1 *is admissible* (*under squared error loss*) *among all unbiased estimators for γ iff X is normally distributed.*

PROOF. For normally distributed $Z_1, \ldots, Z_n$ the unbiased statistic $\tilde{U}$ is sufficient and complete and therefore the minimum variance unbiased estimator for γ. Thus, $\tilde{U}$ is admissible in this case. To prove the only if part, let $\tilde{U}$ be admissible. Since the Pitman estimator for γ is given by $\tilde{U} - E_0^Y(\tilde{U})$, the admissibility of $\tilde{U}$ together with the uniqueness of the Pitman estimator implies $E_0^Y(\tilde{U}) = 0$. Hence, the claim follows from Lemma 2.4. □

Obviously, Theorem 2.5 retains its validity "$\tilde{U}$ is admissible" being replaced by "$\tilde{U}$ is the minimum variance unbiased estimator for γ." The following theorem extends the characterization of the normality of X by $\tilde{U}$ being the minimum variance unbiased estimator to a corresponding characterization for polynomials of $\tilde{U}$.

THEOREM 2.6. *Suppose that $n \geq 3$, condition* (2.3) *is fulfilled and $X_j \in \mathscr{L}^{2k}(P_0)$, $1 \leq j \leq n$, for some $k \in \mathbb{N}$. Further let $Q(s) = \sum_{m=0}^{k} q_m s^m$, $s \in \mathbb{R}$, be a polynomial of degree k (i.e. $q_k \neq 0$) and let $\xi: \mathbb{R} \to \mathbb{R}$ be given by*

$$\xi(\gamma) = E_\gamma(Q \circ \tilde{U}) = \textstyle\sum_{m=0}^{k} d_m \gamma^m, \quad \gamma \in \mathbb{R},$$

where the coefficients d_m may be (*explicitly*) *determined by evaluating $E_\gamma(Q \circ \tilde{U})$. Then $Q \circ \tilde{U}$ is the minimum variance unbiased estimator for $\xi(\gamma)$* iff *X is normally distributed.*

For $A = I_n$ (I_n the $n \times n$ identity matrix) Theorem 2.6 corresponds with Theorem 7.6.1 of Kagan, Linnik and Rao (1973), and since the proof of Theorem 2.6 follows the same pattern (by showing $E_0^Y(U) = 0$ P_0-a.e. and using Lemma 2.4 now), it is omitted.

In Kagan, Linnik and Rao (1973) there are given further characterizations of normality in translation classes referring to the case $A = I_n$ which apparently can be carried over to regular matrices A fulfilling (2.3).

REMARK 2.7. We consider processes fundamental to time series analysis and which are not covered by the ordinary independent case. Without going into details, we briefly indicate a condition under which the above results can be applied to characterize the normality for the processes under consideration.

It is well known (see e.g. Fuller, 1976) that for X being a finite autoregressive process of order p, a moving average process of order q or a (p, q)-*ARMA* a relation $XA = Z$ holds with A and Z satisfying all conditions encountered at the beginning of this section. The entries of A are given by

$$a_{ij} = \begin{cases} d_{j-i} & \text{for } 0 \leq j - i \leq p, \\ 0 & \text{otherwise,} \end{cases}$$

where $1 \leq p \leq n - 1$, $d_k \in \mathbb{R}$ for $0 \leq k \leq p$ and $d_0 \neq 0$. Thus, the assumption (2.3) required for the above results is fulfilled iff

$$\textstyle\sum_{k=0}^{j} d_k \neq 0, \quad 1 \leq j \leq p.$$

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