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INVARIANT CONFIDENCE SETS WITH SMALLEST EXPECTED MEASURE[1]

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A method is given for constructing a confidence set having smallest expected measure within the class of invariant level $1 - \alpha$ confidence sets. The main assumptions are (i) that the invariance group acts transitively on the parameter space and also acts on the parametric function of interest, and (ii) that the measure satisfies a certain equivariance property. When the invariance group satisfies the conditions of the Hunt-Stein Theorem, the optimal invariant confidence set is shown to minimize the maximum expected measure among all level $1 - \alpha$ confidence sets. The method is applied in several estimation problems, including the GMANOVA problem.

1. Introduction. Let X be a random variable taking values in a space $\mathscr{X}$ and having distribution P_θ for some unknown $\theta \in \Theta$. Suppose $\gamma(\theta)$, taking values in Γ, is the parameter of interest and the rest of θ is a nuisance parameter. The symbol γ will be used to denote both the function $\gamma : \Theta \to \Gamma$ and a point in Γ, depending on context. Let $C(X)$ denote a confidence set for γ with confidence level $1 - \alpha$; i.e.,

$$P_\theta[\gamma(\theta) \in C(X)] \ge 1 - \alpha \quad \text{for all} \quad \theta \in \Theta. \tag{1.1}$$

In the frequency theory approach to set estimation due to Neyman, the desirability of C is usually measured by the probability that $C(X)$ covers false values of γ:

$$P_\theta[\gamma \in C(X)] \quad \text{for all} \quad \gamma \in \Gamma_\theta, \theta \in \Theta. \tag{1.2}$$

Here Γ_θ is the set of false γ values relevant to the problem at hand. For example, if γ is real-valued and a lower confidence bound is desired, one has $\Gamma_\theta = (-\infty, \gamma(\theta))$. In a two-sided problem one has $\Gamma_\theta = \Gamma \backslash \{\gamma(\theta)\}$.

An alternative criterion for evaluating C is the expected value of some measure of $C(X)$. For each $\theta \in \Theta$, let $m(\cdot, \theta)$ denote a measure on Γ and consider

$$E_\theta m(C(X), \theta) \quad \text{for all} \quad \theta \in \Theta. \tag{1.3}$$

For example, if $C(X)$ is to provide a lower confidence bound for real-valued γ one might take $m(\cdot, \theta)$ to be Lebesgue measure restricted to $(-\infty, \gamma(\theta))$. If a two-sided interval is desired then Lebesgue measure (unrestricted) is more appropriate.

Pratt (1961) pointed out that the criterion of expected measure is closely related to the criterion of false coverage. By interchanging the order of integration in

$$E_\theta \int_\Gamma I_{C(X)}(\gamma) m(d\gamma, \theta)$$

he showed that

$$E_\theta m(C(X), \theta) = \int_\Gamma P_\theta[\gamma \in C(X)] m(d\gamma, \theta).$$

If the measure satisfies

$$m(\Gamma \backslash \Gamma_\theta, \theta) = 0, \tag{1.4}$$

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then one has

$$E_\theta m(C(X), \theta) = \int_{\Gamma_\theta} P_\theta[\gamma \in C(X)] m(d\gamma, \theta). \tag{1.5}$$

Thus (1.3) can be regarded as a specification of the way in which the probabilities (1.2) are to be made small. It should be stressed that if (1.4) does not hold, then criteria (1.2) and (1.3) are not related. In particular, the expected length of a confidence interval is not relevant in a problem where a one-sided confidence bound is desired. This point is discussed by Madansky (1962). For an example where confusion on this can arise, see Section 6.1.

Usually there is no set estimator that uniformly minimizes (1.3) subject to (1.1). One method for choosing a particular set estimator involves minimizing a weighted average of the expected measures; see Pratt (1961). The present paper presents an alternative method that relies on invariance considerations. Let G be an invariance group that acts transitively on Θ and that also acts on Γ. Let C be an invariant set estimator for γ. (We regard C as a subset of $\mathscr{X} \times \Gamma$, with $C(x)$ denoting the cross section at x, and so use the term invariant rather than equivariant.) Then $C(X)$ has a fixed probability, independent of θ, of covering the true value of γ; see Wijsman (1980, Section 3). In Section 2 below it is shown that the expected measure $E_\theta m(C(X), \theta)$ is determined for all θ by its value at any particular θ, provided the function $m(\cdot, \cdot)$ is equivariant. Consequently criterion (1.3) determines a total ordering on the class of invariant set estimators satisfying (1.1). A method is given for determining the best confidence set in this class.

Suppose C is a uniformly most accurate invariant level $1 - \alpha$ set estimator; i.e., C uniformly minimizes (1.2) among all invariant set estimators satisfying (1.1). It follows from (1.5) that, within the same class, C uniformly minimizes (1.3) for all measures $m(\cdot, \theta)$ satisfying (1.4). In most of the applications in this paper there is no uniformly most accurate invariant set estimator and the optimal procedure derived depends on the choice of the measure $m(\cdot, \theta)$. The results of Cohen and Strawderman (1973) imply that, under mild restrictions on the measure $m(\cdot, \theta)$, the invariant level $1 - \alpha$ set estimator with smallest expected measure is at least almost admissible among all invariant set estimators when the criteria are (1.1) and (1.2).

In Section 3 a version of the Hunt-Stein Theorem is presented. Sections 4, 5, and 6 contain applications. A more detailed exposition and further examples are given in Hooper (1981b). Section 7 discusses a difficulty that arises concerning the shape of optimal confidence regions.

2. Invariant set estimators. Let $\mathscr{A}$ and $\mathscr{F}$ denote σ-fields of subsets of, respectively, $\mathscr{X}$ and Γ. A (possibly randomized) set estimator for γ based on X is defined to be a jointly measurable function ϕ mapping $\mathscr{X} \times \Gamma$ into [0, 1]; see Hooper (1982). Let U be distributed uniformly on [0, 1] and independent of X. The function ϕ is assumed to represent the following nonrandomized confidence set based on (X, U):

$$C_\phi(X, U) = \{\gamma \in \Gamma : U \le \phi(X, \gamma)\}.$$

Observe that $\phi(x, \gamma)$ is the conditional probability that $C_\phi(X, U)$ covers γ given that $X = x$. Probabilities of coverage (true or false) are given by

$$P_\theta[\gamma \in C_\phi(X, U)] = E_\theta \phi(X, \gamma).$$

For each $\theta \in \Theta$, let $m(\cdot, \theta)$ be a σ-finite measure on $(\Gamma, \mathscr{F})$. By applying Tonelli's Theorem, we obtain

$$E_\theta m(C_\phi(X, U), \theta) = \int_\Gamma E_\theta \phi(X, \gamma) m(d\gamma, \theta).$$

Suppose G is an invariance group that also acts on Γ; for definitions see Wijsman (1980, Section 3). The actions will be denoted $x \to gx$, $\theta \to g\theta$, $\gamma \to g\gamma$. In particular we have $\gamma(g\theta) = g\gamma(\theta)$. A set estimator ϕ is invariant provided $\phi(gx, g\gamma) = \phi(x, \gamma)$ for all $x \in \mathscr{X}$, $\gamma \in \Gamma$, $g \in G$. We will say that a function $m: \mathscr{F} \times \Theta \to [0, \infty]$ is *equivariant* provided a linear action of G on $[0, \infty]$ can be defined so that $m(gA, g\theta) = gm(A, \theta)$ for all $A \in \mathscr{F}$, $\theta \in \Theta$, $g \in G$. If ϕ is invariant then we have

$$E_{g\theta}\phi(X, \gamma(g\theta)) = E_\theta \phi(gX, g\gamma(\theta)) = E_\theta\phi(X, \gamma(\theta)),$$

so the probability of covering the true γ is constant on orbits. If ϕ is invariant and $m(\cdot, \cdot)$ is equivariant then we have

$$\int_\Gamma E_{g\theta}\phi(X, \gamma)m(d\gamma, g\theta) = \int_\Gamma E_\theta\phi(gX, g\gamma)m(dg\gamma, g\theta) = g\int_\Gamma E_\theta\phi(X, \gamma)m(d\gamma, \theta), \tag{2.1}$$

so the expected measure is determined on an orbit by its value at a single point in the orbit. Now suppose G acts transitively on Θ; i.e., Θ consists of a single orbit. Then the problem of finding an optimal level $1 - \alpha$ invariant set estimator can be solved by fixing $\theta^* \in \Theta$ and minimizing $\int_\Gamma E_{\theta^*}\phi(X, \gamma)m(d\gamma, \theta^*)$ subject to $E_{\theta^*}\phi(X, \gamma(\theta^*)) \geq 1 - \alpha$ and ϕ invariant.

To do this we need a tractable representation of the class of invariant set estimators. Let $\mathscr{P}^\Gamma = \{P_{\theta,\gamma}: (\theta, \gamma) \in \Theta \times \Gamma\}$ be the family of distributions defined on $\mathscr{X} \times \Gamma$ as follows: for $C \in \mathscr{A} \times \mathscr{F}$, let $C(\cdot, \gamma) \in \mathscr{A}$ denote the cross section at γ and define $P_{\theta,\gamma}(C) = P_\theta(C(\cdot, \gamma))$. Let $T: \mathscr{X} \times \Gamma \to \mathscr{T}$ be invariantly sufficient for $\mathscr{P}^\Gamma$; see Hall et al (1965, page 579). Then the class of invariant set estimators ϕ is equivalent to the class of measurable functions $h: \mathscr{T} \to [0, 1]$, in the sense that, given an invariant ϕ there exists h such that $E_\theta\phi(X, \gamma) = E_\theta h(T(X, \gamma))$ for all $(\theta, \gamma) \in \Theta \times \Gamma$. The function T can be constructed by applying sufficiency and invariance reductions in tandem to $\mathscr{X} \times \Gamma$; see Hooper (1982) for details and examples. Let $\mathscr{P}^T = \{P^T_{\theta,\gamma}: (\theta, \gamma) \in \Theta \times \Gamma\}$ denote the set of distributions induced by $\mathscr{P}^\Gamma$ on $\mathscr{T}$; i.e., for measurable $B \subseteq \mathscr{T}$, we define $P^T_{\theta,\gamma}(B) = P_\theta(T(\cdot, \gamma)^{-1}B)$. We now have the notation needed to present our main result.

THEOREM 1. *Let G be an invariance group that acts on Γ and acts transitively on Θ. For each $\theta \in \Theta$, let $m(\cdot, \theta)$ be a σ-finite measure on Γ and suppose $m(\cdot, \cdot)$ is equivariant. Let $T: \mathscr{X} \times \Gamma \to \mathscr{T}$ be invariantly sufficient for $\mathscr{P}^\Gamma$ and suppose $\mathscr{P}^T$ is dominated by a σ-finite measure μ^T, with densities denoted $p^T(t; \theta, \gamma)$. Fix $\theta^* \in \Theta$ and set $\gamma^* = \gamma(\theta^*)$. Suppose $p^T(t; \theta^*, \gamma)$ is jointly measurable as a function of (t, γ). Define*

$$W(t) = \frac{p^T(t; \theta^*, \gamma^*)}{\int_\Gamma p^T(t; \theta^*, \gamma)m(d\gamma, \theta^*)}. \tag{2.2}$$

Then a level $1 - \alpha$ invariant set estimator with smallest expected measure is given by

$$\phi(x, \gamma) = \begin{cases} 1 & \text{when} \quad W(T(x, \gamma)) > c \\ k & \text{when} \quad W(T(x, \gamma)) = c \\ 0 & \text{when} \quad W(T(x, \gamma)) < c \end{cases} \tag{2.3}$$

where the constants c and k are determined by

$$P^T_{\theta^*,\gamma^*}[W > c] + kP^T_{\theta^*,\gamma^*}[W = c] = 1 - \alpha. \tag{2.4}$$

PROOF. By the discussion preceding the theorem, we know that the optimal invariant set estimator is given by $\phi = h \circ T$, where $h: \mathscr{T} \to [0, 1]$ minimizes

$$\int E_{\theta^*}h(T(X, \gamma))m(d\gamma, \theta^*)$$

subject to $E_{\theta^*}h(T(X, \gamma^*)) \geq 1 - \alpha$. An application of Tonelli's Theorem yields the following expression for the expected measure:

$$\begin{aligned}\int_\Gamma E_{\theta^*}h(T(X,\gamma))m(d\gamma,\theta^*) &= \int_\Gamma\int_{\mathscr{T}} h(t)p^T(t;\theta^*,\gamma)\mu^T(dt)m(d\gamma,\theta^*)\\ &= \int_{\mathscr{T}} h(t)\left\{\int_\Gamma p^T(t;\theta^*,\gamma)m(d\gamma,\theta^*)\right\}\mu^T(dt).\end{aligned} \tag{2.5}$$

The probability of covering γ^* can be written

$$E_{\theta^*}h(T(X,\gamma^*)) = \int_{\mathscr{T}} h(t)p^T(t;\theta^*,\gamma^*)\mu^T(dt).$$

The conclusion follows from the Neyman-Pearson Fundamental Lemma. □

We note that the function W and constant c may depend on the choice of θ^*. It is easy to verify that $W(t;\, g\theta^*) = g^{-1}W(t;\, \theta^*)$, $c(g\theta^*) = g^{-1}c(\theta^*)$, and $k(g\theta^*) = k(\theta^*)$. If $m(\cdot,\, \cdot)$ is invariant then W does not depend on θ^*.

The invariantly sufficient function $T(X,\gamma)$ is a pivotal quantity; i.e., the distribution of $T(X,\gamma(\theta))$ under P_θ does not depend on θ. We will refer to $P^T_{\theta,\gamma(\theta)}$ as the null distribution of T. If the null distribution of $W = W(T)$ is continuous then the optimal invariant set estimator (2.3) is nonrandomized. Otherwise randomization may be avoided by choosing an appropriate value for $1-\alpha$. The α quantile of the null distribution of W can be approximated easily enough using simulation methods when analytic methods are intractible.

From (2.5) we observe that the expected measure under P_{θ^*} of $\phi = h\circ T$ is given by

$$E_{\theta^*,\gamma^*}\{h(T)/W(T)\}. \tag{2.6}$$

This expression can be used to evaluate the expected measure of any invariant set estimator. We note that W equals 0 when the denominator of (2.2) is infinite. If $\alpha^* \equiv \mathrm{P}^T_{\theta^*,\gamma^*}[W=0] > 0$ then, for $\alpha < \alpha^*$, all invariant level $1-\alpha$ confidence sets have infinite expected measure.

When computing the function W, it is convenient to ignore constant multiplicative factors. This does not affect the definition of the set estimator in (2.3) and (2.4). And (2.6) can still be used to compare the expected measures of various invariant set estimators. Thus in our examples the function W given is proportional, but not necessarily equal, to (2.2).

The invariantly sufficient function T sometimes factors into two terms, the second not depending on γ; i.e., $T(X,\gamma) = (T_1(X,\gamma),\, T_2(X))$. Then T_2 is an ancillary statistic since T_2 is invariant and G acts transitively on Θ. The conditionality principle recommends that the conditional confidence level given T_2 is more relevant than the unconditional level. Theorem 1 can be easily modified to give the invariant set estimator with smallest conditional expected measure subject to the conditional level condition. The function W remains the same since the marginal density of T_2 can be factored out of (2.2). The only change is that c and k become functions of T_2 determined by the conditional level condition.

In a number of the examples, $W(T(x,\,\cdot))$ turns out to be a fiducial density for γ with respect to an invariant measure $m_I(d\gamma)$. The $1-\alpha$ fiducial set estimator with smallest $m_I(d\gamma)$ measure is given by

$$\{\gamma : W(T(x,\gamma)) \geq c_\alpha(x)\}, \tag{2.7}$$

provided $W(T(x,\,\cdot))$ is not constant over any region, where $c_\alpha(x)$ is chosen so that (2.7) has fiducial content $1-\alpha$. The $1-\alpha$ level confidence set estimator (2.3) coincides with (2.7) provided $c_\alpha(x)$ equals the α quantile of the null distribution of $W(T)$ for all $x \in \mathscr{X}$. This need not be the case, as is seen in Section 4. We note that $c_\alpha(\cdot)$ must at least be constant if, for each γ, the function $T(\cdot,\gamma)$ depends on x only through a statistic $S(x)$ on which G acts transitively; i.e., $S: \mathscr{X} \to \mathscr{S}$, G acts transitively on $\mathscr{S}$, and $S(gx) = gS(x)$. This statement is proved by observing that

$$\int_{\{\gamma: W(T(x,\gamma)) \ge c\}} W(T(x, \gamma)) m_I(d\gamma)$$

is constant along orbits in $\mathscr{X}$.

3. Minimax set estimators. For a given family of measures $m(\cdot, \theta)$, $\theta \in \Theta$, we will say that a level $1 - \alpha$ set estimator ϕ_0 is minimax if it minimizes

$$\sup_{\theta\in\Theta} \int_\Gamma E_\theta \phi(X, \gamma) m(d\gamma, \theta) \tag{3.1}$$

among all level $1 - \alpha$ set estimators. If $m(\cdot, \cdot)$ is equivariant and the action of G on $[0, \infty)$ is nontrivial, then it follows from (2.1) that (3.1) is infinite for all invariant ϕ. Thus only invariant functions $m(\cdot, \cdot)$ are of interest here. Actually there would be no loss of generality if Theorem 1 were stated only for invariant $m(\cdot, \cdot)$. The problem of finding a set estimator with minimum expected measure is unchanged if the measure $m(\cdot, \cdot)$ is replaced by a measure of the form

$$m_1(\cdot, \theta) = c(\theta) m(\cdot, \theta), \tag{3.2}$$

where $c(\theta)$ is positive for all $\theta \in \Theta$. If G acts transitively on Θ then, given an equivariant $m(\cdot, \cdot)$, we can define an invariant $m_1(\cdot, \cdot)$ satisfying (3.2) by fixing $\theta^* \in \Theta$ and setting $c(g\theta^*) = g^{-1}1$. The function m_1 is uniquely determined up to a positive multiplicative constant.

Theorem 2 below parallels the version of the Hunt-Stein Theorem presented in Lehmann (1959). The following regularity conditions will be assumed. Suppose that $\mathscr{A}$ is countably generated and that the family of distributions $\mathscr{P} = \{P_\theta : \theta \in \Theta\}$ is dominated by a σ-finite measure μ. Denote the densities by $p_\theta(x)$. Let $\mathscr{B}$ be a σ-field of subsets of the group G and, for $\gamma \in \Gamma$, set $G_\gamma = \{g \in G : g\gamma = \gamma\}$. Suppose that the following measurability conditions hold: for some $\gamma^* \in \Gamma$ and each $A \in \mathscr{A} \times \mathscr{B}$, we have $\{(x, g\gamma^*) : (x, g^{-1}) \in A\} \in \mathscr{A} \times \mathscr{F}$ and $G_{\gamma^*} \in \mathscr{B}$; for each $B \in \mathscr{B}$ and $g \in G$, we have $Bg \in \mathscr{B}$; for each $A \in \mathscr{A} \times \mathscr{F}$ we have $\{(x, \gamma, g) : (gx, g\gamma) \in A\} \in \mathscr{A} \times \mathscr{F} \times \mathscr{B}$. Suppose there is a σ-finite measure ν on G such that $\nu(B) = 0$ implies $\nu(Bg) = 0$ for all $B \in \mathscr{B}$, $g \in G$.

THEOREM 2. *Suppose that G acts transitively on Γ and that there exists a sequence of probability distributions ν_n over $(G, \mathscr{B})$ which is asymptotically right invariant in the sense that, for each $g \in G$, $B \in \mathscr{B}$,*

$$\lim_{n\to\infty} |\nu_n(Bg) - \nu_n(B)| = 0. \tag{3.3}$$

Let $m(\cdot, \cdot)$ be invariant. Then, under the above regularity conditions, given any set estimator ϕ there exists an invariant set estimator ϕ_I that satisfies

$$\sup_{\theta\in\Theta} \int_\Gamma E_\theta \phi_I(X, \gamma) m(d\gamma, \theta) \le \sup_{\theta\in\Theta} \int_\Gamma E_\theta \phi(X, \gamma) m(d\gamma, \theta), \tag{3.4}$$

$$\inf_{\theta\in\Theta} E_\theta \phi_I(X, \gamma(\theta)) \ge \inf_{\theta\in\Theta} E_\theta \phi(X, \gamma(\theta)). \tag{3.5}$$

PROOF. For each n, define the set estimator

$$\phi_n(x, \gamma) = \int_G \phi(gx, g\gamma) \nu_n(dg).$$

Then we have $\phi_n(gx, g\gamma) - \phi_n(x, \gamma) \to 0$ for all x, γ, g. The proof is the same as that of Lehmann (1959, page 336, Equation 15). By the Bounded Convergence Theorem it follows that

$$\int_A \{\phi_n(gx, g\gamma) - \phi_n(x, \gamma)\} P_\theta(dx) \to 0 \tag{3.6}$$

for all $\theta \in \Theta$, $A \in \mathscr{A}$.

Fix $\gamma^* \in \Gamma$. By the Weak Compactness Theorem, Lehmann (1959, page 354) there exists a subsequence $\{n_i\}$ and a measurable function $\Psi: \mathscr{X} \to [0, 1]$ such that

$$\lim_{i\to\infty} \int_{\mathscr{X}} \phi_{n_i}(x, \gamma^*) p(x) \mu(dx) = \int_{\mathscr{X}} \Psi(x) p(x) \mu(dx) \tag{3.7}$$

for all μ-integrable functions p. By (3.6) it follows that $\Psi(gx) = \Psi(x)$ (a.e. $\mathscr{P}$) for each $g \in G_{\gamma^*}$. An application of Lehmann (1959, page 225, Theorem 4) shows that there exists a measurable function $\Psi_I: \mathscr{X} \to [0, 1]$ that is invariant under G_{γ^*} and satisfies $\Psi_I(x) = \Psi(x)$ (a.e. $\mathscr{P}$). Now define

$$\phi_I(x, \gamma) = \Psi_I(g^{-1}x)$$

where $\gamma = g\gamma^*$. Then ϕ_I is an invariant set estimator. Note that ϕ_I is well defined since $g_1\gamma^* = g_2\gamma^*$ implies $g_1^{-1}g_2 \in G_{\gamma^*}$ and hence $\Psi_I(g_1^{-1}x) = \Psi_I(g_1^{-1}g_2g_2^{-1}x) = \Psi_I(g_2^{-1}x)$. Setting $\gamma = g\gamma^*$, writing

$$\begin{aligned} &E_\theta\{\phi_{n_i}(X, \gamma) - \phi_I(x, \gamma)\} \\ &\quad = E_\theta\{\phi_{n_i}(X, g\gamma^*) - \phi_{n_i}(g^{-1}X, \gamma^*)\} + E_{g^{-1}\theta}\{\phi_{n_i}(X, \gamma^*) - \phi_I(X, \gamma^*)\}, \end{aligned}$$

and applying (3.6) and (3.7), we obtain

$$\lim_{i\to\infty} E_\theta \phi_{n_i}(X, \gamma) = E_\theta \phi_I(X, \gamma)$$

for all $\theta \in \Theta$, $\gamma \in \Gamma$. Put $G\theta = \{g\theta: g \in G\}$. Then the application of Fatou's Lemma and Tonelli's Theorem yields

$$\begin{aligned} \int_\Gamma E_\theta \phi_I(X, \gamma) m(d\gamma, \theta) &\le \liminf_{i\to\infty} \int_\Gamma E_\theta \phi_{n_i}(X, \gamma) m(d\gamma, \theta) \\ &= \liminf_{i\to\infty} \int_G \int_\Gamma E_\theta \phi(gX, g\gamma) m(d\gamma, \theta) \nu_{n_i}(dg) \\ &= \liminf_{i\to\infty} \int_G \int_\Gamma E_{g\theta} \phi(X, \gamma) m(d\gamma, g\theta) \nu_{n_i}(dg) \\ &\le \sup_{\theta' \in G\theta} \int_\Gamma E_{\theta'} \phi(X, \gamma) m(d\gamma, \theta') \end{aligned}$$

and

$$E_\theta \phi_I(X, \gamma(\theta)) = \lim_{i\to\infty} \int_G E_{g\theta} \phi(X, \gamma(g\theta)) \nu_{n_i}(dg) \ge \inf_{\theta' \in G\theta} E_{\theta'} \phi(X, \gamma(\theta')).$$

Equations 3.4 and 3.5 easily follow. □

COROLLARY 1. *Under the assumptions of Theorems* 1 *and* 2, *the optimal invariant set estimator* (2.3) *is minimax.*

In all of our examples the functions $m(\cdot, \cdot)$ are invariant. Corollary 1 is applicable unless otherwise indicated.

4. Location and scale parameters. Let $(X_1, \cdots, X_n)$ be a random vector with joint density $\sigma^{-n}f((x_1-\beta)/\sigma, \cdots, (x_n-\beta)/\sigma)$ with respect to n-dimensional Lebesque measure, where f is known. First suppose that $\sigma = 1$ is known and $\beta \in \mathbb{R}$ unknown, so that $\theta = \gamma(\theta) = \beta$. Let $G = \mathbb{R}$ with actions $x_i \to x_i + b$, $\beta \to \beta + b$ for $b \in \mathbb{R}$. A maximal invariant on $\mathscr{X} \times \Gamma$ is given by $T(x, \beta) = (x_1 - \beta, \cdots, x_n - \beta)$. Consider a measure of the form $m(d\beta, \beta^*) = \ell_1(\beta - \beta^*)\, d\beta$, for some nonnegative-valued function ℓ_1. Then we obtain

$$W(T(x, \beta)) = \frac{f(x_1-\beta, \cdots, x_n-\beta)}{\int_{-\infty}^{\infty} \ell_1(\beta - b) f(x_1 - b, \cdots, x_n - b)\, db}.$$

If we have $\ell_1 \equiv 1$ then $W(T(x, \cdot))$ is Pitman's fiducial density for β; Pitman (1938, page 396). This does not mean that the confidence interval with shortest expected length is necessarily a fiducial interval. Suppose the X_i are independent and identically distributed. Denote order statistics by $X_{(i)}$. Then an alternative invariantly sufficient function is given by (T_1, T_2), where $T_1 = X_{(1)} - \beta$ and $T_2 = (X_{(2)} - X_{(1)}, \cdots, X_{(n)} - X_{(n-1)})$. Pitman's interval minimizes the expected length in the conditional problem, given T_2.

Next suppose that $\beta = 0$ is known and $\sigma > 0$ unknown, so that $\theta = \gamma(\theta) = \sigma$. Let $G = (0, \infty)$ with actions $x_i \to ax_i$, $\sigma \to a\sigma$ for $a > 0$. A maximal invariant on $\mathscr{X} \times \Gamma$ is given by $T(x, \sigma) = (x_1/\sigma, \cdots, x_n/\sigma)$. Adopting a measure of the form $m(d\sigma, \sigma^*) = \ell_2(\sigma/\sigma^*)\sigma^{-1}\, d\sigma$, we obtain

$$W(T(x, \sigma)) = \frac{\sigma^{-n} f(x_1/\sigma, \cdots, x_n/\sigma)}{\int_0^{\infty} \ell_2(\sigma/a) a^{-n} f(x_1/a, \cdots, x_n/a) a^{-1}\, da}.$$

If we take $\ell_2 \equiv 1$, then $W(T(x, \cdot))$ is Pitman's fiducial density with respect to $\sigma^{-1}\, d\sigma$ (Pitman, 1938, page 404).

Now suppose both parameters are unknown and a joint confidence region is desired; i.e., $\theta = \gamma(\theta) = (\beta, \sigma)$. Let G be the affine group, with actions $x_i \to ax_i + b$, $\beta \to a\beta + b$, $\sigma \to a\sigma$ for $a > 0$, $b \in \mathbb{R}$. A maximal invariant on $\mathscr{X} \times \Gamma$ is $T(x, \beta, \sigma) = ((x_1 - \beta)/\sigma, \cdots, (x_n - \beta)/\sigma)$. The measure $m(d\beta d\sigma, (\beta^*, \sigma^*)) = \ell_1((\beta - \beta^*)/\sigma^*)\ell_2(\sigma/\sigma^*)\sigma^{-2}\, d\beta d\sigma$ yields

$$W(T(x, (\beta, \sigma))) = \frac{\dfrac{1}{\sigma^{n-1}} f\left(\dfrac{x_1 - \beta}{\sigma}, \cdots, \dfrac{x_n - \beta}{\sigma}\right)}{\int_{-\infty}^{\infty}\int_0^{\infty} \ell_1\left(\dfrac{\beta - b}{a}\right)\ell_2\left(\dfrac{\sigma}{a}\right)\dfrac{1}{a^{n-1}} f\left(\dfrac{x_1 - b}{a}, \cdots, \dfrac{x_n - b}{a}\right)\dfrac{1}{a^2}\, da db}.$$

If we take $\ell_1 \equiv 1$ and $\ell_2 \equiv 1$, then $W(T(x, \cdot))$ is Pitman's fiducial density with respect to $\sigma^{-2} d\sigma d\beta$ (Pitman, 1938, page 412).

Finally suppose a confidence interval is desired for β with σ unknown; i.e. $\theta = (\beta, \sigma)$ and $\gamma(\theta) = \beta$. Let G be the affine group with actions defined as above. A maximal invariant on $\mathscr{X} \times \Gamma$ under the translation subgroup is given by $V(x, \beta) = (v_1, \cdots, v_n) = (x_1 - \beta, \cdots, x_n - \beta)$. Let $T(v)$ be a maximal invariant under scale changes on $\mathbb{R}^n$. Consider a measure of the form $m(d\beta, \theta^*) = (1/\sigma^*)\ell((\beta - \beta^*)/\sigma^*)\, d\beta$. Put $\theta^* = (0, 1)$. Applying Wijsman (1967, Theorem 4), we obtain the probability ratio

$$\frac{p^T(T(v); \theta^*, 0)}{p^T(T(v); \theta^*, b)} = \frac{\int_0^{\infty} f(av_1, \cdots, av_n)\, da}{\int_0^{\infty} f(av_1 + b, \cdots, av_n + b)\, da}.$$

Integrating out b in the denominator yields

$$W(T(x,\beta)) = \frac{\int_0^\infty f((x_1-\beta)/a, \cdots, (x_n-\beta)/a)a^{-2}\,da}{\int_{-\infty}^{\infty}\int_0^\infty \ell((\beta-b)/a)f((x_1-b)/a, \cdots, (x_n-b)/a)a^{-3}\,dadb}.$$

5. GMANOVA. The general multivariate analysis of variance model was formulated by Potthoff and Roy (1964) as a convenient generalization of growth curves models. Gleser and Olkin (1970) introduced a canonical form for the model and used invariance reductions in deriving the likelihood ratio test. Kariya (1978) applied further reductions in deriving the locally most powerful invariant test. The results of Hooper (1982) yield corresponding reductions for the set estimation problem. Since we are restricting our attention to invariant set estimators, these reductions imply that we need consider only the following multivariate analysis of covariance model:

$$[X_1\!:\!X_2] \sim N_{m\times(p+q)}([M\!:\!0], I_m\otimes\Sigma),\ S\sim W_{p+q}(\nu,\Sigma)$$

with $[X_1\!:\!X_2]$ and S independent. So the m rows of $[X_1\!:\!X_2]$ are independent multivariate normal with common covariance matrix Σ and S has a Wishart distribution with ν degrees of freedom. We assume that Σ is positive definite and that we have $\nu \ge p+q$. Write $X = (X_1, X_2, S)$. Here $\theta = (M, \Sigma)$ and $\gamma(\theta) = M$.

Let $M(m, p)$ and $GL(p)$ denote, respectively, the set of $m\times p$ matrices and the set of $p\times p$ invertible matrices. Consider the invariance group $G = M(m, p) \times \mathscr{A}$, where $\mathscr{A} \subseteq GL(p+q)$ consists of all lower block-triangular matrices $A = (A_{ij})$ with $A_{11}\in GL(p)$, $A_{22}\in GL(q)$, and $A_{12} = 0$. The group actions are $[X_1\!:\!X_2] \to [X_1\!:\!X_2]A + [F\!:\!0]$, $S \to A'SA$, $M \to MA_{11} + F$, $\Sigma \to A'\Sigma A$, for $(F, A)\in G$. Partition $S = (S_{ij})$ with $S_{11}\!:p\times p$ and $S_{22}\!:q\times q$. A maximal invariant on $\mathscr{X}\times\Gamma$ under G is

$$T(X, M) = (T_1, T_2) = (X_{1\cdot 2}S_{11\cdot 2}^{-1}X_{1\cdot 2}', X_2S_{22}^{-1}X_2'),$$

where

$$X_{1\cdot 2} = (I_m + T_2)^{-1/2}(X_1 - M - X_2S_{22}^{-1}S_{21}) \quad\text{and}\quad S_{11\cdot 2} = S_{11} - S_{12}S_{22}^{-1}S_{21}.$$

Here and elsewhere the exponent ½ denotes the symmetric square root.

Let $\theta^* = (M^*, \Sigma^*) = (0, I)$. It seems difficult to obtain an expression for the density of T when $m > p$ or $m > q$, i.e., when T_1 or T_2 is singular. However only the ratio of the densities (with respect to a measure defined on an appropriate manifold) is needed here. A tractable expression for this ratio is obtained by applying Wijsman (1967, Theorem 4). Put $V_1 = X_{1\cdot 2}S_{11\cdot 2}^{-1/2}$ and $V_2 = X_2S_{22}^{-1/2}$. Let $\mathscr{O}(p)$ denote the set of $p\times p$ orthogonal matrices. Then $(T_1, T_2) = (V_1V_1', V_2V_2')$ is a maximal invariant under $(V_1, V_2) \to (V_1\Omega_1, V_2\Omega_2)$ with $\Omega_1\in\mathscr{O}(p)$ and $\Omega_2\in\mathscr{O}(q)$. Thus we have

$$\frac{p^T(T;\theta^*,0)}{p^T(T;\theta^*,M)} = \frac{\int_{\mathscr{O}(p)}\int_{\mathscr{O}(q)} p^V(V_1\Omega_1, V_2\Omega_2;\theta^*,0)\,d\Omega_2\,d\Omega_1}{\int_{\mathscr{O}(p)}\int_{\mathscr{O}(q)} p^V(V_1\Omega_1, V_2\Omega_2;\theta^*,M)\,d\Omega_2\,d\Omega_1}, \tag{5.1}$$

where $d\Omega_1$ and $d\Omega_2$ denote Haar measure. Under θ^* the conditional distribution of $V_1(M)$ given $(S_{11\cdot 2}, V_2)$ is $N_{m\times p}(-(I_m+T_2)^{-1/2}MS_{11\cdot 2}^{-1/2}, I_m\otimes S_{11\cdot 2}^{-1})$ and, in addition, $S_{11\cdot 2}$ and V_2 are independent; see Kariya (1978, Lemma 3.1). It follows that

$$p^V(V_1, V_2;\theta^*, M)$$
$$= \int_{S_{11\cdot 2}>0} p(V_1\,|\,V_2, S_{11\cdot 2};\theta^*, M)p(S_{11\cdot 2};\theta^*)p(V_2;\theta^*)\,dS_{11\cdot 2}$$

$$(5.2)\qquad = c\cdot p(V_2;\theta^*)\int_{S_{11.2}>0} |S_{11.2}|^{m/2} e\ \mathrm{tr}\left[-\frac{1}{2}\{V_1 + (I_m + T_2)^{-1/2} M S_{11.2}^{-1/2}\} S_{11.2}\{\cdot\}'\right]$$
$$\cdot |S_{11.2}|^{(\nu-q-p-1)/2} e\ \mathrm{tr}\left(-\frac{1}{2} S_{11.2}\right) dS_{11..2}.$$

Now consider the measure

$$(5.3)\qquad m(dM, \theta^*) = |\Sigma_{11.2}^*|^{-m/2}\, dM.$$

Plugging (5.2) into (5.1) and (5.1) into (2.2), then changing the order of integration, we obtain

$$(5.4)\qquad W(T) = |I_m + T_1|^{-(\nu+m-q)/2} |I_m + T_2|^{-p/2}.$$

An alternative measure

$$(5.5)\qquad m(dM, \theta^*) = |\Sigma_{11.2}^*|^{-m/2}\, \mathrm{tr}\{(M - M^*)\Sigma_{11.2}^{*-1}(M - M^*)'\}\, dM$$

yields

$$(5.6)\qquad \begin{aligned} W(T) &= |I_m + T_1|^{-(\nu+m-q)/2} |I_m + T_2|^{p/2} \\ &\quad \times \{(\nu + m - q)\mathrm{tr}(I_m + T_2)T_1 + p\ \mathrm{tr}(I_m + T_2)\}^{-1}. \end{aligned}$$

The invariance group G above does not satisfy the Hunt-Stein condition (3.3) and the set estimators determined by (5.4) and (5.6) are not minimax. Let $LT(p)$ denote the set of $p \times p$ lower triangular matrices with positive diagonal elements. Consider the subgroup $G_1 = M(m, p) \times \mathscr{A}_1$ where $\mathscr{A}_1$ consists of matrices $A = (A_{ij}) \in \mathscr{A}$ with $A_{11}' \in LT(p)$ and $A_{22}' \in LT(q)$. Then $\mathscr{A}_1$ is isomorphic to $LT(p + q)$ and G_1 satisfies (3.3). Let $L_{11\cdot 2} \in LT(p)$ and $L_{22} \in LT(q)$ denote the lower triangular square roots of, respectively, $S_{11\cdot 2}$ and S_{22}; i.e., $S_{11\cdot 2} = L_{11\cdot 2}L_{11\cdot 2}'$ and $S_{22} = L_{22}L_{22}'$. A maximal invariant on $\mathscr{X} \times \Gamma$ under G_1 is $U(X, M) = (U_1, U_2) = (X_{1\cdot 2}L_{11\cdot 2}'^{-1}, X_2L_{22}'^{-1})$. Observe that we have $T_1 = U_1U_1'$ and $T_2 = U_2U_2'$. Under the null distribution, U_1 and U_2 are independent (Kariya, 1978, Lemma 3.1) with distribution given by Olkin and Rubin (1964, Theorem 4.2). Minimax confidence sets for measures (5.3) and (5.5) are determined by, respectively,

$$(5.7)\qquad W(U) = |I_m + T_1|^{-(\nu+m-p-q-1)/2} |I_m + T_2|^{-p/2} \textstyle\prod_{i=1}^p |(I_p + U_1'U_1)^{[i]}|^{-1}$$

and

$$(5.8)\qquad \begin{aligned} W(U) &= |I_m + T_1|^{-(\nu+m-p-q-1)/2} |I_m + T_2|^{-p/2} \textstyle\prod_{i=1}^p |(I_p + U_1'U_1)^{[i]}|^{-1} \\ &\quad \times [\textstyle\sum_{i=1}^p (\nu + m + p - q - 2i + 1)\{U_1'(I_m + T_2)U_1\}_{ii} + p\ \mathrm{tr}(I_m + T_2)]^{-1}, \end{aligned}$$

where $(\ )^{[i]}$ denotes the upper left-hand $i \times i$ submatrix and $(\ )_{ii}$ denotes the ith diagonal element.

The confidence procedure determined by (5.4) corresponds to an improper Bayes test derived by Marden (1980). Our result combined with that of Cohen and Strawderman (1973) shows that this test is admissible among all fully invariant tests. As $\nu \to \infty$, the four confidence sets derived above are asymptotically equivalent to Wilk's determinental criterion, $\{M : |I_m + T_1| \le c\}$, which is the likelihood ratio procedure. Marden and Perlman (1980) show that the likelihood ratio test is admissible among fully invariant tests when $m = 1$; i.e., in the analysis of covariance problem. However, Marden (1980) proves, for $p \ge m$, that the likelihood ratio test is inadmissible among fully invariant tests when $q \ge 1$ and $m \ge 2$; he conjectures this result for $p < m$.

The optimal set estimators derived in this section produce the empty set when $|I_m + T_2|$ is large. The application of the conditionality principle, conditioning on T_2 or U_2, effectively reduces the problem to the case $q = 0$; i.e., the MANOVA problem. In the conditional problem the G-invariant confidence set with smallest expected volume is given by the likelihood ratio procedure.

6. Other applications.

6.1 *Location parameters for exponential distributions.* Let X_{ij}, for $j = 1, \cdots, n_i$ and $i = 1, \cdots, k$, be independent with density $\exp\{-(x_{ij} - \gamma_i)\}$, $x_{ij} \geq \gamma_i$. When $k = 1$ the confidence interval $X_{\min} + \ln(\alpha)/n \leq \gamma \leq X_{\min}$ is uniformly most accurate at level $1 - \alpha$. Littell and Louv (1981, page 126) construct confidence regions for γ, with $k \geq 2$, by inverting tests obtained from popular combination procedures. It turns out that Fisher's combined test determines the translation invariant confidence set with smallest expected volume, appropriate in the combined two-sided problem. However a different region is found to minimize the expected excess, $m(d\gamma, \gamma^*)$ = Lebesgue measure restricted to $\{\gamma : \gamma_i \leq \gamma_i^*, i = 1, \cdots, k\}$, appropriate in the combined lower confidence limit problem.

6.2. *Covariance matrix.* Suppose a confidence region is desired for Σ based on $S \sim W_p(\nu, S)$. We consider a measure of the form $m(d\Sigma, \Sigma^*) = |\Sigma^*|^{-k/2} |\Sigma|^{(k-p-1)/2} d\Sigma$. An invariance reduction under $GL(p)$ produces $T(S, \Sigma)$ = the vector of ordered eigenvalues of $\Sigma^{-1}S$. The optimal confidence set is determined by

$$W(T(S, \Sigma)) = |\Sigma^{-1}S|^{(\nu+k)/2} e\ \mathrm{tr}(-\tfrac{1}{2}\Sigma^{-1}S). \tag{6.1}$$

Note that $W(T(S, \cdot))$ defines, up to a constant factor, a density for Σ with respect to the invariant measure $|\Sigma|^{-(p+1)/2} d\Sigma$. Taking $k = 0$ and $k = p - 1$ yields, respectively, the fiducial distributions of Segal and Cornish-Fisher; see Wilkinson (1977, page 139).

The confidence set determined by (6.1) is not minimax. An invariance reduction under the subgroup $LT(p)$ produces a maximal invariant $U(S, \Sigma) = H^{-1}B$, where $H = (\eta_{ij})$ and $B = (b_{ij})$ denote, respectively, the lower triangular square roots of Σ and S. A minimax confidence set is determined by

$$W(U(S, \Sigma)) = |\Sigma^{-1}S|^{(\nu+k)/2} e\ \mathrm{tr}(-\tfrac{1}{2}\Sigma^{-1}S) \textstyle\prod_{i=1}^{p} (b_{ii}/\eta_{ii})^{p+1-2i}. \tag{6.2}$$

6.3. *Quantiles.* Let $X_1, \cdots, X_n$ be a sample from a continuous distribution F. Here $\theta = F$. Let $0 < p_1 < \cdots < p_m < 1$ be specified. Put $\gamma_i = \min\{x : F(x) \geq p_i\}$ and $\gamma = (\gamma_1, \cdots, \gamma_m)$. Let G consist of all strictly increasing continuous functions g mapping $\mathbb{R}$ onto $\mathbb{R}$, with actions $x_i \to g(x_i)$, $\gamma_i \to g(\gamma_i)$, and $F \to gF$, where $(gF)(gx) = F(x)$. This group does not satisfy the Hunt-Stein condition (3.3). Observe that G does not act transitively on Θ. However the orbit containing the Uniform (0, 1) distribution is dense in the space of continuous distributions with respect to the metric $d(F_1, F_2) = \sup_x |F_1(x) - F_2(x)|$. Consequently a confidence set that is optimal on this orbit must be optimal on Θ. Denote order statistics by $X_{(i)}$, with $X_{(0)} = -\infty$ and $X_{(n+1)} = \infty$. An invariantly sufficient function is given by: $T(X, \gamma) = (t_1, \cdots, t_m)$ if $X_{(t_i)} \leq \gamma_i < X_{(t_i+1)}$ for $i = 1, \cdots, m$. Consider the measure $m(d\gamma, F^*) = \prod_{i=1}^m F^*(d\gamma_i)$. Letting F^* be the Uniform (0, 1) distribution, we compute $W(t) = p^T(t; F^*, \gamma^*)$; i.e., $W(t)$ is the multinomial $(n; p_1, p_2 - p_1, \cdots, p_m - p_{m-1}, 1 - p_m)$ distribution evaluated at $(t_1, t_2 - t_1, \cdots, t_m - t_{m-1}, n - t_m)$. The confidence region obtained is a union of m-dimensional rectangles $X_{i=1}^m [X_{(t_i)}, X_{(t_i+1)})$.

7. Shape of confidence regions. The confidence regions given above are undesirable for some applications because of their shape. When γ is vector-valued one often wants simultaneous confidence sets for a family of parametric functions, say $\{\psi_j(\gamma) : j \in J\}$, rather than a single confidence region for γ, since the latter is difficult to interpret. While each confidence region $C(X)$ for γ determines by projection a family of simultaneous confidence sets

$$A_j(X) = \psi_j C(X), \qquad j \in J, \tag{7.1}$$

the confidence level of $C(X)$ provides only a lower bound on the probability of simultaneous coverage $P_\theta\{\psi_j(\gamma(\theta)) \in A_j(X)\ \forall j \in J\}$. On the other hand, each family of simultaneous confidence sets $\{A_j(X)\}$ determines a confidence region for γ,

$$C_1(X) = \cap_{j\in J}\, \psi_j^{-1} A_j(X), \tag{7.2}$$

and the confidence level of $C_1(X)$ equals the probability of simultaneous coverage of $\{A_j(X)\}$. A confidence region C_1 that satisfies (7.2) for some family $\{A_j\}$ is said to be *exact with respect to* $\{\psi_j : j \in J\}$, and the family $\{A_j\}$ is said to be *exact with respect to* C_1. If $\{A_j\}$ is determined by (7.1) and C_1 by (7.2) then $\{A_j\}$ is smallest exact with respect to C_1 and $C_1(X)$ is the smallest confidence region containing $C(X)$ that is exact with respect to $\{\psi_j\}$; see Wijsman (1980, Section 2). Condition (7.2) imposes a restriction on the shape of the region $C_1(x)$ for each $x \in \mathscr{X}$. And the smaller the family $\{\psi_j\}$ the greater the restriction; see Hooper (1981a, Lemma 2.3).

In the covariance matrix problem one may want simultaneous confidence intervals for all $a'\Sigma a$, $a \in \mathbb{R}^p$. For this family the only G-invariant exact confidence sets are those of the form

$$\{\Sigma : [\lambda_p, \lambda_1] \subseteq A\}, \qquad A \subseteq [0, \infty)\}, \tag{7.3}$$

where $\lambda_1 \geq \cdots \geq \lambda_p$ denote the ordered eigenvalues of $\Sigma^{-1}S$; see Wijsman (1980, Section 4.3). The confidence region determined by (6.1) can be written in the form

$$\{\Sigma : \textstyle\sum_{i=1}^p \{\lambda_i - (\nu + k)\ln(\lambda_i)\} \leq c\}. \tag{7.4}$$

Since $x - (\nu + k)\ln(x)$ is convex in x, the smallest set of the form (7.3) that contains (7.4) has $A = [c_1, c_2]$, where $c_1 < c_2$ are the two solutions of $x - (\nu + k)\ln(x) = c/p$. By Wijsman (1980, Equation 4.3.3), the family of simultaneous confidence intervals that is the smallest exact relative to (7.3) is given by $a'Sa/c_2 \leq a'\Sigma a \leq a'Sa/c_1$.

In the MANOVA and GMANOVA problems, families of parametric functions that have been studied include $\{a'M : a \in \mathbb{R}^m\}$, $\{Mb : b \in \mathbb{R}^p\}$, $\{a'Mb : a \in \mathbb{R}^m,\ b \in \mathbb{R}^p\}$, and $\{\operatorname{tr} N'M : N \in M(m, p)\}$. In the MANOVA problem, for each of these families, Wijsman (1980, Section 4.1) characterizes all exact fully invariant set estimators. Roy's maximum root procedure is exact with respect to each of the families and is the only fully invariant set estimator that is exact with respect to $\{a'Mb\}$. These results are extended to the GMANOVA problem in Hooper (1981a). Here the classes of exact invariant set estimators are larger than in MANOVA. However none of the set estimators of Section 5 are exact with respect to $\{a'Mb\}$. And Roy's maximum root, $\{M : \lambda_1(T_1) \leq c\}$, is the only set estimator based on the eigenvalues of T_1 that is exact with respect to $\{a'Mb\}$.

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