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The issue indicated by the title of Dawid and Stone's paper is substantial and indeed touches on larger questions as to what inference is and what a basis for inference could be. The paper is clearly organized with frequent examples and seems technically correct. The notation is very concise and generally adequate. However, the paper does not touch on the larger questions as to what inference is or what a basis could be, and may indeed be misleading in the pursuit of these questions.

The issue examined in the paper has two parts. The first concerns the particular statistical model, a *structured model* (Fraser, 1971)—although the inappropriate term functional model (Bunke, 1975) is used (Section (ii)). The second concerns an inference procedure, the inversion of the *structural equation* (Fraser, 1966, 1968). This procedure gives a nominal distribution on the parameter space, a *structural distribution* (Fraser, 1966, 1968, 1979; Bunke, 1975)—although the term fiducial distribution is used, and used incorrectly given Fisher's prescriptions (Section (i)). The procedure is examined only for self-consistency and confidence consistency, but not for validity, a primary criterion that should be mentioned and has been examined elsewhere (see Section (ii)). Some comments are given in Sections (i) to (vi).

(i) Fiducial inference. Fiducial probability is clearly the product of R. A. Fisher who gave rules, methods and conditions for its calculation. Fiducial inference is reasonably taken to be the use of fiducial probability as the "inference" from data.

Throughout Fisher's work the fiducial distribution is based on the "sampling frequency distribution" (Fisher's term) or on the "sampling distributions of the data" (Section 4.2). Thus Corollary 4.1 stating that a fiducial distribution is determined by the sampling distribution is presenting what in fact is a premise to fiducial theory. If fiducial is left to be what Fisher attempted and intended, then Corollary 4.1 is a misunderstanding or a mistake.

The key ingredient used by the authors in addition to the sampling distributions is an equation, a pivotal function (Fisher, 1956) for the distribution model or a structural equation (Fraser, 1968) for an error model. Fisher (1956) discusses such an additional ingredient: "It has been proposed that any set of functions having distributions independent of the parameters (i.e., pivotal function) ... can be used to transform the simultaneous frequency distribution of (sample standard deviations and correlation) into the simultaneous distribution of (population standard deviations and correlation) ...." He then asserts that "the (just mentioned) short cut ... has no claim to validity unless it can be proved to be equivalent to a general fiducial argument: The (pivotal functions) cannot indeed be made to supply such an argument." Fisher clearly rejected the *addition* of pivotal functions. With fairness to Fisher, the term fiducial cannot be applied to the procedures in the present paper.

The pivotal functions in the preceding example for Fisher embrace those in Example 2.3 which is then excluded by Fisher's criteria as a fiducial example.

(ii) Error models. Fiducial distributions were developed by Fisher in the context of the distribution (classical/traditional) model. Dawid and Stone focus on *error models*—models given in terms of an *error variable.*

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Error variables have been presented since the time of Gauss and Laplace, as for example in the common $\mathbf{y} = X\boldsymbol{\beta} + \mathbf{e}$. Error models, in which the error is taken as a basic *given*, were presented in Fraser (1966, 1968, 1979): an error variable E, a transformation θ, and the response $X = \theta E$. A *closure* property important for validity was included and the model called structural.

The structural model with error as a basic given plus data gave (ibid) an automatic conditioning, with no need for the ancillarity principle. The automatic conditioning was focal and fundamental—it could have been observed/used since the time of Gauss and Laplace. The conditional model gave (ibid) tests, confidence intervals, estimates—not just the commonly cited structural distributions.

The *structured model* was introduced (Fraser, 1971) for an error model without the above closure property. The identical model was subsequently considered by Bunke (1975) and called a functional model. Discussion in Fraser (1971) focused on validity, and the closure property was found essential, partitioning not being enough (Fraser, 1979; Brenner and Fraser 1979); validity was not addressed in Bunke (1975). The literature supports the term *structural model* rather than functional model, and *structural distribution* rather than fiducial distribution for error models.

(iii) Validity of models. The question as to what is the given with a statistical model is focal to the difference between the distribution and the structural model. What is the given is also fundamental to statistical inference itself (Fraser, 1979); but arbitrary *additives* frequently appear in inference processes (Brenner, Fraser and Monette, 1981).

The validity of a statistical model (what *parts* should be taken as given) has been discussed in Fraser (1979); and *objectivity* is a central criterion concerning *parts* of a model. Validity of an error model thus focuses on objectivity for the error variable (Brenner and Fraser, 1980, 1981). Validity of the model is not addressed by Dawid and Stone.

Criteria for validity/objectivity for error do not support in general an error space larger than the response space. Validity is not then available for material in the present paper, in particular Examples 2.2, 2.3, 2.5, 5.2. The criteria also bear on Example 2.1: what objective input could distinguish between a "Fisherian" family and the SFM structured version in Example 2.1?

(iv) Inference from a structured/structural model. A structural model with data gives many inference methods, of which a structural distribution is only one. If we accept that the structural distribution for a parameter is obtained by means of the structural equation, then the corresponding distribution for a component parameter is obtained by solving the structural equation for the component parameter. This trivial property becomes Lemma 3.1, as a consequence of perhaps misleading notation. Then, unfortunately, the succeeding Wilkinson example only adds confusion (see Section (v)).

A structural model and data imply much more than just a structural distribution. Tests of significance, confidence intervals and regions are available for the primary parameters, and likelihood and, where feasible, tests and confidence regions for the secondary parameters. These are discussed in Fraser (1968), and in more detail in Fraser (1979) where posterior distributions were deliberately downplayed. Most of these procedures derive validity from the automatic conditioning with the structural model.

In contrast to the structured model, the validity of the conditional distribution is, not in general, available, and only a *nominal* inversion onto the parameter space seems left. Does this nominal inversion somehow attain special status by the very nonvalidity for the more common methods of inference? Surely, the demonstrated lack of validity for the nominal error distribution from a general structured model with data should answer the question concerning the basis for such inference: none seems available.

(v) The examples. The range and generality of the examples in the paper seem disproportionately small in relation to the theoretical discussions. The examples are a small subset of those that have had detailed coverage in the structural model form.

The Example 2.2 concerning the normal correlation coefficient mentions an identity between two fiducial/structural distributions, and cites "an explanation of this striking identity" in Section 4.2. The explanation has two parts. The first involves solving for a component variable (ρ) of interest, a simple procedure that is not immediately apparent with the (trivial; see third paragraph of Section (iv)) Lemma 3.1. The second involves a conditional location-relation between ρ and r, although the Corollary 4.1 uses a weaker monotone property (an unneeded extension from the structural to structured case).

The paper cites Fraser (1964) as having reported the identity between the two distributions. In fact, "the explanation of the striking identity" had *also* been presented there: as first part, the solution for the component variable is given by (5.1, loc. sit.); as second part, the conditional location relation is given two displays later.

A Stein-Wilkinson example is presented to illustrate Lemma 3.1 (mentioned in Section (iv)). For this $X_1 = \theta_1 + E_1$, $X_2 = \theta_2 + E_2$, $E_i \sim N(0, 1)$ and interest centers on $\Lambda = \theta_1^2 + \theta_2^2$ and $Z = X_1^2 + X_2^2$. The paper states "there is no inconsistency (with Lemma 3.1), since reduction to Z does not produce an SFM." In fact, the example confounds the essence of Lemma 3.1: there *is* an SFM for the reduced Z; there is, however, *no* reduction of the structural equation to the variable Z.

(vi) Pivotal models. Dawid and Stone discuss pivotal models as if a special case of the structured models. A function $p(X, \theta)$ with a fixed distribution represented by E is summarized as the pivotal model $\langle E = p(x, \theta), E \sim P \rangle$. As presented, the pivotal function $E = p(X, \theta)$ is a solution of the structural equation $X = \theta E$ in the case where the error and data spaces are *bijectively* related by $X = \theta E$; such cases are those where the objectivity for error may obtain (see the validity considerations in Section (iii) and the closure property in Section (ii)).

In a mathematical sense, the pivotal model and the special structured model (with bijective property) are isomorphic. The choice of one model or the other model could be a matter of taste.

If, however, we accept the requirement that the error/pivotal distribution be objective (Section (iii), third paragraph), then the structured/structural model (Fraser, 1966, 1968) presents this directly and explicitly: the data distribution is presented in terms of a function (structural equation) of the basic given, the error variable.

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