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DISTRIBUTIONS OF MAXIMAL INVARIANTS USING QUOTIENT MEASURES[1]

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This paper demonstrates the use of proper actions and quotient measures in representations of non-central distributions of maximal invariants.

1. Introduction. Consider a statistical model with sample space X, parameter set Θ and $\theta \to P_\theta$; $\theta \in \Theta$ as the parameterization of the unknown probability measures on X. For a subset Θ_0 of Θ we have a statistical testing problem of testing the hypothesis $H_0: \theta \in \Theta_0$ versus the hypothesis $H: \theta \in \Theta$. Let $t: X \to Y$ be a statistic related to the test problem, for example a test statistic or estimator under H_0. For the testing problem above the transformed measure $t(P_\theta)$ is called a central distribution if $\theta \in \Theta_0$ and a non-central distribution if $\theta \in \Theta \backslash \Theta_0$. The representation of the non-central distribution is often given in terms of a correction factor, which is simply the density of $t(P_\theta)$, $\theta \in \Theta$, with respect to a $t(P_\theta)$, $\theta \in \Theta_0$. In the description of the correction factor, group actions will often play a fundamental role. This is especially the case in multivariate statistical analysis. See, e.g., James (1964) and the review paper by Muirhead (1978).

The introduction of a group action leads one to the study of a maximal invariant function. For example, an estimator can be a maximal invariant function; or, if a statistical testing problem is invariant under a group action, all invariant test statistics have a unique factorization through a maximal invariant function. For this reason some literature concentrates on the problem of finding the distribution of a maximal invariant function from a general point of view: Bondar (1976), Koehn (1970), Wijsman (1967, 1978).

Let G be a group acting on X and let P be a probability measure on X. Let X/G denote the space of orbits and $\Pi: X \to X/G$ the orbit projection. The main problem of interest is to represent (find) $\Pi(P)$. Any representation of X/G and Π is usually called a maximal invariant function. Often one wants to find a particular maximal invariant function $t: X \to Y$ where Y has some extra structure (e.g. Y might be a nice subset of $\mathbb{R}^n$), such that $t(P)$ can be represented by a density with respect to a measure on Y (e.g. a restriction of a Lebesgue measure). As we shall demonstrate later, many general results about distributions of maximal invariants are simple consequences of the theory of proper actions and quotient measures, which are defined directly on the abstract space X/G. Since this theory seems to be unfamiliar to statisticians, we shall outline some of the background by extracting parts of Bourbaki (1956, 1960, 1963, 1965). See also Tjur (1980).

The idea of using the theory of proper actions and quotient measures to obtain representations of non-central distributions arises from the work to be reported in Andersson, et al. (1982) on a general algebraic theory of normal statistical models. In that paper, proper actions and quotient measures are used to derive the central distributions of the maximal invariants, which occur in this theory of multivariate statistical analysis.

2. The decomposition of a measure. A Radon measure on a locally compact Hausdorff space X is a positive linear form $\mu: \mathcal{K}(X) \to \mathbb{R}$, where $\mathcal{K}(X)$ is the vector-space of continuous real valued functions on X with compact support. The integration theory is

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the extension of μ to a larger class of functions called the μ-integrable functions. The relation to the abstract measure theory and integration theory on the σ-ring generated by the compact sets in X is obtained through Riesz's representation theorem. When X is small, that is, has a denumerable basis for the topology, the difference between the two approaches is only formal.

Let $\mathscr{M}(X)$ denote the space of (Radon) measures on X equipped with the weak topology. For $\mu \in \mathscr{M}(X)$ we denote the support of μ by $\text{supp}(\mu)$. The integral of a μ-integrable function f is denoted by $\int_X f(x)\, d\mu(x)$ or $\int_X f\, d\mu$. For $f \in \mathscr{K}(X)$ we have in addition the expression $\mu(f)$. The definition of measurability with respect to μ of a mapping from X into a topological space T can be found in Bourbaki (1965); in the cases where T has a denumerable basis for the topology, the definition of measurability with respect to μ is the classical one, that is the inverse image of a Borel-set in T is μ-measurable. Otherwise the condition is stronger than the classical one.

Let now ν be a measure on Y and let $(\mu_y)_{y\in Y}$ be a family of measures on X indexed by Y. Suppose that

(1) for every $k \in \mathscr{K}(X)$ the function $y \to \mu_y(k)$, $y \in Y$, is ν-integrable.

In this case we are able to define a measure λ called the *mixture* of the family $(\mu_y)_{y\in Y}$ with respect to ν by the definition

$$\lambda(k) = \int_Y \mu_y(k)\, d\nu(y), \qquad k \in \mathscr{K}(X). \tag{2}$$

The measure λ is also denoted by $\int_Y \mu_y\, d\nu(y)$. To ensure the extension of (2) to λ-integrable functions one must assume that the mapping

$$Y \to \mathscr{M}(X), \qquad y \to \mu_y \tag{3}$$

is measurable with respect to ν and that all spaces are σ-compact. In this case the relation (2) is extended in the following way: Let f be a λ-integrable function; then for ν-almost all $y \in Y$ we have that f is μ_y-integrable and the ν-almost everywhere defined real valued function $y \to \int_X f(x)\, d\mu_y(x)$ on Y is ν-integrable with the integral

$$\int_Y \left(\int_X f(x)\, d\mu_y(x) \right) d\nu(y) = \int_X f(x)\, d\lambda(x). \tag{4}$$

Let now $t: X \to Y$ be λ-measurable. If furthermore for ν-almost all $y \in Y$ we have that

$$\text{supp}(\mu_y) \subseteq t^{-1}(y) \tag{5}$$

then we call the pair $((\mu_y)_{y\in Y}, \nu)$ a *decomposition* of λ with respect to t.

Construction of measures with densities can be considered as a special case of mixtures: Suppose for a moment that $Y = X$ and let p be a non-negative locally ν-integrable function on X. (This means that kp is ν-integrable for every $k \in \mathscr{K}(X)$.) Then the condition (1) is fulfilled for the family $(p(x)\varepsilon_x)_{x\in X}$, where ε_x is the one point measure for $x \in X$. The mixture of this family with respect to ν is denoted $p\nu$ and (2) becomes

$$p\nu(k) = \int_x p(x)k(x)\, d\nu(x),$$

$k \in \mathscr{K}(X)$. It is furthermore seen that (3) is measurable and that $((p(x)\varepsilon_x)_{x\in X}, \nu)$ is a decomposition of $p\nu$ w.r.t. the identity mapping on X. Strictly speaking, it is not necessary for our purpose to define mixtures and decompositions as generally as above. The following definition will be enough:

(1′) For every $k \in \mathscr{K}(X)$ the function $y \to \mu_y(k)$, $y \in Y$, is an element in $\mathscr{K}(Y)$.

Since (1′) does not depend on ν we are able to define the mixture of $(\mu_y)_{y\in Y}$ with respect to every $\nu \in \mathscr{M}(Y)$ and obtain a continuous mapping

$$\mathscr{M}(Y) \to \mathscr{M}(X), \qquad \nu \to \int_Y \mu_y \, d\nu(y) \tag{6}$$

defined by the family $(\mu_y)_{y \in Y}$.

When all spaces are σ-compact, then the extension of (2) to the λ-integrable functions is ensured. To define a decomposition we can furthermore suppose that t is continuous and (5) is valid for all $y \in Y$. Again this more restrictive version of (5) does not contain ν and it will ensure that (6) becomes injective. We shall point out that $t(\lambda)$ is not in general defined since t does not in general transform the measure λ. Nevertheless, if t does transform $\lambda = \int_Y \mu_y \, d\nu(y)$ (that is, for every $k \in \mathscr{K}(Y)$, $k \circ t$ is λ-integrable) then we have

$$t(\lambda)(k) = \lambda(k \circ t) = \int_Y \int_X k(t(x)) \, d\mu_y(x) \, d\nu(y) = \int_Y k(y) \mu_y(X) \, d\nu(y),$$

which shows that $t(\lambda) = f\nu$, where $f(y) = \mu_y(X)$ for ν-almost all $y \in Y$. If all measures are probability measures we have $t(\lambda) = \nu$ and $(\mu_y)_{y \in Y}$ is a version of the conditional distribution given t.

Suppose that we have a decomposition $((\mu_y)_{y \in Y}, \nu)$ of λ. If $P = p\lambda$—that is, P has density p w.r.t. λ—then although t does not transform λ, it is easy to represent $t(P)$ as $t(P) = q\nu$, where

$$q(y) = \int_X p(x) \, d\mu_y(x) \quad \text{for } \nu\text{-almost all} \quad y \in Y. \tag{7}$$

This statement follows from the following calculations: For $h \in \mathscr{K}(Y)$ we have

$$t(P)\,(h) = \int_X h \circ t \, dP = \int_X p(h \circ t) \, d\lambda = \int_Y \int_X p(x) h(t(x)) \, d\mu_y(x) \, d\nu(y)$$
$$= \int_Y h(y) \int_X p(x) \, d\mu_y(x) \, d\nu(y) = \int_Y hq d\nu,.$$

since $\operatorname{supp}(\mu_y) \subseteq t^{-1}(y)$ for ν-almost all $y \in Y$.

It is seen from the above considerations that the problem of describing the non-central distribution is reduced or rather changed to the problem of existence and characterization of a decomposition of the measure λ. In the case where X and Y are Riemannian manifolds, t is a (surjectively) regular transformation, and λ is the geometric measure λ_X on X, then such a decomposition $((\mu_y)_{y \in Y'} \lambda_Y)$ with respect to t exists and is characterized in the following way: λ_Y is the geometric measure on Y and $\mu_y = F\lambda_{X_y}$ where λ_{X_y} is the geometric measure on the sub-Riemannian manifold $t^{-1}(y)$ of X_y and the density F is a differentiable function on X defined by means of the differential of t. Nevertheless we shall not concentrate on this case but on another case, namely, when a group action is present. Under "nice" conditions, group actions ensure a decomposition through the so-called quotient measure.

3. The quotient measure. The relevant references for this section are Bourbaki (1960, 1963). For a comprehensive treatment see Andersson (1978).

Let G be a σ-compact locally compact Hausdorff group and suppose that G acts properly on X. The action of $g \in G$ on $x \in X$ is denoted by gx. Proper action means that the action is continuous and that the mapping

$$G \times X \to X \times X, \qquad (g, x) \to (gx, x) \tag{8}$$

is proper (the inverse image of a compact set is compact). The condition ensures that the final topology on X/G under the orbit projection $\Pi: X \to X/G$ is Hausdorff and locally compact (and of course also σ-compact) so that the notion of Radon measures on X/G can be applied. Furthermore the orbit mapping $\Pi_x: G \to X(g \to gx)$ for $x \in X$ becomes proper.

Since a proper mapping transforms every measure we have that $\beta_x = \Pi_x(\beta)$ is well defined for every measure β on G.

Under the above assumptions, the mapping

$$X \to \mathcal{M}(X), \qquad x \to \beta_x, \tag{9}$$

becomes continuous and $\operatorname{supp}(\beta_x) \subseteq \Pi^{-1}(\Pi(x))$, $x \in X$. If β is a right Haar measure on G, (9) will be a G-invariant function, which then has a unique continuous factorization through X/G. We then have the continuous mapping

$$X/G \to \mathcal{M}(X), \qquad u \to \beta_u, \tag{10}$$

with $\operatorname{supp}(\beta_u) = \Pi^{-1}(u)$, $u \in X/G$. For $k \in \mathcal{K}(X)$ the real valued function $\bar{k}$ on X/G defined by $\bar{k}(u) = \beta_u(k)$ becomes an element in $\mathcal{K}(X/G)$ (see condition (1′)) and the mapping

$$\mathcal{K}(X) \to \mathcal{K}(X/G), \qquad k \to \bar{k}, \tag{11}$$

becomes positive, linear, and onto. This defines an injective "linear" mapping

$$\mathcal{M}(X/G) \to \mathcal{M}(X), \qquad \mu \to \mu^{\#}, \tag{12}$$

where $\mu^{\#}(k) = \mu(\bar{k})$ and $k \in \mathcal{K}(X)$. It can be shown that the image is defined by the condition that λ is in the image if

$$g\lambda = \Delta_G(g)\lambda; \qquad \forall g \in G, \tag{13}$$

where Δ_G is the modular function of the group G. The condition (13) means that λ is relatively invariant with multiplier Δ_G^{-1}; if G is unimodular then (13) implies that λ is invariant. The modular function Δ_G is sometimes called the right hand modulus of G.

Later we shall need the following property of the mapping (12): a non-negative function p on X/G is locally μ-integrable if and only if $p \circ \Pi$ is locally $\mu^{\#}$-integrable, and in this case one has that

$$(p\mu)^{\#} = (p \circ \Pi)\mu^{\#}. \tag{14}$$

The above considerations then show that for a measure λ on X which satisfies (13) there exists one and only one measure denoted by λ/β, called the quotient measure, such that $(\lambda/\beta)^{\#} = \lambda$; that is, for every λ-integrable function we have the relations

$$\begin{aligned} \int_X f(x)\, d\lambda(x) &= \int_{X/G} \left(\int_X f(x)\, d\beta_u(x) \right) d\lambda/\beta(u), \\ \bar{f}(\Pi(z)) &= \int_X f(x)\, d\beta_{\Pi(z)}(x) = \int_G f(gz)\, d\beta(g), \qquad z \in X. \end{aligned} \tag{15}$$

Thus (15) shows that $((\beta_u)_{u \in X/G}, \lambda/\beta)$ is a decomposition of λ with respect to Π.

4. The application of the quotient measure to the distribution of a maximal invariant. Let G act properly on X and let P be a probability density p with respect to λ, where λ satisfies (13) in Section 3. Then it follows directly from the considerations in Sections 2 and 3 that the distribution $Q = \Pi(P)$ of the orbit-projection Π (the maximal invariant function) is given by $Q = q\lambda/\beta$ where the non-negative λ/β-integrable function q with λ/β-integral 1 is given λ/β-almost everywhere by

$$q(\Pi(x)) = \int_G p(gx)\, d\beta(g), \qquad x \in X \tag{16}$$

(Bourbaki, 1963, VII, Section 2, 3°, Proposition 5c). If μ is another measure on X which is relatively invariant under G with multiplier χ_0 (that is, $g^{-1}\mu = \chi_0(g)\mu$ for every $g \in G$, where $\chi_0 : G \to (0, \infty)$ is continuous and $\chi_0(g_1 g_2) = \chi_0(g_1)\chi_0(g_2)$, $g_1, g_2 \in G$) then in the case where $P = p\mu$ we can use the following facts to obtain a representation of $\Pi(P)$. Since

X/G is σ-compact it is also paracompact and it follows from Proposition 7 in Bourbaki (1963, Section 2, 4°), that for every continuous multiplier χ there exists a continuous positive function n on X with the property

$$n(gx) = \chi(g)n(x), \qquad x \in X, \qquad g \in G. \tag{17}$$

Let n be a continuous positive function on X which satisfies (17) with $\chi = \chi_0 \Delta_G$. Then the measure $\lambda = (1/n)\mu$ satisfies (13) and $P = np\lambda$. Therefore $\Pi(P) = q\lambda/\beta$, where now (16) becomes

$$q(\Pi(x)) = n(x) \int_G p(gx)\chi_0(g)\, d\alpha(g), \qquad x \in X, \tag{18}$$

where $\alpha = \Delta_G \beta$ becomes a left Haar-measure on G. For two probability measures P_1 and P_2 on X with densities p_1 and p_2 with respect to μ we then can define the function ρ on X/G by

$$\rho(\Pi(x)) = \frac{\int_G p_1(gx)\chi_0(g)\, d\alpha(g)}{\int_G p_2(gx)\chi_0(g)\, d\alpha(g)} \tag{19}$$

for those $x \in X$ for which the denominator is positive. The function ρ is thus a version of $d\Pi(P_1)/d\Pi(P_2)$.

Under the assumptions that X is an (open) subset of an Euclidean space $\mathbb{R}^n$, that μ is the restriction of the Lebesgue measure to X, that G is a Lie subgroup of the group of $n \times n$ nonsingular matrices and that the action has the linear Cartan property, the result (19) was obtained by Wijsman (1967). His result (3) now follows from our (19) and the remark that the Lebesgue measure is relatively invariant under G with $g \to |\det(g)|$ as the multiplier.

The proper action assumption is nice to work with since we have the following almost trivial result, which together with the remarks below can be considered as a very useful extension of Theorem 2 in Wijsman (1967).

PROPOSITION 1. *Let G act properly on X, let H be a closed subgroup of G and let $Y \subseteq X$ be closed with the property $HY = Y$. Then the restriction of the proper action $G \times X \to X$ to $H \times Y \to Y$ is proper.*

PROOF. Bourbaki (1960, Chapter 3, Section 4.1, Examples 1 and 2).

REMARK 1. Every continuous action of a compact group is proper.

REMARK 2. The classical (transitive) action of the group of $n \times n$ nonsingular matrices on the set of $n \times n$ positive definite matrices is proper. One only has to show that the inverse image of a bounded set by the mapping (8) is bounded. The fact that this action is proper was first pointed out in the thesis by Tolver Jensen (1971) written under the supervision of H. K. Brøns.

REMARK 3. The (transitive and free) action of the translation group on an affine space is proper.

By combining the remarks and the proposition above, one easily obtains the proper action condition needed for the distribution of maximal invariants in multivariate analysis. A new application is given in Andersson and Perlman (1981).

5. Characterization of the quotient measure by invariance. Suppose that G is a closed subgroup of a locally compact group K and that the proper action of G on X is a restriction of a continuous action of K on X. Furthermore let H be a closed subgroup of K (H is then a locally compact group in the induced topology) with the properties that for every $g \in G$ and $h \in H$ there exists a $g' \in G$ such that

$$hg = g'h \tag{20}$$

and that $K = HG$. The property described by (20) determines a group homomorphism

$$H \to \mathrm{Aut}(G), \qquad h \to \phi_h = (g \to hgh^{-1} = g'),$$

where $\mathrm{Aut}(G)$ is the group of automorphisms of G. If the decomposition of $k \in K$ into $k = hg$ is unique then K is a semidirect product of H and G. For every $h \in H$ we have the mapping $x \to \Pi(hx)$ from X into X/G, which is seen to be G-invariant because of (20). Then $h \in H$ uniquely defines $\bar{h}: X/G \to X/G$, and it is easily seen that we have a continuous action of H on X/G given by

$$H \times X/G \to X/G, \qquad (h, u) \to \bar{h}(u) = hu. \tag{21}$$

The continuity follows from the facts that the diagram

$$\begin{array}{ccc} H \times X & \to & X \\ \downarrow 1 \times \Pi & & \downarrow \Pi \\ H \times X/G & \to & X/G \end{array} \tag{22}$$

commutes, where the horizontal mappings are the actions, and that $1 \times \Pi$ and Π are open and onto. We remark that the action of K on X is transitive if and only if the action of H on X/G is transitive.

Let $\mathrm{mod}\phi_h$ denote the modulus of ϕ_h, $h \in H$. For $f \in \mathscr{K}(X)$ and $h \in H$ we have (see (15))

$$(\overline{hf})(\Pi(x)) = \int_G f(h^{-1}gx)\, d\beta(g) = \int_G f(\phi_h^{-1}(g)h^{-1}x)\, d\beta(g)$$
$$= (\mathrm{mod}\phi_h) \int_G f(gh^{-1}x)\, d\beta(g) = (\mathrm{mod}\phi_h)\bar{f}(\Pi(h^{-1}x)) = (\mathrm{mod}\phi_h)h\bar{f}(\Pi(x)).$$

Thus one has

$$(\overline{hf}) = (\mathrm{mod}\phi_h)h\bar{f}, \qquad h \in H, \qquad f \in \mathscr{K}(X). \tag{23}$$

From (23) and (12) it follows that

$$(h\mu)^{\#} = (\mathrm{mod}\phi_h)h\mu^{\#}, \qquad \mu \in \mathscr{M}(X/G). \tag{24}$$

It now follows from (23) and (14) and the fact that (12) is one to one, that a measure $\mu \in \mathscr{M}(X/G)$ has the property

$$h\mu = p_h\mu, \qquad h \in H, \tag{25}$$

where p_h is a non-negative locally μ-integrable function, $h \in H$, if and only if

$$h\mu^{\#} = (\mathrm{mod}\phi_h)^{-1}(p_h \circ \Pi)\mu^{\#}, \qquad h \in H. \tag{26}$$

It follows from (13) that (26) is equivalent to

$$k\mu^{\#} = \Delta_G(g)(\mathrm{mod}\phi_h)^{-1}(p_h \circ \Pi)\mu^{\#}, \qquad k = hg \in K. \tag{27}$$

Under continuity assumptions one has the following version of the equivalence between (25) and (26).

PROPOSITION 2. *A measure $\mu \in \mathscr{M}(X/G)$ has the property* (25), *where p_h is a non-negative continuous function, if and only if*

(28) $$h\mu^{\#} = q_h\mu^{\#}, \quad h \in H,$$

where q_h is a non-negative continuous function. In this case one has

(29) $$q_h(x) = (\text{mod}\phi_h)^{-1}p_h(\Pi(x)), \qquad x \in \text{supp}(\mu^{\#}), \qquad h \in H.$$

PROOF. One only has to show that q_h is an invariant function on the G-invariant set $\text{supp}(\mu^{\#})$ under the action of G. This follows from

$$(q_h \circ g)\mu^{\#} = g^{-1}(q_h g\mu^{\#}) = \Delta_G(g)g^{-1}(q_h\mu^{\#}) = \Delta_G(g)g^{-1}(h\mu^{\#})$$
$$= h\mu^{\#} = q_h\mu^{\#}, \qquad g \in G, \qquad h \in H.$$

An application of Proposition 2 is given in Andersson, et al. (1981).

Thus one has especially that $\mu \in \mathscr{M}(X/G)$ is relatively invariant under the action of H with multiplier χ if and only if $\mu^{\#}$ is relatively invariant under the action of K with multiplier $k \rightarrow \Delta_G(g)^{-1}(\text{mod}\phi_h)\chi(h)$, $k = hg \in K$.

The above considerations show how invariance properties under the action of K of a measure λ, which has the property (13), transform into invariance properties of λ/β under the action of H.

When K acts transitively and properly on X, then for every multiplier χ_0 on K there exists one and only one (up to multiplication by a positive constant) relatively invariant measure on X with multiplier χ_0.

Although in this case the action of H on X/G generally is not proper (but of course transitive and continuous) the consideration above shows that for every multiplier χ on H there exists one and only one (up to multiplication by a positive constant) relatively invariant measure on X/G with multiplier χ.

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