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ADMISSIBILITY IN LINEAR ESTIMATION

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Necessary and sufficient conditions for a linear estimator to be admissible among linear estimators are described. The model assumed is general, allowing for relations between elements of the mean vector and covariance matrix, and allowing the covariance matrix to vary in an arbitrary subset of nonnegative definite symmetric matrices.

1. Introduction. The work of Olsen, Seely and Birkes (1976) provided seminal results in the characterization of admissible linear estimators in the general linear model. They described necessary conditions for the admissibility of unbiased linear estimators and showed that the admissible unbiased linear estimators form a minimal complete class of unbiased linear estimators. Their necessary conditions are demonstrably not sufficient. LaMotte (1977b) noted an obvious extension of their characterization.

Without the restriction to unbiasedness, admissible linear estimators have been characterized only in special linear models. Cohen (1966) characterized admissible linear estimators of the mean vector while assuming a covariance matrix of the form $\sigma^2 I$. C. R. Rao (1976) accomplished the same characterization for models with mean vectors varying through a linear subspace and covariance matrices of the form $\sigma^2 V$ with V known (i.e., restricted to a subspace of one dimension). Neither of these efforts appears to generalize to models in which the covariance matrix varies over more than one dimension, or in which the mean vector and covariance matrix are functionally related, or in which restrictions on the parameters of the model restrict attention to a subset of the natural parameter space. For example, in the simple linear regression model, C. R. Rao's results guarantee that the least squares estimator is admissible among linear estimators. But Marquardt (1970) and Perlman (1972) observed that if the parameter space is restricted in certain ways, then the least squares estimator is not admissible. Olsen, Seely and Birkes (1976) established a relation between admissibility and bestness (defined below) which allowed them to establish necessary conditions for admissibility in any given parameter space. The same sort of relation is used here to characterize admissible linear estimators.

2. Definitions and Summary. Let Y be a random n-vector with mean vector μ and variance-covariance matrix V, with (μ, V) contained in an arbitrary subset $\mathscr{P}$ of the Cartesian product of Euclidean n-space R^n and the set of $n \times n$ symmetric nonnegative definite matrices. Let C be an $n \times t$ matrix of constants and consider estimating $C'\mu$ by linear functions $L'Y$ with L an $n \times t$ matrix of constants. Total mean squared error will be used as the risk function:

$$\text{(2.1)} \quad \text{TMSE}_L(V, \mu\mu') = E\{(L'Y - C'\mu)'(L'Y - C'\mu)\} = \text{tr}\{L'VL + (L-C)'\mu\mu'(L-C)\}.$$

For a matrix M, denote the transpose of M, the linear subspace spanned by the columns of M, and the null space $\{x : Mx = 0\}$ of M by M', $\mathbf{R}(M)$, and $\mathbf{N}(M)$, respectively. Denote the trace of a square matrix M by $\text{tr}(M)$. We will frequently deal with linear subspaces $\mathscr{A}$ of $r \times s$ matrices, in which case the trace inner product $\text{tr}(MH')$ will be used, along with the corresponding squared norm $\text{tr}(MM')$. If $\mathscr{U}$ is a linear subspace of a vector space $\mathscr{A}$, denote by $\bar{\mathscr{U}}$ a linear subspace such that $\mathscr{A}$ is the direct sum of $\mathscr{U}$ and $\bar{\mathscr{U}}$. The minimal linear subspace containing a set $\mathscr{Q}$ of vectors will be denoted by $\text{sp}(\mathscr{Q})$. For a subset $\mathscr{Q}$ of $\mathscr{A}$, $[\mathscr{Q}]$ will denote the minimal closed convex cone containing $\mathscr{Q}$.

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We shall be concerned with admissibility of estimators $L'Y$ of $C'\mu$ among non-empty affine subsets of $n \times t$ matrices of the form $\mathscr{L} = \{L:AL = B\}$. We shall often refer to estimators $L'Y$ in terms of L, e.g., as "estimators $L \in \mathscr{L}$." The set of all $n \times t$ matrices is such an affine set, as is the set $\{L:X'L = X'C\}$ of unbiased linear estimators of $C'\mu$ when $\text{sp}\{\mu:(\mu, V) \in \mathscr{P}\} = \mathbf{R}(X)$ and X is an $n \times p$ matrix. With $L_0 \in \mathscr{L}$ and N_0 a matrix such that $\mathbf{R}(N) = \mathbf{N}(A)$, every matrix in $\mathscr{L}$ has a representation $L = L_0 + NZ$ for some matrix Z.

Noting that TMSE_L is a real-valued linear function on $\mathscr{T} = \{(S_1, S_2):S_2 = \mu\mu', (\mu, S_1) \in \mathscr{P}\}$, let $\mathscr{W}$ be a linear subspace of $\{(S_1, S_2):S_1$ and S_2 are $n \times n$ symmetric matrices$\}$ containing $\mathscr{T}$; once $\mathscr{W}$ is so chosen, it remains fixed in subsequent developments. Let $\mathscr{W}_+ = \{(S_1, S_2):S_1$ and S_2 are nnd$\}$; $\mathscr{T} \subset \mathscr{W}_+$. Let $\mathscr{X} \subset \mathscr{W}$. Extend TMSE_L to $\mathscr{W}$ as $\text{TMSE}_L(S) = \text{tr}[L'S_1L + (L - C)'S_2(L - C)]$. The linear estimator K will be said to be *as good as* L on $\mathscr{X}$ iff $\text{TMSE}_K(S) \le \text{TMSE}_L(S)$ for all $S \in \mathscr{X}$; and K will be said to be *better than* L on $\mathscr{X}$ iff K is as good as L on $\mathscr{X}$ and K has less TMSE than L at some point in $\mathscr{X}$. An estimator L is *admissible among* $\mathscr{L}$ on $\mathscr{X}$ iff $L \in \mathscr{L}$ and no estimator in $\mathscr{L}$ is better than L on $\mathscr{X}$. Given $S = (S_1, S_2)$ in $\mathscr{W}$, $L \in \mathscr{L}$ is *best among* $\mathscr{L}$ *at* S iff $\text{TMSE}_L(S) \le \text{TMSE}_K(S)$ for every $K \in \mathscr{L}$.

Proposition 3.6 of Olsen, Seely and Birkes (1976) may be extended fairly easily to establish that if L_* is admissible among $\mathscr{L}$ on $\mathscr{T}$ then there exists a non-zero point S_* in $[\mathscr{T}]$ such that L_* is best among $\mathscr{L}$ at S_*. Further, it may be seen that if L_* is best among $\mathscr{L}$ at S_* in $[\mathscr{T}]$ then admissibility of L_* among $\mathscr{L}$ on $\mathscr{T}$ is equivalent to admissibility of L_* among $\mathscr{L}_1 = \{L \in \mathscr{L}:L$ is best among $\mathscr{L}$ at $S_*\}$ on $\mathscr{T}$. $\mathscr{L}_1$ is of the form $\{L:A_1L = B_1\}$, is an affine subset of $\mathscr{L}$, and is a proper subset of $\mathscr{L}$ unless every estimator in $\mathscr{L}$ is best among $\mathscr{L}$ at S_*. This suggests that a characterization of admissible estimators among $\mathscr{L}$ can be obtained by repeated applications of these results, reducing the dimension of $\mathscr{L}_1$ at each step. However, if $[\mathscr{T}]$ contains a nonzero point at which all members of $\mathscr{L}$ are best, then it cannot be guaranteed that the dimension of $\mathscr{L}_1$ is less than the dimension of $\mathscr{L}$. Upon applying these results to $\mathscr{L}_1$, every member of $\mathscr{L}_1$ is best among $\mathscr{L}_1$ at S_*, so that the suggested procedure is stymied at the next step.

Define *trivial points* for $\mathscr{L}$ in $\mathscr{W}$ as points at which every member of $\mathscr{L}$ is best among $\mathscr{L}$. Let $\mathscr{S}$ denote the set of trivial points for $\mathscr{L}$. The following results are proved in the next section. $\mathscr{S}$ is a linear subspace. If $\mathscr{T} \subset \mathscr{S}$ then every member of $\mathscr{L}$ is admissible among $\mathscr{L}$ on $\mathscr{T}$. If $\mathscr{T} \not\subset \mathscr{S}$, L_* is admissible among $\mathscr{L}$ on $\mathscr{T}$ only if there exists a point S_* in $[\mathscr{T} + \mathscr{S}]$ but not in $\mathscr{S}$ such that L_* is best among $\mathscr{L}$ at S_*. Let $\mathscr{S}_1$ denote the set of trivial points for $\mathscr{L}_1$ in $\mathscr{W}$. Since $S_* \in [\mathscr{T} + \mathscr{S}]$ is not a trivial point for $\mathscr{L}$, then $\mathscr{S}_1$ contains $\mathscr{S}$ as a proper subset and hence has dimension greater than the dimension of $\mathscr{S}$. Thus, given $L_* \in \mathscr{L}$, in a finite number of steps, the admissiblity of L_* is determined: L_* is admissible among $\mathscr{L}$ on $\mathscr{T}$ iff ultimately $\mathscr{T} \subset \mathscr{S}_r$, with $0 \le r \le$ dimension of $\mathscr{W} -$ dimension of $\mathscr{S}$.

It is noted that if L_* is best among $\mathscr{L}$ at a point S_* in $[\mathscr{T} + \mathscr{S}]$ such that the rank of $N'(S_{1*} + S_{2*})N$ is maximal among $\{N'(S_1 + S_2)N:(S_1, S_2) = S \in [\mathscr{T} + S]\}$ then L_* is admissible among $\mathscr{L}$ on $\mathscr{T}$. This leads to the conclusion that L_* is admissible among $\mathscr{L}$ on $\mathscr{T}$ iff ultimately L_* is best among $\mathscr{L}_r$ at a point S_* in $[\mathscr{T} + \mathscr{S}_r]$ such that $N_r'(S_{1*} + S_{2*})N_r$ has maximal rank among $\{N_r'(S_1 + S_2)N_r:S \in [\mathscr{T} + \mathscr{S}_r]\}$.

The class $\mathscr{L}$ may be replaced initially by an essentially complete class in $\mathscr{L}$ in such a way that matrices of maximal rank in $\{N_r'(S_1 + S_2)N_r:S \in [\mathscr{T} + \mathscr{S}_r]\}$ are positive definite. Among this essentially complete class, L_* is admissible iff ultimately there exists an S_* in $[\mathscr{T} + \mathscr{S}_r]$ such that L_* is the only member of $\mathscr{L}_r$ best at S_*.

3. Propositions and Proofs. Theorem 3.1 provides results on bestness.

THEOREM 3.1. *Let* $S = (S_1, S_2) \in \mathscr{W}$ *and define*

$$S_{11} = N'(S_1 + S_2)N, \quad S_{12} = N'S_1L_0 + N'S_2(L_0 - C),$$

$$S_{22} = L_0'S_1L_0 + (L_0 - C)'S_2(L_0 - C).$$

(i) *In order that there exist an L best among $\mathscr{L}$ at S, it is necessary and sufficient that S_{11} be* nnd *and every column of S_{12} be in $\mathbf{R}(S_{11})$.*

(ii) *In order that $L \in \mathscr{L}$ be best among $\mathscr{L}$ at S, it is necessary and sufficient that S_{11} be* nnd *and $N'(S_1 + S_2)L = N'S_2C$.*

(iii) *If S_{11} is* nnd *and H is a matrix such that $S_{11}H + S_{12} = 0$, then for every $L \in \mathscr{L}$,*

$$\text{TMSE}_L(S) \geq \operatorname{tr}(S_{22} - H'S_{11}H). \tag{3.1}$$

(iv) *In order that every $L \in \mathscr{L}$ be best among $\mathscr{L}$ at S, it is necessary and sufficient that $S_{11} = 0$ and $S_{12} = 0$.*

(v) *There exists exactly one L best among $\mathscr{L}$ at S* iff *S_{11} is positive definite.*

(vi) *If there exists an $L \in \mathscr{L}$ best among $\mathscr{L}$ at S then the set of members of $\mathscr{L}$ best among $\mathscr{L}$ at S is an affine subset of the form $\mathscr{L}_1 = \{L : A_1L = B_1\}$, and $\mathscr{L}_1$ is a proper subset of $\mathscr{L}$* iff *S_{11} is nonzero.*

Proof. For any $S \in \mathscr{W}$, with S_{11}, S_{12} and S_{22} defined as above, there exists a matrix H, whose columns are in $\mathbf{R}(S_{11})$, and a matrix T, whose columns are in $\mathbf{N}(S_{11})$, such that $-S_{12} = S_{11}H + T$. With $L = L_0 + NZ$,

$$\begin{aligned}\text{TMSE}_L(S) &= \operatorname{tr}(Z'S_{11}Z + Z'S_{12} + S_{12}'Z + S_{22}) \\ &= \operatorname{tr}\{(Z - H)'S_{11}(Z - H) - Z'T - T'Z\} + \operatorname{tr}(S_{22} - H'S_{11}H).\end{aligned}$$

If S_{11} is nnd and columns of S_{12} are in $R(S_{11})$, then $T = 0$ and the sufficiency of (i) is clear upon noting that $L_0 + NH$ is best among $\mathscr{L}$ at S. If not all columns of S_{12} are in $R(S_{11})$, then T is nonzero: in this case, with $L_\alpha = L_0 + N(H + \alpha T)$, $\{\text{TMSE}_{L_\alpha}(S) : \alpha > 0\}$ is unbounded below so there is no L in $\mathscr{L}$ best at S. If $T = 0$ and S_{11} is not nnd, let P_- be an eigenvector of S_{11} corresponding to a negative eigenvalue of S_{11} and let $Z_\alpha = \alpha(P_-, 0, \cdots 0) + H$ and $L_\alpha = L_0 + NZ_\alpha$, so that $\{\text{TMSE}_{L_\alpha}(S) : \alpha > 0\}$ is unbounded below, and thus there exists no $L \in \mathscr{L}$ best at S.

Proofs of (ii) through (v) are straightforward. With $L = L_0 + NZ$, (ii) is equivalent to S_{11} nnd and $S_{11}Z + S_{12} = 0$. For (vi), if S permits a best estimator among $\mathscr{L}$, then the set of $n \times t$ matrices best among $\mathscr{L}$ at S is $\mathscr{L}_1 = \{L : AL = B$ and $N'(S_1 + S_2)L = N'S_2C\} = \{L : A_1L = B_1\}$ with

$$A_1 = \begin{pmatrix} A \\ N'(S_1 + S_2) \end{pmatrix}, \qquad B_1 = \begin{pmatrix} B \\ N'S_2C \end{pmatrix}. \tag{3.2}$$

$\mathscr{L}_1$ is a proper subset of $\mathscr{L}$ iff the rank of A_1 is greater than the rank of A: it may be seen that this condition is equivalent to the condition that $N'(S_1 + S_2)N = S_{11}$ be nonzero. □

By part (iv) of Theorem 3.1, the set of trivial points for $\mathscr{L}$ in $\mathscr{W}$ is $\mathscr{S} = \{S \in \mathscr{W} : N'(S_1 + S_2)N = 0,\ N'S_1L_0 + N'S_2(L_0 - C) = 0\}$. The set of points in $\mathscr{W}$ which admit a best estimator among $\mathscr{L}$ is a convex cone because the conditions of Theorem 3.1(i) are closed under sums and positive multiples: this set is contained in the closed convex cone $\{S \in \mathscr{W} : N'(S_1 + S_2)N$ is nnd$\}$ and contains the closed convex cone $\mathscr{W}_+$. Given $L \in \mathscr{L}$, the set $\{S \in \mathscr{W} : L$ is best among $\mathscr{L}$ at $S\}$ is a closed convex cone containing $\mathscr{S}$, being the intersection of $\{S \in \mathscr{W} : N'(S_1 + S_2)N$ is nnd$\}$ and the linear subspace $\{S \in \mathscr{W} : N'(S_1 + S_2)L = N'S_2C\}$.

The following lemma notes that relations (as good as, better than) among $\mathscr{L}$ are equivalent on certain sets.

Lemma 3.2. *Let $\mathscr{X} \subset \mathscr{W}$ and let $\mathscr{S}_0$ be a linear subspace in $\mathscr{S}$. Let K and L be members of $\mathscr{L}$.*

(i) *K is as good as (better than) L on $\mathscr{X}$* iff *K is as good as (better than) L on $[\mathscr{X}]$. (Relations in $\mathscr{L}$ are equivalent on $\mathscr{X}$ and $[\mathscr{X}]$.)*

(ii) *Relations in $\mathscr{L}$ are equivalent on $\mathscr{X}$ and $\mathscr{X} + \mathscr{S}_0$.*

(iii) *If $\mathscr{M}$ is a subset of $\mathscr{W}$ such that $[\mathscr{M}] + \mathscr{S}_0 = [\mathscr{X} + \mathscr{S}_0]$ then relations in $\mathscr{L}$ are equivalent on $\mathscr{M}$ and $\mathscr{X}$.*

PROOF. To prove (i), note that $\{S \in \mathscr{W} : \text{TMSE}_K(S) \leq \text{TMSE}_L(S)\}$ is a closed half-space, hence a closed convex cone. To prove (ii), note that for any $S \in \mathscr{W}$ and $S_0 \in \mathscr{S}_0$, $\text{TMSE}_K(S + S_0) - \text{TMSE}_L(S + S_0) = \text{TMSE}_K(S) - \text{TMSE}_L(S)$: that is, TMSE's are parallel along $\mathscr{S}_0$. The last statement follows from (i) and (ii). □

LEMMA 3.3. *There exists a linear subspace $\mathscr{S}_0$ in $\mathscr{S}$ and a closed convex cone $\mathscr{M}$ such that $\mathscr{M} + \mathscr{S}_0 = [\mathscr{T} + \mathscr{S}_0]$ and $\mathscr{M} \cap \mathscr{S} = \{0\}$. If $\mathscr{S}$ contains a nonzero point, then $\mathscr{S}_0$ may be chosen to contain a nonzero point.*

PROOF. Let $\mathscr{S}_0 = \mathscr{S}$ and $\mathscr{M} = [\mathscr{T} + \mathscr{S}] \cap \bar{\mathscr{S}}$. $\mathscr{M}$ is a closed convex cone. For any set $\mathscr{Q}$ in $\mathscr{W}$, the projection of $\mathscr{Q}$ onto $\bar{\mathscr{S}}$ along $\mathscr{S}$ is $(\mathscr{Q} + \mathscr{S}) \cap \bar{\mathscr{S}}$ and $\mathscr{Q} + \mathscr{S} = \mathscr{S} + ((\mathscr{Q} + \mathscr{S}) \cap \bar{\mathscr{S}})$. But $[\mathscr{T} + \mathscr{S}] \supset \mathscr{S}$ because $[\mathscr{T} + \mathscr{S}]$ is a closed convex cone, so, with $\mathscr{Q} = [\mathscr{T} + \mathscr{S}]$, $[\mathscr{T} + \mathscr{S}] + \mathscr{S} = [\mathscr{T} + \mathscr{S}]$ and hence $[\mathscr{T} + \mathscr{S}] = \mathscr{S} + \mathscr{M}$. Because $\mathscr{M} \subset \bar{\mathscr{S}}$ and $\mathscr{S}$ and $\bar{\mathscr{S}}$ meet only at 0, $\mathscr{M} \cap \mathscr{S} = \{0\}$. □

Our objective is to show that any estimator admissible among $\mathscr{L}$ on $\mathscr{T}$ must be best among $\mathscr{L}$ at a point which is not trivial for $\mathscr{L}$ (unless every point in $\mathscr{T}$ is trivial, in which case Lemma 3.2(ii) establishes that every $L \in \mathscr{L}$ is admissible among $\mathscr{L}$ on $\mathscr{T}$), and to describe the points in $\mathscr{W}$ at which admissible members of $\mathscr{L}$ must be best. Toward this end, Lemmas 3.2 and 3.3 show that $\mathscr{T}$ may be replaced by a suitable closed convex cone $\mathscr{M}$, containing no nonzero trivial point for $\mathscr{L}$, such that relations in $\mathscr{L}$ are equivalent on $\mathscr{T}$ and $\mathscr{M}$. Note that $\mathscr{M} \subset [\mathscr{T} + \mathscr{S}_0]$.

The following simple result is stated separately because of its importance in the sequel.

LEMMA 3.4. *If $S = (S_1, S_2)$ is a point in $[\mathscr{W}_+ + \mathscr{S}]$ then $N'(S_1 + S_2)N$ is* nnd.

PROOF. $\mathscr{W}_+ + \mathscr{S}$ is contained in the closed convex cone $\{S \in \mathscr{W} : N'(S_1 + S_2)N \text{ is nnd}\}$ so also is $[\mathscr{W}_+ + \mathscr{S}]$. □

The following result follows from the observation that TMSE's are parallel along $\mathscr{S}$.

LEMMA 3.5. *If $\mathscr{T} \subset \mathscr{S}$ then every $L \in \mathscr{L}$ is admissible among $\mathscr{L}$ on $\mathscr{T}$.*

THEOREM 3.6. *If L_* is best among $\mathscr{L}$ at $S_* \in [\mathscr{T} + \mathscr{S}]$ and $N'(S_{1*} + S_{2*})N$ has maximal rank among $\{N'(S_1 + S_2)N : (S_1, S_2) = S \in [\mathscr{T} + \mathscr{S}]\}$ then L_* is admissible among $\mathscr{L}$ on $\mathscr{T}$ and $\mathscr{T} \subset \mathscr{S}_1 = \{S \in \mathscr{W}$: every $L \in \mathscr{L}_1$ is best among $\mathscr{L}_1$ at $S\}$.*

PROOF. With L_* best among $\mathscr{L}$ at $S_* \in [\mathscr{T} + \mathscr{S}]$, admissibility of L_* among $\mathscr{L}$ on $\mathscr{T}$ is equivalent to admissibility of L_* among $\mathscr{L}_1 = \{L \in \mathscr{L} : L$ is best among $\mathscr{L}$ at $S_*\}$ on $\mathscr{T}$. (Necessity of this statement is immediate. To show sufficiency, suppose $L \in \mathscr{L}$ is better than L_* on $\mathscr{T}$. Then L is better than L_* on $[\mathscr{T} + \mathscr{S}]$, by Lemma 3.2, so that $L \in \mathscr{L}_1$ and L_* is not admissible among $\mathscr{L}_1$ on $[\mathscr{T} + \mathscr{S}]$, hence on $\mathscr{T}$.) By the proof of Theorem 3.1(vi), $\mathscr{L}_1$
$-$ ' $A_1 L = B_1\}$ with A_1 and B_1 from (3.2) at S_*. Let N_1 be a matrix such that $\mathbf{R}(N_1) =$
$\subset \mathbf{N}(A) = \mathbf{R}(N)$; thus there exists a matrix P such that $N_1 = NP$. Because $N'(S_{1*}$
$N_1 = 0$, columns of P are in $\mathbf{N}[N'(S_{1*} + S_{2*})N]$. By the proof of Lemma 1 in LaMotte
.) and Lemma 3.4, because $\{N'(S_1 + S_2)N : S \in [\mathscr{T} + \mathscr{S}]\}$ is a convex subset of nnd
es, $\mathbf{N}[N'(S_{1*} + S_{2*})N] \subset \mathbf{N}[N'(S_1 + S_2)N]$ for all $S \in [\mathscr{T} + \mathscr{S}]$. Thus $P'N'(S_1 +$
$= N_1'(S_1 + S_2)N_1 = 0$ for all $S \in [\mathscr{T} + \mathscr{S}]$. If $S \in \mathscr{T}$ then $N_1'(S_1 + S_2)N_1 = 0$ implies
$'_1 S_1 = N_1' S_2 = 0$, so that $\mathscr{T} \subset \mathscr{S}_1 = \{S \in \mathscr{W}$: every $L \in \mathscr{L}_1$ is best among $\mathscr{L}_1$ at $S\}$. By
a 3.5, every $L \in \mathscr{L}_1$ is admissible among $\mathscr{L}_1$ on $\mathscr{T}$. By the first statement of this proof,
$L \in \mathscr{L}_1$ is admissible among $\mathscr{L}$ on $\mathscr{T}$. □

MA 3.7. *Let $\mathscr{M}$ be a closed convex cone and let $\mathscr{U}$ be the maximal linear subspace*
If $\mathscr{M}$ contains a point not in $\mathscr{U}$, then there exists a compact convex subset $\mathscr{C}$ of $\mathscr{M}$
hat $0 \notin \mathscr{C}$, $[\mathscr{C}] \cap \mathscr{U} = \{0\}$, and $\mathscr{M} = [\mathscr{C}] + \mathscr{U}$.

PROOF. Because $\mathscr{M}$ is a closed convex cone, $\mathscr{M} = (\mathscr{M} \cap \bar{\mathscr{U}}) + \mathscr{U}$. $\mathscr{M}$ contains a point not in $\mathscr{U}$ so $\mathscr{M} \cap \bar{\mathscr{U}}$ contains a nonzero point. Further, $\mathscr{M} \cap \bar{\mathscr{U}}$ is a closed convex cone containing no linear subspace of positive dimension. We shall find a compact convex subset $\mathscr{C}$ such that $[\mathscr{C}] = \mathscr{M} \cap \bar{\mathscr{U}}$. Let $\mathscr{R} = \mathscr{M} \cap \bar{\mathscr{U}}$.

Let $\mathscr{B} = \{S \in \mathscr{W}: \text{tr}(SS') = 1\}$. $\mathscr{B} \cap \mathscr{R}$ is compact. Let $\mathscr{C}$ be the minimal convex set containing $\mathscr{B} \cap \mathscr{R}$. By Theorem 17.2 in Rockafellar (1970), $\mathscr{C}$ is compact. For any nonzero $S \in \mathscr{R}$, $S/\sqrt{\text{tr}(SS')} \in \mathscr{B} \cap \mathscr{R} \subset \mathscr{C}$, so $[\mathscr{C}] \supset \mathscr{R}$; and $\mathscr{R}$ is a closed convex cone containing $\mathscr{C}$ so $\mathscr{R} \supset [\mathscr{C}]$; therefore $[\mathscr{C}] = \mathscr{R}$. If $0 \in \mathscr{C}$ then there exist points $S_1, \cdots, S_r$ in $\mathscr{B} \cap \mathscr{R}$ and positive numbers $\lambda_1, \cdots, \lambda_r$, $\lambda_1 + \cdots + \lambda_r = 1$, such that $0 = \lambda_1 S_1 + \cdots + \lambda_r S_r$. Clearly, $r > 1$ since 0 is not in $\mathscr{B} \cap \mathscr{R}$. Thus $S_1 = -(\lambda_2 S_2 + \cdots + \lambda_r S_r)/\lambda_1$ is nonzero and in $\mathscr{R}$ and $-S_1$ is in $\mathscr{R}$ because $\mathscr{R}$ is a convex cone, so $\text{sp}(S_1) \subset \mathscr{R}$ contrary to the fact that $\mathscr{R}$ contains no nontrivial linear subspace. Therefore $0 \notin \mathscr{C}$. □

LEMMA 3.8. *If $\mathscr{C}$ is a compact convex subset of matrices and if $C_0 \in \mathscr{C}$ has minimum norm in $\mathscr{C}$, then* $\text{tr}(C_0'C) \geq \text{tr}(C_0'C_0)$ *for all $C \in \mathscr{C}$.*

PROOF. Show that the orthogonal projection λC_0 of $C \in \mathscr{C}$ onto $\text{sp}(C_0)$ has $\lambda \geq 1$, or see Rockafellar (1970, page 271). □

THEOREM 3.9. *Let $\mathscr{M}$ be a closed convex cone in $[\mathscr{W}_+ + \mathscr{S}]$ and let $\mathscr{U}$ be the maximal linear subspace in $\mathscr{M}$. Suppose $\mathscr{U} \neq \mathscr{M}$. If $L \in \mathscr{L}$ is not best among $\mathscr{L}$ at any nonzero $S \in \mathscr{M}$ then there exists $L_* \in \mathscr{L}$ which is better on $\mathscr{M}$ than L and strictly better on $\mathscr{M} \setminus \mathscr{U}$.*

PROOF. Suppose $L = L_0 + NZ$ is best among $\mathscr{L}$ in $\mathscr{M}$ only at 0. Then $F_Z(S) = S_{11}Z + S_{12}$ is nonzero in $\mathscr{M}$ except at 0. With $\mathscr{U} \neq \mathscr{M}$, from Lemma 3.7, $\mathscr{M} = [\mathscr{C}] + \mathscr{U}$, where $\mathscr{C}$ is a compact convex subset, not containing 0, such that $[\mathscr{C}] = \mathscr{M} \cap \bar{\mathscr{U}}$. By Lemma 3.4, S_{11} is nnd throughout $\mathscr{U}$; with $-S$ also in $\mathscr{U}$, this implies that $S_{11} = 0$ throughout $\mathscr{U}$.

In order to find $L_* \in \mathscr{L}$ which is better on $\mathscr{M}$ than L, we shall find a positive scalar γ and a matrix H in $F_Z(\mathscr{U})^\perp$ (the orthogonal complement of the linear subspace $F_Z(\mathscr{U})$) such that, with $L_* = L_0 + N(Z - \gamma H)$,

$$0 \leq \text{TMSE}_L(S) - \text{TMSE}_{L_*}(S) = 2\gamma \,\text{tr}[H'F_Z(S)] - \gamma^2 \text{tr}(H'S_{11}H) \tag{3.3}$$

is satisfied throughout $\mathscr{C}$, hence throughout $[\mathscr{C}]$.

The orthogonal projection of $F_Z(\mathscr{C})$ onto $F_Z(\mathscr{U})^\perp$ is a compact convex set (a linear mapping) and does not contain 0. (Let $S \in \mathscr{C}$. $F_Z(S) = X_1 + X_2$ with $X_1 \in F_Z(\mathscr{U})$ and $X_2 \in F_Z(\mathscr{U})^\perp$. If $X_2 = 0$ then $F_Z(S) = X_1 = F_Z(U)$ for some $U \in \mathscr{U}$. But $\mathscr{M} \supset \mathscr{U}$ so $S - U \in \mathscr{M}$; and $F_Z(S - U) = 0$ implies, under the hypothesis that F_Z is zero only at 0 in $\mathscr{M}$, that $S = U \in \mathscr{U}$, contrary to the fact that $S \in \mathscr{C} \subset \mathscr{M} \cap \bar{\mathscr{U}}$ and $\mathscr{C}$ does not contain 0.) Using Lemma 3.8, let H be the element of minimum norm in the orthogonal projection of $F_Z(\mathscr{C})$ onto $F_Z(\mathscr{U})^\perp$. Because $\text{tr}(H'S_{11}H)$, with $S_{11} = N'(S_1 + S_2)N$, is continuous on $\mathscr{C}$, it is bounded above on $\mathscr{C}$ by $M > 0$. *Let* $\gamma = tr(H'H)/M$.

If $S \in \mathscr{C}$ then $F_Z(S) = X_1 + X_2$ with $X_1 \in F_Z(\mathscr{U})$ and $X_2 \in F_Z(\mathscr{U})^\perp$, so

$$2\gamma \,\text{tr}[H'F_Z(S)] - \gamma^2 \text{tr}(H'S_{11}H) = 2\gamma \,\text{tr}(H'X_2) - \gamma^2 \text{tr}(H'S_{11}H)$$
$$\geq 2\gamma \,\text{tr}(H'H) - \gamma^2 M = \gamma^2 M > 0.$$

Thus $L_* = L_0 + N(Z - \gamma H)$ is better on $\mathscr{C}$ than $L = L_0 + NZ$, hence L_* is better than L on $[\mathscr{C}] = \mathscr{M} \cap \bar{\mathscr{U}}$ (strictly better except at 0). Because $\mathscr{M} = \mathscr{U} + \mathscr{M} \cap \bar{\mathscr{U}}$ and $S_{11} = 0$ in $\mathscr{U}$ and $H \in F_Z(\mathscr{U})^\perp$, it follows that L_* is better than L on $\mathscr{M}$. □

COROLLARY 3.10. *In order that $L_* \in \mathscr{L}$ be admissible among $\mathscr{L}$ on $\mathscr{T}$ it is necessary and sufficient that either $\mathscr{T} \subset \mathscr{S}$ or there exist a point S_* in $[\mathscr{T} + \mathscr{S}]$, which is not trivial for $\mathscr{L}$, such that L_* is best among $\mathscr{L}$ at S_* and is admissible among $\mathscr{L}_1 = \{L \in \mathscr{L}: L$ is best among $\mathscr{L}$ at $S_*\}$ on $\mathscr{T}$.*

PROOF. By Lemma 3.5, if $\mathscr{T} \subset \mathscr{S}$ then every $L \in \mathscr{L}$ is admissible among $\mathscr{L}$ on $\mathscr{T}$.

If L_* is best among $\mathscr{L}$ at $S_* \in [\mathscr{T} + \mathscr{S}]$ and admissible among $\mathscr{L}_1$, any $L \in \mathscr{L}$ better than L_* on $\mathscr{T}$ must also be best at S_*, contrary to the admissibility of L_* among $\mathscr{L}_1$ on $\mathscr{T}$.

To show necessity, suppose $\mathscr{T}$ contains a point not in $\mathscr{S}$ and either L_* is not best among $\mathscr{L}$ at any nontrivial point in $[\mathscr{T} + \mathscr{S}]$ or L_* is best among $\mathscr{L}$ at a nontrivial point S_* in $[\mathscr{T} + \mathscr{S}]$ but L_* is not admissible among $\mathscr{L}_1$.

As in Lemma 3.3, let $\mathscr{M} = [\mathscr{T} + \mathscr{S}] \cap \bar{\mathscr{S}}$. Let $\bar{S} = (\bar{S}_1, \bar{S}_2)$ be the projection onto $\bar{\mathscr{S}}$ along $\mathscr{S}$ of a point S in $\mathscr{T}$ and not in $\mathscr{S}$. Then $N'(S_1 + S_2)N = N'(\bar{S}_1 + \bar{S}_2)N$ is nnd and nonzero (otherwise $S \in \mathscr{S}$), so $\mathscr{M}$ is not a linear subspace (otherwise $-\bar{S} \in \mathscr{M}$ implies that $N'(\bar{S}_1 + \bar{S}_2)N = 0$). Finally, $\mathscr{M} + \mathscr{S} = [\mathscr{T} + \mathscr{S}] \subset [\mathscr{W}_+ + \mathscr{S}]$, and relations in $\mathscr{L}$ are equivalent on $\mathscr{M}$ and $\mathscr{T}$, by Lemma 3.2(iii). By Theorem 3.9, if L_* is not best among $\mathscr{L}$ at any nonzero point in $\mathscr{M}$ (i.e., any nontrivial point in $[\mathscr{T} + \mathscr{S}]$) then L_* is not admissible among $\mathscr{L}$ on $\mathscr{M}$, hence on $\mathscr{T}$. If L_* is best among $\mathscr{L}$ at a point S_* in $[\mathscr{T} + \mathscr{S}]$ which is not trivial for $\mathscr{L}$, but L_* is not admissible among $\mathscr{L}_1$ on $\mathscr{T}$, then L_* is not admissible among $\mathscr{L}$ on $\mathscr{T}$ because $\mathscr{L}_1 \subset \mathscr{L}$. □

COROLLARY 3.11. *Let $\mathscr{M}$ be a closed convex cone, and let $\mathscr{S}_0$ be a linear subspace in $\mathscr{S}$, such that $\mathscr{M} + \mathscr{S}_0 = [\mathscr{T} + \mathscr{S}_0]$. In order that $L_* \in \mathscr{L}$ be admissible among $\mathscr{L}$ on $\mathscr{T}$, it is necessary and sufficient that either $\mathscr{T} \subset \mathscr{S}$ or there exist a nonzero point S_* in $\mathscr{M}$ such that L_* is best among $\mathscr{L}$ at S_* and is admissible among $\mathscr{L}_1 = \{L \in \mathscr{L}: L$ is best among $\mathscr{L}$ at $S_*\}$ on $\mathscr{T}$.*

PROOF. Relations are equivalent on $\mathscr{M}$ and $\mathscr{T}$, so sufficiency is obvious. Proof of necessity follows the argument in Corollary 3.10 after observing that, if $\mathscr{T}$ contains a point not in $\mathscr{S}$, then $\mathscr{M}$ is not a linear subspace. □

THEOREM 3.12. *If $L_* \in \mathscr{L}$ is best among $\mathscr{L}$ at $S_* \in [\mathscr{T} + \mathscr{S}]$ and $\mathscr{T} \subset \mathscr{S}_1 = \{S \in \mathscr{W}$: every $L \in \mathscr{L}_1$ is best among $\mathscr{L}_1$ at $S\}$, then $N'(S_{1*} + S_{2*})N$ has maximal rank among $\{N'(S_1 + S_2)N: S \in [\mathscr{T} + \mathscr{S}]\}$. (Here, $\mathscr{L}_1 = \{L \in \mathscr{L}: L$ is best among $\mathscr{L}$ at $S_*\} = \{L: A_1 L = B_1\}$.)*

PROOF. Note that $\mathscr{T} \subset \mathscr{S}_1$ implies that $[\mathscr{T} + \mathscr{S}] \subset \mathscr{S}_1$ because $\mathscr{S} \subset \mathscr{S}_1$. Let N_1 be a matrix such that $\mathbf{R}(N_1) = \mathbf{N}(A_1)$. Let $z \in \mathbf{N}\{N'(S_{1*} + S_{2*})N\}$. Then $ANz = 0$ and $N'(S_{1*} + S_{2*})Nz = 0$ so $Nz \in \mathbf{R}(N_1)$. But every point in $[\mathscr{T} + \mathscr{S}]$ is a trivial point with respect to $\mathscr{L}_1$ and $N_1'(S_1 + S_2)N_1 = 0$ for all $S \in \mathscr{S}_1$, so for any $S \in [\mathscr{T} + \mathscr{S}]$, $z'N'(S_1 + S_2)Nz = 0$ and $N'(S_1 + S_2)N$ is nnd, so $z \in \mathbf{N}\{N'(S_1 + S_2)N\}$. Thus $\mathbf{N}\{N'(S_{1*} + S_{2*})N\}$ is a subset of $\mathbf{N}\{N'(S_1 + S_2)N\}$ for every S in $[\mathscr{T} + \mathscr{S}]$, therefore $N'(S_{1*} + S_{2*})N$ has maximal rank among $\{N'(S_1 + S_2)N: S \in [\mathscr{T} + \mathscr{S}]\}$. □

LEMMA 3.13. *If $S_* \in [\mathscr{T} + \mathscr{S}]$ and $S_* \notin \mathscr{S}$ and L_* is best among $\mathscr{L}$ at $S_* = (S_{1*}, S_{2*})$, then $\mathscr{S}$ is a proper subset of $\mathscr{S}_1 = \{S \in \mathscr{W}$: every $L \in \mathscr{L}_1$ is best among $\mathscr{L}_1$ at $S\}$.*

PROOF. $\mathscr{L}_1 \subset \mathscr{L}$ implies that $\mathscr{S}_1 \supset \mathscr{S}$ and the inclusion is proper because S_* is in $\mathscr{S}_1$ and not in $\mathscr{S}$. □

Given $L_* \in \mathscr{L}$, Corollary 3.10 or Corollary 3.11 can be applied repeatedly to determine the admissibility of L_* in a finite number of steps (by Lemma 3.13, at most the lesser of dimension of $\mathscr{W}$-dimension of $\mathscr{S}$ and dimension of $\mathscr{L}$ steps). This may be seen with the following procedure.

$\mathscr{T} \subset \mathscr{S}$ then L_* is admissible among $\mathscr{L}$ on $\mathscr{T}$.

$\mathscr{T} \not\subset \mathscr{S}$:

If there is no point in $[\mathscr{T} + \mathscr{S}]$ and not in $\mathscr{S}$ at which L_* is best among $\mathscr{L}$, then L_* is not admissible among $\mathscr{L}$ on $\mathscr{T}$.

2.2. If there exists a point S_* in $[\mathscr{T} + \mathscr{S}]$ and not in $\mathscr{S}$ at which L_* is best among $\mathscr{L}$, admissibility of L_* among $\mathscr{L}$ on $\mathscr{T}$ is equivalent to admissibility of L_* among $\mathscr{L}_1 = \{L \in \mathscr{L}: L \text{ is best among } \mathscr{L} \text{ at } S_*\} = (L: A_1 L = B_1\}$ on $\mathscr{T}$. Replace $\mathscr{L}$ and $\mathscr{S}$ accordingly and return to 1.

Each time through, from step 2.2 to step 1, the rank of N decreases, the dimension of $\mathscr{L}$ decreases and the dimension of $\mathscr{S}$ increases (Lemma 3.13), so that in a finite number of times through this procedure, either 1 or 2.1 must hold. This is summarized as follows without need of further proof.

Let $\mathscr{L}_{(0)}$ be $\mathscr{L}$, $\mathscr{S}_{(0)}$ be $\mathscr{S}$, $N_{(0)}$ be N, $A_{(0)} = A$, and $B_{(0)} = B$. Let $S_{*(1)}, \cdots, S_{*(r)}$ be a sequence of points in $\mathscr{W}$. For each $i = 1, \cdots, r$, let

$$A_{(i)} = \begin{pmatrix} A_{(i-1)} \\ N'_{(i-1)}(S_{1*(i)} + S_{2*(i)}) \end{pmatrix}, \qquad B_{(i)} = \begin{pmatrix} B_{(i-1)} \\ N'_{(i-1)} S_{2*(i)} C \end{pmatrix}$$

where $N_{(i)}$ is a matrix such that $\mathbf{R}(N_{(i)}) = \mathbf{N}(A_{(i)})$. Let

$$\begin{aligned} \mathscr{L}_{(i)} &= \{L \in \mathscr{L}_{(i-1)}: L \quad \text{is best among} \quad \mathscr{L}_{(i-1)} \quad \text{at} \quad S_{*(i)}\} \\ &= \{L: A_{(i)} L = B_{(i)}\}. \end{aligned}$$

Let

$$\mathscr{S}_{(i)} = \{S \in \mathscr{W}: \quad \text{every} \quad L \in \mathscr{L}_{(i)} \quad \text{is best among} \quad \mathscr{L}_{(i)} \quad \text{at} \quad S\}.$$

THEOREM 3.14. *In order that L_* be admissible among $\mathscr{L}$ on $\mathscr{T}$, it is necessary and sufficient that either $\mathscr{T} \subset \mathscr{S}_{(0)}$ or there exist a finite sequence $S_{*(1)}, \cdots, S_{*(r)}$ of points in $\mathscr{W}$ such that $L_* \in \mathscr{L}_{(i)}$, $S_{*(i)} \in [\mathscr{T} + \mathscr{S}_{(i-1)}]$ and $S_{*(i)}$ is not in $\mathscr{S}_{(i-1)}$ for each $i = 1, \cdots, r$, and $N'_{(r-1)}(S_{1*(r)} + S_{2*(r)})N_{(r-1)}$ has maximal rank among $\{N'_{(r-1)}(S_1 + S_2)N_{(r-1)}: (S_1, S_2) \in [\mathscr{T} + \mathscr{S}_{(r-1)}]\}$.*

When points S in $[\mathscr{T} + \mathscr{S}]$ having maximal rank of $N'(S_1 + S_2)N$ have $N'(S_1 + S_2)N$ singular, the elements of $\mathscr{L}$ best among $\mathscr{L}$ at S form an affine subset of positive dimension, and all such estimators have identical TMSEs on $\mathscr{T}$. The simplest way to systematically choose one of each such set is to transform the problem so that $N'(S_1 + S_2)N$ is positive definite at maximal points in $[\mathscr{T} + \mathscr{S}]$, following LaMotte (1977a). This may be accomplished in several ways; following are two. In each, let columns of N form an orthonormal basis for $\mathbf{N}(A)$ and let L_0 be the unique matrix in $\mathscr{L}$ whose columns are in $\mathbf{R}(A')$.

Let S_* have maximal rank in $[\mathscr{T}]$. Let columns of F form a basis for $\mathbf{R}(S_{1*} + S_{2*})$. Every n-vector x has a unique representation $x = Fy + z$ with $z \in \mathbf{N}(S_{1*} + S_{2*})$, so that $y = (F'F)^{-1}F'x$. For any $L \in \mathscr{L}$, $F(F'F)^{-1}F'L$ has the same TMSE on $\mathscr{T}$ as L. For any S of maximal rank in $[\mathscr{T}]$, $F'(S_1 + S_2)F$ is positive definite. (To see this, observe that $y'F'(S_1 + S_2)Fy = 0$ implies that $Fy \in \mathbf{N}(S_1 + S_2)$, but $Fy \in \mathbf{R}(S_{1*} + S_{2*}) = \mathbf{R}(S_1 + S_2)$ because S has maximal rank in $[\mathscr{T}]$. Thus $Fy = 0$ so that $y = 0$.) Replace "estimate $C'\mu$ by $L'Y$, $L \in \mathscr{L}$, on $\mathscr{T}$" by "estimate $\{(F'F)^{-1}F'C\}'(F'\mu)$ by $M'(F'Y)$, $M \in \{(F'F)^{-1}F'L: L \in \mathscr{L}\}$, on $\{(F'S_1F, F'S_2F): (S_1, S_2) \in \mathscr{T}\}$." Here, we replace Y by $F'Y$, with corresponding replacements of C, $\mathscr{L}$, and $\mathscr{T}$.

Another way to modify the problem so that maximal points in $[\mathscr{T} + \mathscr{S}]$ admit a unique best estimator in $\mathscr{L}$ is to augment the constraints describing $\mathscr{L}$. Let S_* be maximal in $[\mathscr{T}]$, in the sense that $\mathbf{N}\{N'(S_{1*} + S_{2*})N\} \subset \mathbf{N}\{N'(S_1 + S_2)N\}$ for all $S \in [\mathscr{T}]$. Let q be the column dimension of N. Let P be the orthogonal projection matrix onto $\mathbf{N}\{N'(S_{1*} + S_{2*})N\}$. Replace $\mathscr{L}$ by $\{L: AL = B \text{ and } PN'L = 0\} = \mathscr{L}_*$. Let columns of N_* form a basis for $\mathbf{N}(\begin{smallmatrix} A \\ PN' \end{smallmatrix})$. Then $\mathbf{R}(N_*) \subset \mathbf{R}(N)$ so $N_* = NQ$ for some Q. Suppose $z'N'_*(S_{1*} + S_{2*})N_*z = 0$. Then $Qz \in \mathbf{N}\{N'(S_{1*} + S_{2*})N\}$, and $PN'N_*z = PQz = 0$ so $Qz = 0$; but columns of Q are linearly independent (otherwise N'_*N_* is singular), so $z = 0$. Therefore $N'_*(S_{1*} + S_{2*})N_*$ is positive definite; hence for every maximal point S in $[\mathscr{T}]$ (hence $[\mathscr{T} + \mathscr{S}]$), $N'_*(S_1 + S_2)N_*$ is positive definite. Note that for each $L \in \mathscr{L}$ there is an $L_* \in \mathscr{L}_*$ with identical TMSE on $\mathscr{T}$, so that $\mathscr{L}_*$ forms an essentially complete class in $\mathscr{L}$.

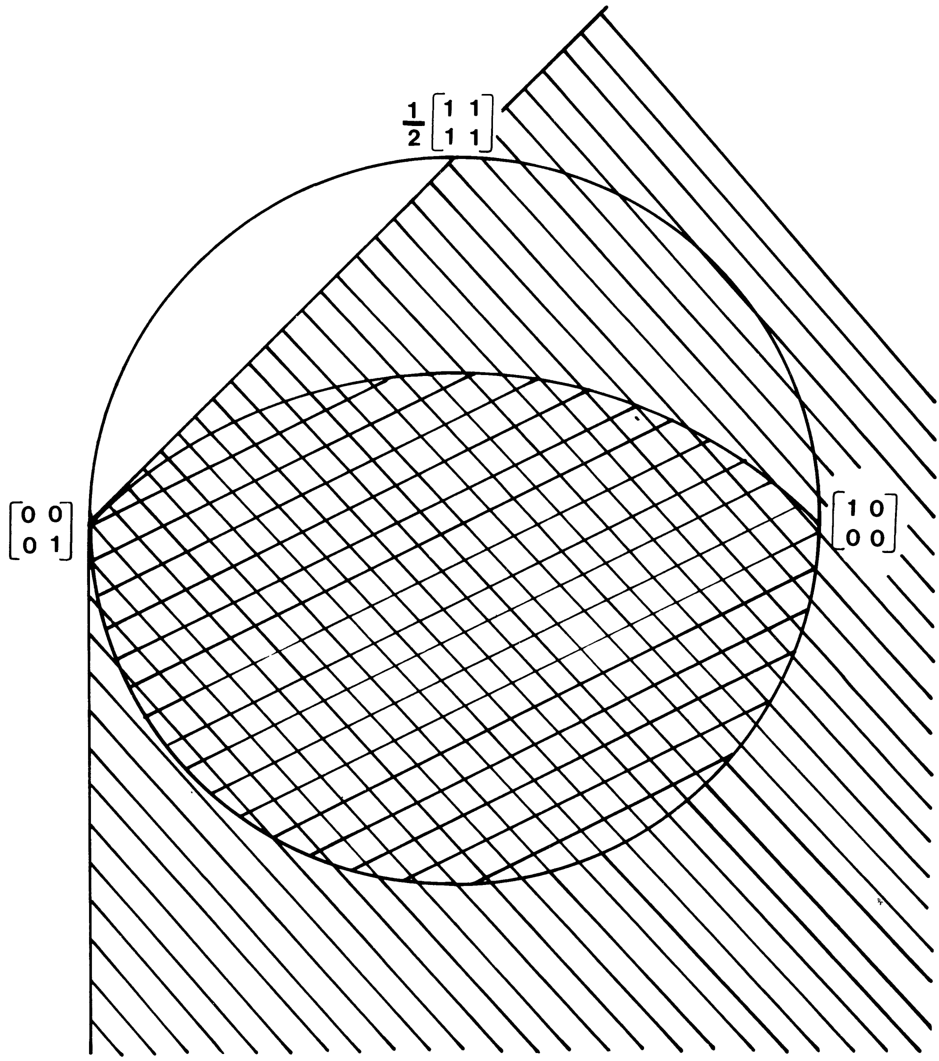

FIG. 1 2×2 *symmetric matrices in the plane* $\{S: \mathrm{tr}(S) = 1\}$

4. Examples. Two examples are discussed here. The purpose of the first example is to illustrate the machinery of Section 3. The second example is the regression model, first with its natural parameter set, then with a restricted parameter set in which the inadmissibility of some biased linear estimators is demonstrated.

The thesis by Azzam (1980) has independently developed similar results and applied them to unbiased estimation when $\mathscr{T}$ is contained in a finitely generated convex cone in $\mathscr{W}_+$.

4.1. *Example* 1. With $n = 2$, $\mu = (\mu_1, \mu_2)'$ and the set of all 2×2 nnd symmetric matrices is the minimal closed convex cone containing the large circle and its interior in Figure 1. The circle and its interior form the intersection of the nnd matrices with the plane $\{S: \mathrm{tr}(S) = 1\}$. Suppose $\mathscr{H} = (V: (V, \mu\mu') \in \mathscr{P}\}$ is the doubly-shaded region in Figure 1 and the cone it generates: this region is tangent to the line through $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ and $\frac{1}{2}\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ at $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$. Consider unbiased linear estimation of μ_2 when $\{\mu: (\mu, V) \in \mathscr{P}\} = \mathbf{R}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. Let $L_0 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. One application of Lemma 3.2(ii) allows us to replace $\mathscr{T}$ with $\mathscr{H}$. With $\mathscr{W}$ the set of 2 × 2 symmetric matrices, $\mathscr{S} = \{S : s_{11} = s_{12} = 0\}$, and $\mathscr{H}$ intersects $\mathscr{S}$ in the ray through $\binom{0\,0}{0\,1}$. Every point S in $\mathscr{H}$ not on this ray has $N'SN$ nonsingular. $[\mathscr{H} + \mathscr{S}]$ intersects the plane of Figure 1 in the singly-shaded area. For any $z > -1$, it may be seen that $L_0 + Nz$ is best at a point S in $\mathscr{H}$ with $N'SN$ nonsingular, and hence is admissible among $\mathscr{L}$. The estimator $L = L_0 - N = \binom{-1}{1}$ is best among $\mathscr{L}$ in $\mathscr{H}$ only on the ray of trivial points but it is best among $\mathscr{L}$ in $[\mathscr{H} + \mathscr{S}]$ at $\frac{1}{2}\binom{1\,1}{1\,1}$, which has $N'SN > 0$, hence $\binom{-1}{1}$ is admissible among $\mathscr{L}$ on $\mathscr{T}$.

4.2. *The regression model.* In this model $\mathscr{P} = \{(X\beta, \sigma^2 I) : \beta \in R^p \text{ and } \sigma^2 \geq 0\}$, where X is an $n \times p$ matrix of rank $p < n$. Then

$$[\mathscr{T}] = \{(\gamma I, X\Phi X') : \gamma \geq 0, \Phi \text{ is a } p \times p \text{ symmetric nnd matrix}\}.$$

Let $\mathscr{W} = \{(aI, X\Gamma X') : a \text{ real}, \Gamma \text{ symmetric}\}$. We shall examine the class of linear estimators of $C'X\beta$, where C is $n \times t$, so $\mathscr{L} = \{L : L \text{ is } n \times t\}$. Points of maximal rank in $[\mathscr{T}]$ have $\gamma > 0$, so if L is best among $\mathscr{L}$ at $(\gamma I, X\Phi X')$ with $\gamma > 0$ then L is admissible among $\mathscr{L}$.

Suppose L_* is best among $\mathscr{L}$ at $(0, X\Phi_0 X')$. Let columns of N_1 form an orthonormal basis for $\mathbf{N}(X\Phi_0 X')$. Then

$$\mathscr{L}_1 = \{L : X\Phi_0 X' L = X\Phi_0 X' C\} \quad \text{and}$$
$$\mathscr{S}_1 = \{(aI, X\Phi X') : N_1'(aI + X\Phi X')N_1 = 0,\ N_1' X\Phi X'(L_0 - C) = 0\}$$

with L_0 the unique element of $\mathscr{L}_1$ with columns in $\mathbf{R}(X\Phi_0 X')$. In $[\mathscr{T} + \mathscr{S}_1]$, it may be seen that $N_1'S_1N_1$ and $N_1'S_2N_1$ are nnd: that is, that γI and $N_1'X\Phi X'N_1$ are nnd for any $(\gamma I, X\Phi X') \in [\mathscr{T} + \mathscr{S}_1]$; note that $S_1 = 0$ for all $S \in \mathscr{S}_1$.

Suppose L_* is best among $\mathscr{L}_1$ at $(0, X\Phi_1 X')$ in $[\mathscr{T} + \mathscr{S}_1]$ with $N_1'X\Phi_1 X'N_1 \neq 0$. With $\Phi_* = \Phi_0 X'X\Phi_0 + \Phi_1 X'N_1N_1'X\Phi_1$, which is nnd, note that $X\Phi_* X'$ has greater rank than $X\Phi_0 X'$, and L_* is best among $\mathscr{L}$ at $(0, X\Phi_* X') \in [\mathscr{T}]$. Given that L_* is best among $\mathscr{L}$ at a point in $[\mathscr{T}]$, we may take such a point to be of maximal rank among those at which L_* is best. With $X\Phi_0 X'$ so chosen, it is clear that if L_* is best among $\mathscr{L}_1$ at $(\gamma_1 I, X\Phi_1 X')$ in $[\mathscr{T} + \mathscr{S}_1]$ then $\gamma_1 > 0$ and $N_1'(\gamma_1 I + X\Phi_1 X')N_1$ is pd and hence L_* is admissible among $\mathscr{L}$.

Following the procedure described in Section 3, L_* is admissible among $\mathscr{L}$ iff L_* is best at $(\gamma I, X\Phi X') \in [\mathscr{T}]$ with $\gamma > 0$, or L_* is best at $(0, X\Phi_0 X') \in [\mathscr{T}]$ with $X\Phi_0 X' \neq 0$ and best among $\mathscr{L}_1$ at $(\gamma_1 I, X\Phi_1 X') \in [\mathscr{T} + \mathscr{S}_1]$ with $\gamma_1 > 0$. In the first case,

$$L_* = (\gamma I + X\Phi X')^{-1} X\Phi X' C. \tag{4.1}$$

In the second case, L_* is the unique solution to

$$\begin{pmatrix} X\Phi_0 X' \\ N_1'(\gamma_1 I + X\Phi_1 X') \end{pmatrix} L = \begin{pmatrix} X\Phi_0 X' \\ N_1' X\Phi_1 X' \end{pmatrix} C. \tag{4.2}$$

In this case it may be shown that

$$L_* = \lim_{\lambda \to 0^+} (\lambda\gamma_1 I + X\Phi_\lambda X')^{-1} X\Phi_\lambda X' C \tag{4.3}$$

where $\Phi_\lambda = (1 - \lambda)\Phi_0 + \lambda\Phi_1$. That is, L_* is a limit of matrices best among $\mathscr{L}$ at points along the line segment joining $(0, X\Phi_0 X')$ and $(\gamma_1 I, X\Phi_1 X')$. To show that (4.3) holds, note that, though $X\Phi_1 X'$ may not be nnd, there exists a $\lambda_* > 0$ such that $\lambda\gamma_1 I + X\Phi_\lambda X'$ is pd for all $\lambda \in (0, \lambda_*)$.

We conclude that L is admissible among $\mathscr{L}$ on $\mathscr{T}$ iff L satisfies (4.1) or (4.2) and hence (4.3). If L satisfies (4.1), a little algebra shows that $L = XP\Lambda^{-1/2}D\Lambda^{-1/2}P'X'C$, where $X'X = P\Lambda P'$ is a spectral decomposition of $X'X$ and

$$D = \Lambda^{1/2}P'\Phi P\Lambda^{1/2}(\gamma I + \Lambda^{1/2}P'\Phi P\Lambda^{1/2})^{-1} \tag{4.4}$$

is a symmetric nnd matrix with all its eigenvalues in [0, 1). If L satisfies (4.3) then $L = XP\Lambda^{-1/2}D\Lambda^{-1/2}P'X'C$ for some symmetric nnd D with eigenvalues in [0, 1]. On the other hand, for any symmetric nnd D with all its eigenvalues in [0, 1], it may be shown that there exists a nnd Φ and $\gamma > 0$ such that (4.4) is true; and, for such a D with all its eigenvalues in [0, 1], there exist nnd Φ_0 and Φ_1 such that

$$D = \lim_{\delta \to 0^+} \Lambda^{1/2} P' \Phi_\delta P \Lambda^{1/2} (\delta I + \Lambda^{1/2} P' \Phi_\delta P \Lambda^{1/2})^{-1} \tag{4.5}$$

with $\Phi_\delta = (1 - \delta)\Phi_0 + \delta\Phi_1$. Thus we see that L is admissible among $\mathscr{L}$ on $\mathscr{T}$ iff $L = XP\Lambda^{-1/2}D\Lambda^{-1/2}P'X'C$ for some nnd symmetric D with all its eigenvalues in [0, 1]. This is precisely the characterization of admissible linear estimators obtained by specializing the results of Rao (1976) and Cohen (1966) to this model. Note that the least squares estimator ($D = I$), the generalized ridge estimators (Hoerl and Kennard, 1970), with

$$D = (I + \Lambda^{-1}K)^{-1}, \qquad K = \text{diag}(k_i \geq 0), \tag{4.6}$$

the fractional rank estimators (Marquardt 1970), with

$$D = \begin{pmatrix} I_f & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad 0 \leq c < 1,$$

and fixed-scalar shrinkage estimators with $D = cI$, $0 \leq c \leq 1$, are all admissible among $\mathscr{L}$ on $\mathscr{T}$.

Now consider the restricted parameter space $\mathscr{P}_M = \{(X\beta, \sigma^2 I) : \sigma^2 \geq 0, \beta \in R^p, \beta'\beta \leq M\sigma^2\}$, for fixed $M > 0$. Consider estimating β, so that $C'X\beta = \beta$. Here, $[\mathscr{T}_M] = \{(\gamma I, X\Phi X') : \gamma \geq 0, \Phi \text{ nnd}, \text{tr } \Phi \leq M\gamma\}$. In $[\mathscr{T}_M]$, $\gamma = 0$ implies that $\Phi = 0$. But any $L_* \in \mathscr{L}$ which is admissible among $\mathscr{L}$ on $\mathscr{T}_M$ must be best among $\mathscr{L}$ at a nonzero point in $[\mathscr{T}_m]$, so that

$$L_* = (\gamma I + X\Phi X')^{-1} X\Phi$$

for some $(\gamma I, X\Phi X') \in [\mathscr{T}_M]$ with $\gamma > 0$. Thus, if L_* is admissible among $\mathscr{L}$ on $\mathscr{T}_M$ then $L'_* X\beta = \beta$ only if $\beta = 0$. This means that no unbiased or conditionally unbiased estimator in $\mathscr{L}$ is admissible among $\mathscr{L}$ on $\mathscr{T}_M$. In particular, the least squares estimator, the fractional rank estimators and ridge estimators with some $k_i = 0$ are all not admissible on $\mathscr{T}_M$. Further, it may be seen that the ridge estimator (4.6) is admissible among $\mathscr{L}$ on $\mathscr{T}_M$ only if $k_i \geq 1/M$, $i = 1, \cdots, p$.

$\mathscr{T}_M$ corresponds to Perlman's (1972) conditions, which we may state as $\mathscr{T}_M \subset \{(S_1, S_2) : MS_1 - S_2 \text{ nnd}\}$. Perlman showed that no unbiased linear estimator is admissible on $\mathscr{T}_M$. LaMotte (1979, 1980) noted that linear models for invariant quadratics in multivariate normal random variables have this structure and that therefore no such invariant quadratic estimator (except 0) is admissible among invariant quadratic estimators of its expectation.

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