The Annals of Statistics
1981, Vol. 9, No. 5, 1114–1118

THE CALCULATION OF THE LIMITING DISTRIBUTION OF THE LEAST SQUARES ESTIMATOR OF THE PARAMETER IN A RANDOM WALK MODEL[1]

BY G. B. A. EVANS[2] AND N. E. SAVIN

University of Cambridge

The limit distribution of the least squares estimator $\hat{\alpha}$ of the parameter α of the first order stochastic difference equation, in the boundary case $\alpha = 1$, is calculated.

Rao (1978) has obtained an expression for the limiting density of the least squares estimator of the parameter α for the case $|\alpha| = 1$ in the model

$$y_t = \alpha y_{t-1} + \epsilon_t \qquad (t = 0, 1, 2, \cdots),$$

where y_0 is fixed and the ϵ_t are independently and identically distributed with zero expectation and variance σ^2.

We were interested in calculating numerical values for this density because of the growing use of the random walk model ($\alpha = 1$) in econometric studies. Two recent papers that use this model are Hall (1978) and Altonji and Ashenfelter (1979). Also in a recent paper Hendry and Mizon (1978) describe a procedure for testing for the presence of common factors in a stochastic difference equation; an important application here is to test for the presence of unit roots. Further in many econometric studies the estimated value of α is often close to unity and hence it is of interest to test the hypothesis $\alpha = 1$ against the alternative that $\alpha < 1$.

In the course of carrying out these calculations we discovered an error in Professor Rao's paper which is corrected in Rao (1980). To calculate the limiting cumulative distribution and density function of the least squares estimator of α in the model for the case $\alpha = 1$ we proceeded as follows.

Let

$$\hat{\alpha}T = \textstyle\sum_1^T y_t y_{t-1} / \sum_1^T y_{t-1}^2,$$

where

$$\sigma^2 U_T = \sqrt{2} \textstyle\sum_1^T y_{t-1}(y_t - \alpha y_{t-1})/T,$$

$$\sigma^2 V_T = 2 \textstyle\sum_1^T y_{t-1}^2 / T^2$$

Then

$$T(\hat{\alpha}_T - \alpha)/\sqrt{2} = U_T/V_T.$$

White (1958) shows that when the ϵ_t are normally distributed the joint limiting characteristic function of U_T and V_T as $T \to \infty$ is

$$\xi(u, v) = \exp(-\omega/2)\left(\cos\theta - \frac{\omega \sin\theta}{\theta}\right)^{-1/2},$$

where

$$\omega = 2^{1/2}\alpha i u, \theta = 2(iv)^{1/2}.$$

Received May 1979; revised April 1980.

[1] Research supported in part by Social Science Research Council Grant HR 7048/1.

[2] Deceased.

AMS 1970 *subject classifications*. Primary 62E20, 62M10; secondary 60F05.

Key words and phrases. Limit distribution, boundary parameter estimator, unstable process.

Gurland (1948) shows that if the joint characteristic function of the random variables U and V is $\psi(u, v)$ and if $P(V \leq 0) = 0$, then at points of continuity of $F(x)$

$$P(U < x\ V) = F(x) = \frac{1}{2} - \frac{1}{2\pi i} \lim_{\epsilon \to 0, \delta \to \infty} \int_{\epsilon < |u| < \delta} \frac{\psi(u, -ux)}{u} du.$$

Further

$$\frac{dF(x)}{dx} = \frac{1}{2\pi i} \lim_{\epsilon \to 0, \delta \to \infty} \int_{\epsilon < |u| < \delta} \left[\frac{\partial}{\partial v} \psi(u, v) \right]_{v = -ux} du$$

provided that the integral is uniformly convergent with respect to x. Hence

$$\begin{aligned} H(x, \alpha) &= \lim_{T \to \infty} P[T(\hat{\alpha}_T - \alpha)/\sqrt{2} < x] \\ &= \frac{1}{2} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\xi(u, -ux)}{u} du = \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \frac{\text{Imag}[e^{-\omega/2} A^{-1/2}]}{u} du; \end{aligned}$$

and

$$h(x, \alpha) = \frac{d}{dx} H(x, \alpha) = \frac{1}{\pi} \int_0^{\infty} \text{Real}[e^{-\omega/2} A^{-3/2} B]\, du,$$

where

$$\omega = 2^{1/2} iu, \qquad A = \cos\theta - \omega \frac{\sin\theta}{\theta}, \qquad \theta = 2(iv)^{1/2}, \qquad v = -ux,$$

$$B = \left(-\frac{1}{2i} \frac{\partial A}{\partial v} \right) = \frac{\sin\theta}{\theta} + \frac{\omega}{\theta^2} \left(\cos\theta - \frac{\sin\theta}{\theta} \right).$$

It will be noted that the integrand in the expression for $h(x)$ in equation (20) of Rao (1978) tends to ∞ as $x \to 0$ if t is non-zero, but the above expression for $h(x, \alpha)$ tends to a limit as $x \to 0$.

Define $z = (2|ux|)^{1/2}$, $y = 2^{1/2} \alpha u S_x$, where $S_x = 1$ if $x > 0$, $S_x = -1$ if $x < 0$,

and let $a = \cos z \cosh z$, $d = \sin z \sinh z$, $b = (\sin z \cosh z - \cos z \sinh z)/2z$, $e = (\sin z \cosh z + \cos z \sinh z)/2z$, $G = a + by$, $H = d - ey$, $U = G^2 + H^2$, $\beta = \tan^{-1}(H/G)$, $P = e - \alpha/(2\sqrt{2x})(a - e)$, $Q = b - \alpha/(2\sqrt{2x})(d - b)$, $V = P^2 + Q^2$, $\gamma = \tan^{-1}(Q/P)$.

Then

$$h(x, \alpha) = \frac{1}{\pi} \int_0^{\infty} V^{1/2} U^{-3/4} \cos\{\gamma - (3\beta + y)/2\}\, du \qquad (x \neq 0)$$

and

$$h(0, \alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is}\{(1 + 2is/3)/(1 + 2is)^{3/2}\}\, ds.$$

That this is the correct expression for $h(x)$ in terms of real variables was confirmed by showing by calculation that for a large number of values of x and u

$$\text{Real}(e^{-\omega/2} A^{-3/2} B) = V^{1/2} U^{-3/4} \cos\{\gamma - (3\beta + y)/2\};$$

this can be done directly since Fortran allows one to calculate the values of expressions involving complex variables. (Indeed it is not clear that from a computational point of view there is any great advantage in evaluating the integrand in terms of real variables.)

It can be shown that

$$H(x, \alpha) + H(-x, -\alpha) = 1$$

and

$$H(0, 1) = 2\,\Phi(1) - 1, H(0, -1) = 2\{1 - \Phi(1)\},$$

where Φ is the standard Normal cumulative distribution function. We also have

$$h(0, \alpha) = \frac{1}{2^{1/2}\pi}\int_{-\infty}^{\infty} \frac{e^{iu}(1 + \frac{2iu}{3})}{(1 + 2iu)^{3/2}}\, du.$$

Using the Laplace inversion formula for s^{ν} we find that

$$h(0, \alpha) = (\pi\, e)^{-1/2}.$$

For $\alpha = 1$ we calculated the limiting density, $h(x, 1)$, and the limiting cumulative distribution function, $H(x, 1)$, for a number of values of x; the results are given in Table 1. The limiting density is graphed in Figure 1. For both $h(x, 1)$ and $H(x, 1)$ care is needed in the evaluation of the relevant integrands. Each integrand involves the square root of a complex quantity; the evaluation turns on the value used for the argument of the quantity and it was found that to use the principal value of this argument (the value which lies in the interval $[-\pi, \pi]$) does not necessarily preserve the continuity of the integrand. In consequence special measures have to be taken in order to maintain this continuity. We are grateful to D. Kreps for pointing out this problem.

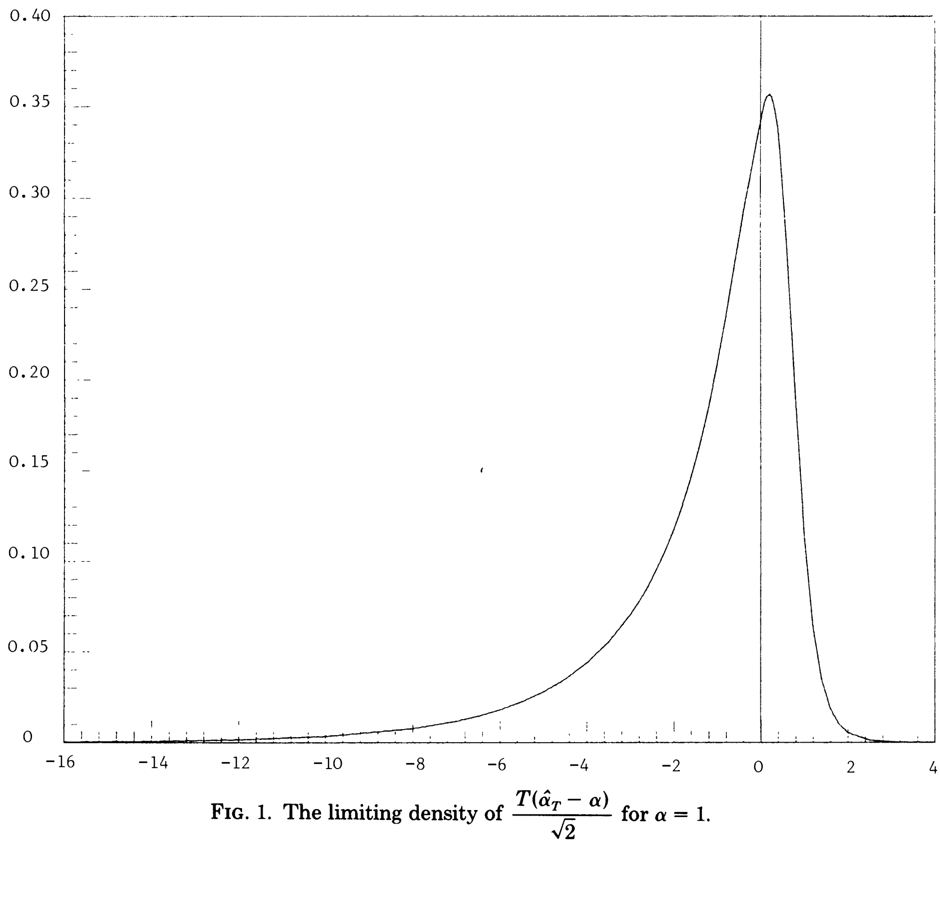

FIG. 1. The limiting density of $\frac{T(\hat{\alpha}_T - \alpha)}{\sqrt{2}}$ for $\alpha = 1$.

TABLE 1

The Density Function, $h(x, 1)$ and the Cumulative Distribution Function, $H(x, 1)$ of the Limiting Distribution of $T(\infty_T - \alpha)/\sqrt{2}$ for $\alpha = 1$.

x	Density Function $h(x, 1)$	Cumulative Distribution Function $H(x, 1)$
−16	0.0003	0.001
−14	0.0007	0.002
−12	0.0016	0.004
−10	0.0035	0.009
−8	0.0078	0.019
−7	0.0118	0.029
−6.5	0.0146	0.036
−6	0.0181	0.044
−5.5	0.0224	0.054
−5	0.0279	0.067
−4.5	0.0349	0.082
−4	0.0439	0.102
−3.5	0.0554	0.126
−3	0.0705	0.158
−2.8	0.0778	0.173
−2.6	0.0860	0.189
−2.4	0.0952	0.207
−2.2	0.1057	0.227
−2.0	0.1175	0.249
−1.8	0.1310	0.274
−1.6	0.1464	0.302
−1.4	0.1641	0.333
−1.2	0.1845	0.368
−1.0	0.2079	0.407
−0.8	0.2343	0.451
−0.6	0.2629	0.501
−0.4	0.2907	0.556
−0.2	0.3158	0.617
0	0.3413	0.683
0.2	0.3566	0.753
0.4	0.3381	0.823
0.6	0.2757	0.885
0.8	0.1897	0.931
1.0	0.1143	0.962
1.2	0.0639	0.979
1.4	0.0347	0.989
1.6	0.0186	0.994
1.8	0.0100	0.997
2.0	0.0054	0.999
2.5	0.0012	1.000
3.0	0.0003	1.000
3.5	0.0001	1.000
4.0	0.0000	1.000

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FACULTY OF ECONOMICS AND POLITICS,
SIDGWICK AVENUE,
CAMBRIDGE, CB3 9DD,
ENGLAND.