The Annals of Statistics
1981, Vol. 9, No. 5, 1079–1086

THE INADMISSIBILITY OF LINEAR RANK TESTS UNDER BAHADUR EFFICIENCY

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Hájek (1974) has shown that in the two-sample problem the best exact slope for a test of randomness against any particular member of a large class of alternative hypotheses is attained by a linear rank test. Here a new class of two-sample rank tests is constructed, and it is shown that for each linear test there exists a test within the new class which is always at least as efficient in terms of exact Bahadur efficiency irrespective of which alternative hypothesis, is tested. Conditions are provided under which the new test is strictly more efficient than the linear rank test. Some comments are made about the practical applicability of the new class of tests.

1. Introduction. Suppose we have a sample of $N = m + n$ independently drawn observations $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ where the X's have distribution function $F(x)$ and the Y's have distribution function $G(x)$. It is assumed that F and G are continuous so that ties among the observations of the combined sample occur with probability zero. The usual linear rank test statistics for testing the null hypothesis H that $F(x) = G(x)$ for all x are defined by

$$S_{m,n} = S_{m,n}(Z) = \sum_{i=1}^{N} a_N(i) Z_{Ni} \tag{1.1}$$

where $a_N(1), \cdots, a_N(N)$ are real numbers and for $1 \le i \le N$, $Z_{Ni} = 1$ if the ith observation in the ordered combined sample is an X, otherwise $Z_{Ni} = 0$. These test statistics are particularly appropriate for testing H against alternatives like location-shift and change-of-scale when the test is to reject H for large values of $S_{m,n}$. In this context they have been studied extensively, for instance by Hájek and Šidák (1967).

The motivation for calculating the exact slope of a sequence of test statistics and using these slopes in order to compare tests is now well-known. It is fully discussed in Bahadur (1967). The resulting Bahadur efficiencies have often been employed as a basis for comparing different tests. Examples of this are given by Abrahamson (1967), Woodworth (1970) and Killeen and Hettmansperger (1974) as well as several others. Hájek (1974) proved that if we wished to test H against any alternative hypothesis for which F and G were absolutely continuous, then the usual linear rank tests defined using (1.1) were sufficient in the sense that the best exact slope possible for the alternative hypothesis (whose value is derived by Raghavachari (1970)) is attained by a linear rank test. In this note rank tests are constructed which are always at least as efficient as the linear rank tests over this range of alternative hypotheses and which are strictly more efficient for certain alternative hypotheses.

Define the mapping

$$g: (Z_{N1}, \cdots, Z_{NN}) \to (Z_{N2}, \cdots, Z_{NN}, Z_{N1})$$

and let $\mathscr{G}$ be the group of order N generated by g. Define the non-linear rank statistic

$$\nu_{m,n} = \max_{\sigma \in \mathscr{G}} \{S_{m,n}(\sigma(Z))\}. \tag{1.2}$$

Received October 27, 1978; revised February 12, 1981.
AMS 1970 *subject classifications*: Primary 62G10; secondary 62G20.
Key words and phrases: Linear rank statistic, Bahadur efficiency, inadmissibility.

These statistics provide tests of H against the general alternative hypothesis that $F(x) \neq G(x)$ for some x when H is rejected for large values of $\nu_{m,n}$. Their practical applicability is discussed in Section 3. In Section 2 we show that the Bahadur efficiency of $\{\nu_{m,n}\}$ relative to $\{S_{m,n}\}$ for particular F and G, written $e_{F,G}(\nu, S)$, satisfies $e_{F,G}(\nu, S) \geq 1$ for all F, G absolutely continuous.

2. Bahadur efficiency. The actual computation of the exact slope is performed in two stages. First a law of large numbers for $\nu_{m,n}$ is proved.

Let $F_m(x)$ and $G_n(x)$ denote the empirical distribution functions of the X's and Y's respectively and define $H_N(x) = (m/N)F_m(x) + (n/N)G_n(x)$. Define a random function on $[0, 1]$ by

$$\hat{S}_{m,n}(t) = m/N \int_0^1 a_N(1 + [(u - t)N]) \, d\, F_m(H_N^{-1}(u)), \tag{2.1}$$

where the definition of $a_N(i)$ is extended beyond $1 \leq i \leq N$ by defining $a_N(i)$ to be periodic with period N. All integrals are assumed to be from 0 to 1 unless specified otherwise. Then

$$N^{-1}\nu_{m,n} = \sup_{0 \leq t \leq 1} \{\hat{S}_{m,n}(t)\}. \tag{2.2}$$

Assume that F, G are absolutely continuous with densities f, g respectively. Suppose that as $N \to \infty$, $m/N \to \lambda$, $0 < \lambda < 1$. Then define $H(x) = \lambda F(x) + (1 - \lambda)G(x)$ and take $\bar{F}(x) = F(H^{-1}(x))$ with $\bar{f}(x) = d\bar{F}(x)/dx$. The following assumption is made about the scores: there exists an integrable function $\phi(u)$ for $0 < u < 1$ for which

$$\lim_{N \to \infty} \int_0^1 | a_N(1 + [uN]) - \phi(u) | \, du = 0. \tag{2.3}$$

The following result generalizes Theorem 1 of Hájek (1974).

PROPOSITION 2.1. *Suppose that* $a_N(1 + [uN])$ *satisfies* (2.3) *and also*

$$\limsup_{N \to \infty} \sup_{0 < a < b < 1} V_a^b(a_N(1 + [uN])) < \infty, \tag{2.4}$$

where $V_a^b(f)$ *denotes the total variation of* f *over* $[a, b]$. *Then with probability* 1,

$$\hat{S}_{m,n}(t) \to S(t)$$

uniformly for $0 \leq t \leq 1$, *where*

$$S(t) = \lambda \int_0^1 \phi(u - t)\bar{f}(u) \, du \tag{2.5}$$

with $\phi(u)$ *defined outside* $(0, 1)$ *by making it periodic with period* 1.

PROOF. First we show that we may assume that $a_N(1 + [uN])$ is of bounded variation on $[0, 1]$. There exist numbers ϵ_1, ϵ_2 satisfying $0 < \epsilon_1 < \epsilon_2 < 1$ so that if

$$\begin{aligned} \tilde{a}_N(1 + [uN]) &= a_N(1 + [uN]) \qquad u \in [\epsilon_1, \epsilon_2] \\ &= 0 \qquad \text{otherwise}, \end{aligned}$$

then for a given $\delta > 0$ there exists N_0 such that for $N \geq N_0$, $N^{-1} \sum_{i=1}^N | a_N(i) - \tilde{a}_N(i) | < \delta$. Furthermore if $0 < u < 1$ and

$$\begin{aligned} \tilde{\phi}(u) &= \phi(u) \qquad u \in [\epsilon_1, \epsilon_2] \\ &= 0 \qquad \text{otherwise}, \end{aligned}$$

then $\int | \phi(u) - \tilde{\phi}(u) | \, du < \delta$. Define $\tilde{S}_{m,n}(Z) = \sum_{i=1}^N \tilde{a}_N(i) Z_{Ni}$. If $N \geq N_0$, then $N^{-1} | S_{m,n}(Z)$

$- \tilde{S}_{m,n}(Z) | < \delta$ for any Z and for $0 \le t \le 1$, $\lambda | \int \tilde{\phi}(u - t)\bar{f}(u)\, du - \int \phi(u - t)\bar{f}(u)\, du | < \delta$($\tilde{\phi}$ extended outside (0, 1) by periodicity, period 1). This implies that it suffices to prove the proposition in the case where $a_N(1 + [uN])$ is of bounded variation on [0, 1].

For $0 \le t \le 1$,

$$(2.6)\quad \hat{S}_{m,n}(t) = \frac{m}{N}\int_0^1 \phi(u - t)\bar{f}(u)\, du + \frac{m}{N}\int_0^1 \{a_N(1 + [(u - t)N]) - \phi(u - t)\}\, \bar{f}(u)\, du + \frac{m}{N}\int_0^1 a_N(1 + [(u - t)N])\, d(\bar{F}_m - F)(u)$$

where $\bar{F}_m(u) = F_m(H_N^{-1}(u))$. By applying the definition of $\bar{f}(u)$ to the second term on the right hand side of (2.6), it follows that in modulus this term is less than

$$\frac{m}{N}\lambda^{-1}\int | a_N(1 + [uN]) - \phi(u) |\, du$$

for all $0 \le t \le 1$ and so it tends to 0 uniformly for $0 \le t \le 1$ by (2.3). Furthermore, by (2.4) there exists N_1 such that for $N \ge N_1$,

$$\Big| \frac{m}{N}\int a_N(1 + [(u - t)N])\, d(\bar{F}_m - \bar{F})(u) \Big| = \Big| \frac{m}{N}\int_0^1 \{\bar{F}_m(u) - \bar{F}(u)\}\, d(a_N(1 + [(u - t)N])) \Big| \le \frac{mK}{N} \sup_{0<u<1} \{ | \bar{F}_m(u) - \bar{F}(u) | \},$$

where for $N \ge N_1$, $V_0^1[a_N(1 + [uN])] < K$. Consequently applying the Glivenko-Cantelli lemma to the third term on the right hand side of (2.6) it follows that this term tends to 0 uniformly for $0 \le t \le 1$. This proves the proposition.

Directly from proposition 2.1 it follows that with probability 1,

$$(2.7)\quad N^{-1}\nu_{m,n} \to \sup_{0\le t\le 1} \{\lambda \int_0^1 \phi(u - t)\bar{f}(u)\, du\}.$$

In order to find the exact slope for the sequence $\{\nu_{m,n}\}$ it remains to calculate the large deviation

$$I(\nu, x) = \lim_{N\to\infty} - N^{-1} \log P\{\nu_{m,n} \ge Nx_N | F = G\}$$

where x_N is any sequence of real numbers tending to x. The key observation here is lemma 2.2.

LEMMA 2.2. *If $I(S, x)$ denotes the large deviation of $\{S_{m,n}\}$, then*

$$(2.8)\quad I(\nu, x) = I(S, x).$$

PROOF. When H is true, the statistics $S_{m,n}(\sigma(R))$ for the different $\sigma \in \mathscr{G}$ are identically distributed. The lemma proved by Killeen and Hettmansperger (1972, page 1509) therefore applies and (2.8) is proved.

THEOREM 2.3. *Under the conditions of Proposition* 2.1, $e_{F,G}(\nu, S) \ge 1$ *for all absolutely continuous F and G.*

PROOF. The Bahadur efficiency of one sequence of tests relative to another is given by the ratio of their exact slopes. The exact slope of $\{N^{-1}\nu_{m,n}\}$ for given F and G equals $2I(\nu, b_\nu(F, G))$ where $b_\nu(F, G)$ is the almost sure limit of $N^{-1}\nu_{m,n}$ obtained from (2.7). Similarly, the exact slope of $\{N^{-1}S_{m,n}\}$ for given F, G equals $2I(S, b_s(F, G))$ where $b_s(F, G)$

is the almost sure limit of $N^{-1}S_{m,n}$ under F and G. Clearly however, $b_\nu(F, G) \geq b_s(F, G)$ and $I(S, x)$ is non-decreasing so that the theorem follows by Lemma 2.2.

In order to study the exact slope of $\{\nu_{m,n}\}$ further and find conditions for which $e_{F,G}(\nu, S) > 1$, the following result is needed.

PROPOSITION 2.4. *Assume that* $\phi(u)$ *is not almost everywhere constant. Then* $I(S, x)$ *is a strictly increasing function for* $x \in [r_1, r_2]$ *where* $r_1 = \lambda\int\phi(u)\,du$ *and* $r_2 = \sup\{\lambda\int\phi(u)f(u)\,du\colon 0 \leq f(u) \leq \lambda^{-1}$ *for* $0 < u < 1$ *and* $\int f(u)\,du = 1\}$.
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PROOF. Results from Woodworth (1970) are needed. From Theorem 4 of that paper it follows that for $r_1 \leq x \leq r_2$,

$$I(S, x) = \rho(x - \lambda s) - \int_0^1 \log((1-\lambda) + \lambda\exp[\rho\{\phi(u) - s\}])\,du \tag{2.9}$$

where (ρ, s) is the unique solution to the pair of equations

$$\int_0^1 \exp[\rho\{\phi(u) - s\}]/(1 - \lambda + \lambda\exp[\rho\{\phi(u) - s\}])\,du = 1 \tag{2.10}$$

and

$$\int_0^1 \lambda\phi(u)\exp[\rho\{\phi(u) - s\}]/(1 - \lambda + \lambda\exp[\rho\{\phi(u) - s\}])\,du = l(\rho) = x. \tag{2.11}$$

Woodworth (1970, pp. 280–282) shows that $l(\rho)$ is strictly increasing for $\rho \geq 0$ and that $r_1 = l(0)$ and $l(\rho) \to r_2$ as $\rho \to \infty$. It is sufficient, therefore, to show that $I(S, l(\rho))$ is a strictly increasing function for $\rho \geq 0$. But $I(S, l(\rho)) = \rho k'(\rho) - k(\rho)$ where $k(\rho) = \int_0^1 \log(1 - \lambda + \lambda\exp[\rho\{\phi(u) - s\}])\,du$ and is therefore strictly increasing because for $\rho > 0$,

$$\rho k''(\rho) = \rho\lambda(1-\lambda)\int_0^1 \frac{\{\phi(u) - s\}^2\exp[\rho\{\phi(u) - s\}]}{(1 - \lambda + \lambda\exp[\rho\{\phi(u) - s\}])^2}\,du > 0, \tag{2.12}$$

since $\phi(u)$ is not almost everywhere constant.

From Proposition 2.4 it follows that in order to prove $e_{F,G}(\nu, S) > 1$ for particular F and G, it is sufficient to show that $b_\nu(F, G) > b_s(F, G)$. The following theorem shows that l[illegible]ar rank tests are inadmissible since by part (*i*) the inequality $e_{F,G}(\nu, S) \geq 1$, for all absolutely continuous F and G proved in Theorem 2.3 is strict for some F and G. In part (*ii*) of Theorem 2.5 we have taken a particular set of conditions for $\phi(u)$ and studied the range of alternative hypotheses (F, G) for which $e_{F,G}(\nu, S) > 1$ holds. By considering different $\phi(u)$ further results in the same vein may be obtained.

THEOREM 2.5.

(*i*) *Suppose that* $\phi(u)$ *is not almost everywhere constant on* $[0, 1]$. *Then there exist absolutely continuous distributions* F, G *for which* $e_{F,G}(\nu, S) > 1$.

(*ii*) *Suppose that* $\phi(u) = au + b$, *a and b constants. Then for any pair* F, G *of absolutely continuous distributions for which* $F(x) > G(x)$ *for some* x, *it follows that* $e_{F,G}(\nu, S) > 1$.

PROOF. Suppose $F(x)$ is some distribution on $[0, 1]$. The translate $F_a(x)$ is defined for $0 < a < 1$ as follows:

$$\begin{aligned} F_a(x) &= F(x + a) - F(a) && 0 \leq x < 1 - a, \\ &= 1 - F(a) + F(x + a - 1) && 1 - a \leq x < 1. \end{aligned}$$

If $F(x)$ is absolutely continuous then $F_a(x)$ is the distribution corresponding to $f(x - a)$, the translate of the density which is defined to be periodic with period 1.

In order to prove part (*i*) we first show that for each $0 < t < 1$ there exists an interval

$E \subset [0, 1]$ for which $\int_E \phi(u - t)\, du < \int_E \phi(u)\, du$. Assume that there is no interval $D \subset [0, 1]$ for which $\int_D \phi(u - t)\, du > \int_D \phi(u)\, du$, otherwise the result is clearly true. Consequently for every $\epsilon > 0$, $\epsilon^{-1} \int_a^{a+\epsilon} [\phi(u - t) - \phi(u)]\, du = 0$ which by Lebesgue's differentiation theorem implies that $\phi(u - t) = \phi(u)$ almost everywhere and so $\phi(u)$ is constant almost everywhere, contradicting the hypothesis. Therefore the required E exists. Assume that $E = [\xi, 1]$ where without loss of generality we may suppose $\frac{1}{2} < \xi < 1$. Define $\bar{f}(u) = 1/\xi$, for $0 \le u \le \xi$ and $\bar{f}(u) = 0$ for $\xi < u \le 1$. Then $\int \phi(u - t)\bar{f}(u)\, du > \int \phi(u)\bar{f}(u)\, du$. It is easy to check that $\bar{F}(u)$, the distribution corresponding to $\bar{f}(u)$ can be written in the form $F(H^{-1}(u))$ where $F(u) = u/\xi$, $0 \le u \le \xi$, $F(u) = 1$, $\xi < u \le 1$ and $G(u) = \xi^{-1}(1 - \lambda)^{-1}(\xi - \lambda)u$, $0 \le u \le \xi$, $G(u) = (1 - \lambda)^{-1}(u - \lambda)$, $\xi < u \le 1$. If $E = [\xi', \xi'']$ where $0 \le \xi' < \xi'' < \xi' + \frac{1}{2}$, then the previous arguments may be applied to $\phi(u + 1 - \xi'')$. The pair of distributions for $\phi(u)$ is then obtained by taking $F_{1-\xi''}(u)$ and $G_{1-\xi''}(u)$.

In view of Proposition 2.4, and (2.7) it follows that $e_{F,G}(\nu, S) > 1$. Notice that the result we have proved here is stronger than is necessary in order to prove part (i) for we have shown that for *each* t, $0 < t < 1$, there exists F, G for which $\int \phi(u - t)\bar{f}(u)\, du > \int \phi(u)\bar{f}(u)\, du$. Consequently for any linear rank test obtained using translated scores there exists F, G for which the test with the translated scores is more efficient.

In order to prove part (ii) assume without loss of generality that F, G are distributions on $[0, 1]$. To prove the result it is sufficient to show that for some $0 < t < 1$,

$$\int_0^1 (u - t)\bar{f}(u)\, du > \int_0^1 u\bar{f}(u)\, du, \tag{2.13}$$

or equivalently that for some $0 < a < 1$,

$$\int_0^1 u dF_a(H_a^{-1}(u)) > \int_0^1 u dF(H^{-1}(u)).$$

After integration by parts, we find that it is sufficient to show that

$$\int_0^1 F(H^{-1}(u))\, du - \int_0^1 F_a(H_a^{-1}(u)) > 0 \tag{2.14}$$

for some $0 < a < 1$. This is true because

$$\begin{aligned}\int_0^1 \{F(H^{-1}(u)) - F_a(H_a^{-1}(u))\}\, du &= \int_0^1 F(u)\, dH(u) - \int_0^1 F_a(u)\, dH_a(u) \\ &= (1 - \lambda)\left\{\int_0^1 F(u)\, dG(u) - \int_0^1 F_a(u)\, dG_a(u)\right\} \\ &= (1 - \lambda)\{F(a) - G(a)\} > 0\end{aligned}$$

for some $a \in (0, 1)$ by hypothesis. This proves (2.14) and hence part (ii) of the theorem.

Theorems 2.3 and 2.5 show that $S_{m,n}$ is inadmissible, but what can be said about the admissibility of $\nu_{m,n}$? It certainly need not be admissible. Enlarge $\mathscr{G}$ by defining the transformation

$$h: (Z_{N1}, \cdots, Z_{NN}) \to (Z_{NN}, \cdots, Z_{N1})$$

and take $\mathscr{H}$ to be the group generated by g and h. Define

$$\mu_{m,n} = \max_{\sigma \in \mathscr{H}}\{S_{m,n}(\sigma(Z))\}.$$

The arguments used for Proposition 2.1 and Lemma 2.2 extend readily to show that $e_{F,G}(\mu, \nu) \geqslant 1$ for all F, G absolutely continuous. There are cases when $e_{F,G}(\mu, \nu) = 1$ for all F and G, for instance if $\phi(u) = \phi(1 - u)$ for $0 < u < 1$. In other cases, however, it is possible to find F, G for which $e_{F,G}(\mu, \nu) > 1$, and this is certainly true when $\phi(u) = u$, the generating function of the Mann-Whitney test. This was verified by actual computation of efficiencies when F, G are normal distributions.

The basic ideas behind Theorem 2.3 with the Killeen-Hettmansperger lemma as the key also shed light on other results. Let $D^+_{m,n} = \sup_z\{F_m(z) - G_n(z)\}$ and $D^-_{m,n} = \sup_z\{G_n(z) - F_m(z)\}$. Define the Smirnov test statistic by $D_{m,n} = \max(D^+_{m,n}, D^-_{m,n})$ and the Kuiper test statistic by $V_{m,n} = D^+_{m,n} + D^-_{m,n}$. On the basis of actual calculation of the exact slopes Abrahamson (1967) showed that for all F and G, $e_{F,G}(V, D) \geqslant 1$. However, Barr and Shudde (1973) show that $V_{m,n} = \max_{\sigma\in\mathscr{G}}\{D_{m,n}(\sigma(Z))\}$ and so this is a consequence of the Killeen-Hettmansperger lemma. Likewise the derivation of the exact slope for Ajne's N-test given by Rao (1972) can be simplified by noting that the test statistic has the form required by the lemma; this is similar to the discussion of the Hodges' sign test in Killeen and Hettmansperger (1972). It also seems reasonable to surmise that the class of one-sample linear rank tests is also inadmissible under Bahadur efficiency, although to confirm this a detailed investigation of the exact slopes for this class of tests would be required.

3. The new tests in practice. This section contains a brief discussion of how the tests which reject H for large values of $\nu_{m,n}$ may be used and gives some idea of their performance against certain kinds of alternative hypothesis. Detailed proofs are not included. However, as doubt has been expressed as to whether these test statistics are of more than theoretical interest, it seems to be appropriate to discuss that question here.

The test statistic obtained when $a_N(i) = i$, $1 \leqslant i \leqslant N$, in (1.2) is studied by Eplett (1979, 1980). For this particular case

$$\nu_{m,n} = S_{m,n} + 2mnD^+_{m,n} \tag{3.1}$$

and consequently $\nu_{m,n}$ is easy to evaluate from the data. Although the expression obtained is not as elegant as (3.1), a similar kind of result holds when $\phi(u)$ is continuous and piecewise-linear, providing a reasonably quick method of calculating $\nu_{m,n}$.

The following result is used to obtain rejection regions for large sample sizes:

THEOREM 3.1. *Assume that $\phi(u)$ is square integrable and is extended outside* (0, 1) *by defining it to be periodic with period* 1. *If*

(i) $$m/n \to \lambda,\ 0 < \lambda < 1, \text{ as } N \to \infty,$$

(ii) $$\int_0^1 \{a_N(1 + [uN]) - \phi(u)\}^2\, du \to 0 \text{ as } N \to \infty,$$

then under H,

$$\{N/(mn)\}^{1/2}\{\nu_{m,n} - mN^{-1}\textstyle\sum_{i=1}^N a_N(i)\} \xrightarrow{\mathscr{D}} \sup_{0<t<1}\{S(t)\}, \tag{3.2}$$

where $S(t)$ is a Gaussian process on [0, 1] *having $E\{S(t)\} = 0$, $0 \leqslant t \leqslant 1$, $E\{S(s)S(t)\} = \int \{\phi(u-t) - \bar\phi\}\{\phi(u-s) - \bar\phi\}\, du$ $0 \leqslant s, t \leqslant 1$, where $\bar\phi = \int \phi(u)\, du$ and $P\{S(t) \in C([0, 1])\} = 1$, while $\xrightarrow{\mathscr{D}}$ denotes convergence in distribution.*

A systematic approach towards obtaining approximations for the tail probabilities of $\sup\{S(t)\}$ is to approximate the sample functions by simpler functions which only assume a finite number of different values (such as step-functions or piecewise-linear, continuous functions). For instance in Eplett (1980), the vector $\max\{S(i/100), 0 \leqslant i \leqslant 100\}$ is simulated when $\phi(u) \equiv u$. This furnishes a reasonable approximation to $\sup\{S(t)\}$ provided that m and n are both larger than 25, and it seems that in general they provide upper bounds to the exact probabilities. This approach may be used for any choice of $\phi(u)$.

Simulation has been used to study the power of the tests. Results are given here for the case where $a_N(i) = i$, $1 \leqslant i \leqslant N$. In both Tables 3.1 and 3.2, the alternative hypotheses F and G correspond to $N(0, 1)$ and $N(\mu, \sigma^2)$ distributions. Table 3.1. is a small sample size comparison of the performance of the Mann-Whitney based test against the Smirnov test using exact null probabilities. This is a condensed version of a table given in Eplett (1980).

Table 3.2 is a large sample size comparison of the relative performance of the Smirnov, Mann-Whitney and the Mann-Whitney based test defined by (3.1). The rejection region for the last test is obtained using the table in Eplett (1980) based on Theorem 3.1.

TABLE 3.1

Comparison of powers of $\nu_{m,n}$ and $D_{m,n}$ for sample sizes $m = n = 12$, nominal significance levels 0.05, F and G respectively $N(0, 1)$ and $N(\mu, \sigma^2)$. Obtained from 20,000 simulations.

μ	σ	Power of $\nu_{m,n}$	Power of $D_{m,n}$
0.5	0.25	.73	.56
0.5	1.00	.08	.13
0.5	2.00	.24	.13
0.5	4.00	.68	.24
−0.5	0.25	.78	.50
−0.5	1.00	.10	.13
−0.5	2.00	.28	.13
−0.5	3.00	.53	.18
1.5	1.0	.39	.82
1.5	4.0	.70	.42
−1.0	0.25	.92	.93
−1.0	1.0	.32	.46
−1.0	3.0	.57	.29

TABLE 3.2

Comparison of powers of $D_{m,n}$, $S_{m,n}$ and $\nu_{m,n}$ for sample sizes $m = n = 40$, nominal significance levels 0.05, F and G respectively $N(0, 1)$ and $N(\mu, \sigma^2)$ obtained from 5000 simulations

μ	σ	Power of $D_{m,n}$	Power of $S_{m,n}$	Power of $\nu_{m,n}$
0.00	1.00	0.053	0.05	0.046
0.10	1.30	0.093	0.059	0.140
0.10	0.70	0.149	0.077	0.238
−0.10	1.30	0.092	0.067	0.158
−0.10	0.70	0.137	0.078	0.255
0.25	1.20	0.158	0.152	0.122
0.25	1.40	0.205	0.138	0.228
0.25	1.60	0.267	0.120	0.384
0.25	1.80	0.341	0.116	0.555
−0.25	1.20	0.165	0.158	0.146
−0.25	1.40	0.217	0.149	0.276
−0.25	1.60	0.260	0.124	0.446
−0.25	1.80	0.341	0.112	0.611
0.50	1.50	0.481	0.382	0.366
0.50	2.50	0.752	0.206	0.916
0.50	3.00	0.868	0.171	0.978
−0.50	1.50	0.486	0.388	0.495
−0.50	2.00	0.610	0.281	0.797
−0.50	2.50	0.755	0.213	0.945
0.75	1.50	0.754	0.700	0.456
0.75	2.00	0.788	0.520	0.745
0.75	3.00	0.915	0.310	0.983
−0.75	1.50	0.760	0.714	0.684
−0.75	2.50	0.790	0.520	0.857

The values when $\sigma < 1$ are for the most part omitted in these tables since the power of the respective tests behaves in basically the same way as for the case $\sigma > 1$. This may be explained by considering what happens when the roles of the two samples are interchanged.

These two tables highlight the motivation behind the new tests. The alternative hypothesis of location shift and change in scale is a stylized expression of the fact that the alternative hypothesis of location shift (or equally, change in scale) by itself is rather restrictive in many situations where rank tests are useful. For better or for worse, omnibus tests like the Smirnov test have an important role to play in testing the more complicated hypotheses which arise in practice. Unfortunately the Smirnov test, with its analogies to the median test, performs best when F and G have relatively heavy tails and the alternative hypothesis is dominated by a shift in location. By using different scores $a_N(i)$ in (1.2) one obtains a variety of omnibus tests which in specific cases are going to be substantially more powerful than the Smirnov test and still retain that power over a wide range of alternative hypotheses; in other words they will be power-robust. As Table 3.2 indicates, linear rank tests lose power very quickly against a combination of location shift and change in scale unless the particular change they are designed to handle strongly dominates the alternative hypothesis.

Tests obtained from (1.2) may be motivated through the method of union-intersection, see Eplett (1980). This may be used to prove the consistency of these tests for testing H against the general alternative that $F(x) \neq G(x)$ for some x(F and G continuous), provided certain assumptions are made about $\phi(u)$, e.g. that it is strictly monotone. Much of this discussion applies equally to tests which reject H for large values of $\mu_{m,n}$. Such tests are particularly appropriate for testing data on the circumference of a circle since the test statistics are invariant under the different possible choices of starting point and direction of measurement used in measuring the angular displacements of the observations.

In conclusion, therefore, the tests discussed here possess notable efficiency and power properties. It has been suggested to the author that the presentation equates good power with good Bahadur efficiency. This is not the case as the basic definitions are very different; an illustration of this difference in practice may be found in Eplett (1980).

ACKNOWLEDGEMENT. Mrs. A. Mayho programmed the simulations in Section 3 and I am grateful for her assistance. I am also indebted to an associate editor who pointed out an error in a previous version of the paper.

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