The Annals of Statistics
1981, Vol. 9, No. 3, 673–677

THE SHORTCOMING OF LOCALLY MOST POWERFUL TESTS IN CURVED EXPONENTIAL FAMILIES

BY WILBERT C. M. KALLENBERG

Vrije Universiteit, Amsterdam

Comparison of tests with respect to contiguous alternatives is mostly concerned with fixed levels. Properties of locally most powerful (LMP) tests in this sense are well-known in statistical literature. In this note the behaviour of LMP tests is studied for local (not necessarily contiguous) alternatives and vanishing levels of significance. It turns out that the shortcoming of the LMP test tends to zero at the rate $n^{-1}|\log \alpha_n|^{3/2}$.

1. Introduction. Let $X_1, X_2, \cdots$ be i.i.d. k-dimensional random variables (rv's) with density

$$\exp\{\gamma_\theta' x - \psi(\gamma_\theta)\}, \qquad \theta \in \Theta, \tag{1.1}$$

with respect to a σ-finite measure μ on $\mathbb{R}^k$. Here Θ is an interval in $\mathbb{R}^1$, γ_θ a three times differentiable bijection from Θ onto $\gamma(\Theta) \subset \Gamma = \{\gamma \in \mathbb{R}^k; \int \exp(\gamma' x)\, d\mu(x) < \infty\}$ and $\psi(\gamma) = \log \int \exp(\gamma' x)\, d\mu(x)$. So the distribution of X_i belongs to a curved exponential family in the terminology of Efron (1975). This means that our one-parameter family is smooth in the sense that it can be embedded in an exponential family in a suitable way. We consider the testing problem H: $\theta = \theta_0$ against K: $\theta > \theta_0$ with level of significance $\alpha_n \in (0, 1)$, where n denotes the number of available observations and $\theta_0 \in \Theta$ is given.

In Efron (1975) and Pfanzagl (1975) some properties of locally most powerful (LMP) tests are mentioned for this kind of testing problems. If $\alpha_n = \alpha$ is fixed, asymptotic expansions of the power function of LMP tests for $\theta \to \theta_0$ are obtained e.g., by Pfanzagl (1973, 1975), Chibisov (1973) and Albers (1974). The LMP tests turn out to be nonoptimal even under contiguous alternatives. According to Pfanzagl (1975) the shortcoming of the LMP test tends to zero at the rate n^{-1} for contiguous alternatives if $\alpha_n = \alpha$ is fixed. For nonlocal alternatives the performance of LMP tests can be expressed by its Bahadur slope. Intuitively it is obvious that LMP tests are not optimal from this nonlocal point of view. Indeed, indicating differentiation with respect to θ by a dot, the slope is not optimal at θ unless the vectors $\gamma_\theta - \gamma_{\theta_0}$ and $\dot\gamma_{\theta_0}$ have the same direction.

Nonlocal comparison of tests in the sense of Bahadur requires levels of significance tending to zero at an exponential rate. Comparison of tests with respect to contiguous alternatives is mostly concerned with fixed levels. This note attempts to fill in the gap: the behaviour of LMP tests is studied for local (not necessarily contiguous) alternatives and vanishing levels of significance. It turns out that the shortcoming of the LMP test tends to zero at the rate $n^{-1}|\log \alpha_n|^{3/2}$. This agrees with the well-known results for fixed α.

2. Main results. Consider the probability space $(\mathbb{R}^k, \mathscr{B}^k, P_\gamma)$, where $\mathscr{B}^k$ is the σ-field of Borel sets in $\mathbb{R}^k$ and $dP_\gamma = \exp\{\gamma' x - \psi(\gamma)\}\, d\mu(x)$ for all $\gamma \in \Gamma$, and suppose X_i has distribution P_{γ_θ}, $i = 1, \cdots, n$. Since with n observations $X_1, \cdots, X_n$ the sample mean $\bar X_n = n^{-1} \sum_{i=1}^n X_i$ is sufficient, LMP tests and most powerful (MP) tests only depend on $\bar X_n$. If $Y_1, \cdots, Y_n$ are i.i.d. rv's each distributed according to P_γ, the distribution of $\bar Y_n$ is denoted by $\bar P_\gamma^n$, and the expectation and covariance matrix of Y_i by $\lambda(\gamma)$ and Σ_γ, respec-

Received November, 1979; revised May, 1980.

AMS 1970 *subject classification.* 62F05.

Key words and phrases. Locally most powerful tests, shortcoming, curved exponential families.

tively.

The size-α_n LMP test of H against K based on n observations is given by

$$\phi_n^L(\bar{x}_n) = \begin{cases} 1 & > \\ \delta_n & \text{if } \dot{\gamma}'_{\theta_0}\bar{x}_n = d_n, \\ 0 & < \end{cases} \tag{2.1}$$

where the constants d_n and δ_n satisfy $E_{\theta_0}\phi_n^L(\bar{X}_n) = \alpha_n$, cf. Efron (1975). Let β_n^+ be the level-α_n envelope power function. Then the *shortcoming* R_n of the LMP test is defined by

$$R_n(\theta) = \beta_n^+(\theta) - E_\theta\phi_n^L, \qquad \theta > \theta_0.$$

To obtain the required asymptotic expansions we present condition (C), which is of the same kind as Cramér's condition (C), cf. Cramér (1962). For some $\varepsilon > 0$

$$\lim_{|t|\to\infty} \sup_{\|\gamma_1\|\le\varepsilon, \|\gamma_2\|\le\varepsilon} | E_{\gamma_{\theta_0}+\gamma_1} e^{it(\dot{\gamma}_{\theta_0}+\gamma_2)'X_1} | < 1,$$

where $\|\cdot\|$ denotes the Euclidean norm. This effectively rules out discrete random variables.

THEOREM 2.1. *Assume that* $\gamma_{\theta_0} \in \text{int } \Gamma$, *condition* (C) *holds and that the Fisher information of* X_i *at* θ_0 *is positive. Let* $\{\alpha_n\}$ *be a sequence of levels satisfying* $\alpha_n \le \alpha < 1$, *then*

$$\theta_n \to \theta_0 \textit{ implies } R_n(\theta_n) = O(n^{-1}|\log \alpha_n|^{3/2}) \quad \textit{as} \quad n \to \infty. \tag{2.2}$$

Note that a positive Fisher information at θ_0 implies $\dot{\gamma}_{\theta_0} \neq 0$.

REMARK 2.1. If $\alpha_n = \alpha \in (0, 1)$ is fixed the well-known order $O(n^{-1})$ is obtained. For sequences of alternatives $\{\theta_n\}$ tending to θ_0 at a rather slow or a rather fast rate we have $R_n(\theta_n) = O(n^{-1})$ even if α_n is not fixed, cf. Lemma 3.6.

The following example indicates that the order term $O(n^{-1}|\log \alpha_n|^{3/2})$ is sharp.

EXAMPLE 2.1. Let $X_1, X_2, \cdots$ be i.i.d. 2-dimensional rv's with normal $N(\gamma_\theta; I_2)$ distributions, where $\gamma_\theta = (\theta, \frac{1}{2}\theta^2)$ $(-\infty < \theta < \infty)$ and I_2 the 2×2 identity matrix. Let $\theta_0 = 0$, $\theta_n = n^{-1/2}\Phi^{-1}(1 - \alpha_n)$, where α_n is such that $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} n^{-1}|\log \alpha_n|^{3/2}$; Φ denotes the standard normal distribution function, ϕ denotes the standard normal density. Then $\lim_{n\to\infty} n|\log \alpha_n|^{-3/2}R_n(\theta_n) = \frac{1}{4}\pi^{-1/2}$.

Without condition (C) expansions can be made up to order $O(n^{-1/2})$. In that case we obtain

THEOREM 2.2. *Assume that* $\gamma_{\theta_0} \in \text{int } \Gamma$ *and that the Fisher information of* X_i *at* θ_0 *is positive. Let* $\{\alpha_n\}$ *be a sequence of levels satisfying* $\alpha_n \le \alpha < 1$, *then*

$$\theta_n \to \theta_0 \quad \textit{implies} \quad R_n(\theta_n) = O(n^{-1/2} + n^{-1}|\log \alpha_n|^{3/2}) \quad \textit{as} \quad n \to \infty. \tag{2.3}$$

3. Proofs. In the sequel we assume $\gamma_{\theta_0} \in \text{int } \Gamma$, $\dot{\gamma}_{\theta_0} \neq 0$ (since the Fisher information of X_i at θ_0 is positive), $\alpha_n \le \alpha < 1$ and $\lim_{n\to\infty} n^{-1}\log \alpha_n = 0$. Note that if $n_k^{-1}|\log\alpha_{n_k}| \to a \in (0, \infty]$ for some subsequence $\{n_k\}$ with $\lim_{k\to\infty} n_k = \infty$, then $R_{n_k}(\theta_{n_k}) \le 1 = O(n_k^{-1}|\log \alpha_{n_k}|^{3/2})$ if $k \to \infty$. (In fact $\beta_{n_k}^+(\theta_{n_k})$ and hence $R_{n_k}(\theta_{n_k})$ tend to zero at an exponential rate in this case.) Without loss of generality let $\theta_0 = \gamma_0 = \lambda(0) = 0$ and let Σ_0 be nonsingular.

The size-α_n LMP test of H against K is given by (2.1) with $\theta_0 = 0$. Since $\alpha_n \le \alpha < 1$ it holds that $\lim \inf_{n\to\infty} n^{1/2} d_n > -\infty$. Moreover,

$$n^{-1}\log \alpha_n \to 0 \quad \text{implies} \quad d_n \to 0. \tag{3.1}$$

We use the following notation

$$s^2 = \mathrm{Var}_0\, \dot{\gamma}_0' X_1 = \dot{\gamma}_0' \Sigma_0 \dot{\gamma}_0, \qquad \Delta = E_0(\dot{\gamma}_0' X_1)^3 \qquad \text{and} \quad \xi_t = \Phi^{-1}(t),\ 0 < t < 1.$$

Relations between α_n and d_n are given in the following two lemmas.

LEMMA 3.1.

$$n^{1/2} d_n s^{-1} = \xi_{1-\alpha_n} + 6^{-1}\Delta s^{-3} n^{-1/2} \xi_{1-\alpha_n}^2 + O(n^{-1/2} + \xi_{1-\alpha_n}^3 n^{-1}) \qquad as \quad n \to \infty. \tag{3.2}$$

PROOF. Let $a_n = n^{1/2} d_n s^{-1}$, then $a_n = o(n^{1/2})$ as $n \to \infty$ in view of (3.1). By Theorem 1 on page 218 in Petrov (1975) we have

$$\log \alpha_n = \log(1 - \Phi(a_n)) + 6^{-1}\Delta s^{-3} n^{-1/2} a_n^3 + O((|a_n| + 1)n^{-1/2} + a_n^4 n^{-1}) \tag{3.3}$$

as $n \to \infty$. It easily follows that $a_n \xi_{1-\alpha_n}^{-1} \to 1$ as $n \to \infty$. Taylor expansion of $\log(1 - \Phi(\cdot))$ at $\xi_{1-\alpha_n}$ yields

$$\begin{aligned} \log(1 - \Phi(a_n)) = \log \alpha_n - (a_n - \xi_{1-\alpha_n})\phi(\xi_{1-\alpha_n})(1 - \Phi(\xi_{1-\alpha_n}))^{-1} \\ + O(|a_n - \xi_{1-\alpha_n}|^2) \end{aligned} \tag{3.4}$$

as $n \to \infty$. In view of (3.3), (3.4) and $1 - \Phi(x) = x^{-1}\phi(x)(1 + O(x^{-2}))$ as $x \to \infty$ the result is established. □

LEMMA 3.2. *If condition* (C) *holds*

$$n^{1/2} d_n s^{-1} = \xi_{1-\alpha_n} + 6^{-1}\Delta s^{-3} n^{-1/2}(\xi_{1-\alpha_n}^2 - 1) + O(n^{-1}|\log \alpha_n|^{3/2}) \tag{3.5}$$

as $n \to \infty$.

PROOF. Again let $a_n = n^{1/2} d_n s^{-1}$, then $a_n = o(n^{1/2})$ as $n \to \infty$ in view of (3.1). By Saulis (1969) (cf. also Petrov (1975) chapter 8, Section 4, Number 3 on page 249; note that $P(S_n \geq \sigma(nx)^{1/2})$ has to be replaced there by $P(S_n \geq \sigma n^{1/2} x)$) we have

$$\log \alpha_n = \log(1 - \Phi(a_n)) + 6^{-1}\Delta s^{-3} n^{-1/2}(a_n^2 - 1)\phi(a_n)(1 - \Phi(a_n))^{-1} + O((a_n^4 + 1)n^{-1})$$

as $n \to \infty$. By Taylor expansion of $\log(1 - \Phi(\cdot))$ at $\xi_{1-\alpha_n}$ the result is established. □

An expansion of the power of the LMP test is given in the following

LEMMA 3.3 *If* $\lim_{n\to\infty} \theta_n = 0$

$$E_{\theta_n}\phi_n^L = 1 - \Phi(b_n) + O(n^{-1/2}) \qquad as \quad n \to \infty; \tag{3.6}$$

if $\lim_{n\to\infty} \theta_n = 0$ *and condition* (C) *holds, then*

$$E_{\theta_n}\phi_n^L = 1 - \Phi(b_n) - 6^{-1}\rho_n n^{-1/2}(1 - b_n^2)\phi(b_n) + O(n^{-1}) \qquad as \quad n \to \infty, \tag{3.7}$$

where

$$b_n = n^{1/2}[d_n s^{-1} - s\theta_n - \tfrac{1}{2}\theta_n s^{-3}\Delta\, d_n - \tfrac{1}{2}\dot{\gamma}_0' \Sigma_0 \ddot{\gamma}_0 s^{-1}\theta_n^2 + O(\theta_n^2(\theta_n + d_n))]$$

and

$$\rho_n = (\dot{\gamma}_0' \Sigma_{\gamma_{\theta_n}} \dot{\gamma}_0)^{-3/2} E_{\theta_n}\{\dot{\gamma}_0'(X_1 - \lambda(\gamma_{\theta_n}))\}^3.$$

PROOF. Since $\dot{\gamma}_0' \lambda(\gamma_{\theta_n}) = s^2\theta_n + \tfrac{1}{2}(\dot{\gamma}_0' \Sigma_0 \ddot{\gamma}_0 + \Delta)\theta_n^2 + O(\theta_n^3)$ and $\dot{\gamma}_0' \Sigma_{\gamma_{\theta_n}} \dot{\gamma}_0 = s^2 + \theta_n\Delta + O(\theta_n^2)$, the Berry-Esseen theorem and Theorem 1 on page 159 in Petrov (1975) imply (3.6) and (3.7), respectively. □

To study the envelope power function we introduce the critical function ϕ_n^+ of the size-α_n MP test of H against $\theta = \theta_n$, given by

$$\phi_n^+(\bar{x}_n) = \begin{cases} 1 & > \\ \varepsilon_n & \text{if } \theta_n^{-1}\gamma'_{\theta_n}\bar{x}_n = e_n, \\ 0 & < \end{cases}$$

where the constants e_n and ε_n satisfy $E_{\theta_n}\phi_n^+(\bar{X}_n) = \alpha_n$. By a similar argument as above we obtain

LEMMA 3.4. *If* $\lim_{n\to\infty}\theta_n = 0$

$$n^{1/2}e_n s_n^{-1} = \xi_{1-\alpha_n} + 6^{-1}\Delta_n s_n^{-3} n^{-1/2}\xi_{1-\alpha_n}^2 + O(n^{-1/2} + \xi_{1-\alpha_n}^3 n^{-1}) \quad as \quad n \to \infty; \tag{3.8}$$

if $\lim_{n\to\infty}\theta_n = 0$ *and condition* (C) *holds*

$$n^{1/2}e_n s_n^{-1} = \xi_{1-\alpha_n} + 6^{-1}\Delta_n s_n^{-3} n^{-1/2}(\xi_{1-\alpha_n}^2 - 1) + O(n^{-1}|\log \alpha_n|^{3/2}) \tag{3.9}$$

as $n \to \infty$, *where*

$$s_n^2 = \mathrm{Var}_0\, \theta_n^{-1}\gamma'_{\theta_n}X_1 = \theta_n^{-2}\gamma'_{\theta_n}\Sigma_0\gamma_{\theta_n} \text{ and } \Delta_n = E_0(\theta_n^{-1}\gamma'_{\theta_n}X_1)^3.$$

LEMMA 3.5. *If* $\lim_{n\to\infty}\theta_n = 0$

$$E_{\theta_n}\phi_n^+ = 1 - \Phi(c_n) + O(n^{-1/2}) \quad as \quad n \to \infty; \tag{3.10}$$

if $\lim_{n\to\infty}\theta_n = 0$ *and condition* (C) *holds, then*

$$E_{\theta_n}\phi_n^+ = 1 - \Phi(c_n) - 6^{-1}\rho_n^+ n^{-1/2}(1 - c_n^2)\phi(c_n) + O(n^{-1}) \text{ as } n \to \infty, \tag{3.11}$$

where

$$c_n = n^{1/2}[e_n s_n^{-1} - s\theta_n - \tfrac{1}{2}\theta_n s^{-3}\Delta e_n - \tfrac{1}{2}(\dot{\gamma}_0'\Sigma_0\ddot{\gamma}_0)s^{-1}\theta_n^2 + O(\theta_n^2(\theta_n + e_n))]$$

and

$$\rho_n^+ = (\theta_n^{-2}\gamma'_{\theta_n}\Sigma_{\gamma_{\theta_n}}\gamma_{\theta_n})^{-3/2}E_{\theta_n}\{\theta_n^{-1}\gamma'_{\theta_n}(X_1 - \lambda(\gamma_{\theta_n}))\}^3.$$

In the next lemma the shortcoming is determined for sequences $\{\theta_n\}$ tending to zero. Note that $\theta_n \sim s^{-1}n^{-1/2}\{-2\log\alpha_n\}^{1/2}$ is the "contiguous" case.

LEMMA 3.6. *Assume that condition* (C) *holds and* $\lim_{n\to\infty}\theta_n = 0$.
If $\lim_{n\to\infty} s\theta_n n^{1/2}\{-2\log\alpha_n\}^{-1/2} < 1$ *then* $R_n(\theta_n) = O(n^{-1})$ *as* $n \to \infty$.
If $\lim_{n\to\infty} s\theta_n n^{1/2}\{-2\log\alpha_n\}^{-1/2} = 1$ *then* $R_n(\theta_n) = O(n^{-1}|\log\alpha_n|^{3/2})$ *as* $n \to \infty$.
If $\lim_{n\to\infty} s\theta_n n^{1/2}\{-2\log\alpha_n\}^{-1/2} > 1$ *then* $R_n(\theta_n) = O(n^{-1})$ *as* $n \to \infty$.

PROOF. By lemma 3.3, 3.4 and 3.5 we have $b_n - c_n = O(n^{-1}|\log\alpha_n|^{3/2} + \theta_n^3 n^{1/2})$ and $\rho_n = \rho_n^+ + O(\theta_n)$.

Defining $f(x) = 1 - \Phi(x) - 6^{-1}\rho_n n^{-1/2}(1 - x^2)\phi(x)$ and hence $f'(x) = -\phi(x) - 6^{-1}\rho_n n^{-1/2}(x^3 - 3x)\phi(x)$ it follows by (3.7) and (3.11) that

$$R_n(\theta_n) = E_{\theta_n}\phi_n^+ - E_{\theta_n}\phi_n^L = f(c_n) - f(b_n) + 6^{-1}n^{-1/2}(1 - c_n^2)\phi(c_n)(\rho_n - \rho_n^+) + O(n^{-1}).$$

If $\lim_{n\to\infty} s\theta_n n^{1/2}\{-2\log\alpha_n\}^{-1/2} = 1 - \varepsilon < 1$ and $\alpha_n \to 0$, then $c_n \sim \varepsilon\{-2\log\alpha_n\}^{1/2}$ and $b_n \sim \varepsilon\{-2\log\alpha_n\}^{1/2}$. Hence $6^{-1}n^{-1/2}(1 - c_n^2)\phi(c_n)(\rho_n - \rho_n^+) = o(n^{-1})$ and in view of the mean value theorem $f(c_n) - f(b_n) = (b_n - c_n)f'(\eta_n) = O(n^{-1}|\log\alpha_n|^{3/2}f'(\eta_n)) = O(n^{-1}\eta_n^3 f'(\eta_n)) = o(n^{-1})$, where η_n lies between b_n and c_n. So in this case $R_n(\theta_n) = O(n^{-1})$. Suppose $\theta_n = O(n^{-1/2}|\log\alpha_n|^{1/2})$ then $b_n - c_n = O(n^{-1}|\log\alpha_n|^{3/2})$ and hence $f(c_n) - f(b_n) = O(n^{-1}|\log\alpha_n|^{3/2})$. Moreover, $6^{-1}n^{-1/2}(1 - c_n^2)\phi(c_n)(\rho_n - \rho_n^+) = O(\theta_n n^{-1/2}) = O(n^{-1}|\log\alpha_n|^{3/2})$. This completes the proof of the first and second statement.

If $\lim_{n\to\infty} s\theta_n n^{1/2}\{-2 \log \alpha_n\}^{-1/2} > 1$ then $\lim_{n\to\infty} b_n s^{-1}\theta_n^{-1}n^{-1/2} = \lim_{n\to\infty} c_n s^{-1}\theta_n^{-1}n^{-1/2} < 0$.

Hence $6^{-1}n^{-1/2}(1 - c_n^2)\phi(c_n)(\rho_n - \rho_n^+) = O(n^{-1})$ and in view of the mean value theorem $f(c_n) - f(b_n) = (b_n - c_n)f'(\eta_n) = O(\theta_n^3\, n^{1/2}f'(\eta_n)) = O(n^{-1}\theta_n^3\, n^{3/2}\, gh_n^{-3}\eta_n^3\, f'(\eta_n)) = O(n^{-1})$, where η_n lies between b_n and c_n. This completes the proof of the lemma. □

THEOREM 2.1 is an immediate consequence of lemma 3.6. Similarly one obtains

LEMMA 3.7. *If* $\lim_{n\to\infty} s\theta_n n^{1/2}\{-2 \log \alpha_n\}^{-1/2} < 1$ *then* $R_n(\theta_n) = O(n^{-1/2})$ *as* $n \to \infty$. *If* $\lim_{n\to\infty} s\theta_n n^{1/2}\{-2 \log \alpha_n\}^{-1/2} = 1$ *then* $R_n(\theta_n) = O(n^{-1/2} + n^{-1}|\log \alpha_n|^{3/2})$ *as* $n \to \infty$. *If* $\lim_{n\to\infty} s\theta_n n^{1/2}\{-2 \log \alpha_n\}^{-1/2} > 1$ *and* $\lim_{n\to\infty} \theta_n = 0$ *then* $R_n(\theta_n) = O(n^{-1/2})$ *as* $n \to \infty$.

THEOREM 2.2 is an immediate consequence of lemma 3.7.

REFERENCES

ALBERS, W. (1974). Asymptotic expansions and the deficiency concept in statistics. Mathematical Centre Tracts 58, Amsterdam.

CHIBISOV, D. M. (1973). Asymptotic expansions for some asymptotically optimal tests. *Proc. Prague Symp. Asymptotic Statist.* **2** 37–68.

CRAMÉR, H. (1962). *Random Variables and Probability Distributions.* 2nd Ed. Cambridge Univ.

EFRON, B. (1975). Defining the curvature of a statistical problem. *Ann. Statist.* **3** 1189–1242.

PETROV, V. V. (1975). *Sums of Independent Random Variables.* Springer, New York.

PFANZAGL, J. (1973). Asymptotically optimum estimation and test procedures. *Proc. Prague Symp. Asymptotic Statist.* **1** 201–272.

PFANZAGL, J. (1975). *On Asymptotically Complete Classes.* Statistical Inference **2** 1–43. (M. Puri, ed.) Academic, New York.

SAULIS, L. I. (1969). Asymptotic expansions of probabilities of large deviations. (in Russian). *Litovskiĭ matem. sb.* **9** 605–625.

WISKUNDIG SEMINARIUM
VRIJE UNIVERSITEIT
DE BOELELAAN 1081
1007 MC AMSTERDAM
THE NETHERLANDS