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SOME PROPERTIES OF THE ASYMPTOTIC RELATIVE PITMAN EFFICIENCY

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A general approach to Pitman efficiency as a limit of the ratio of sample sizes is presented. The results can be used especially to derive the Pitman efficiency of tests based on asymptotically χ^2-distributed statistics with different degrees of freedom.

1. Introduction. The concept of asymptotic relative Pitman efficiency (ARPE) is a useful tool for the comparison of test sequences. However, the available techniques generally allow only the investigation of ARPE of tests based on test statistics which under the hypothesis have the same asymptotic distribution, especially both a normal or a χ^2-distribution with the same number of degrees of freedom. Using a very general definition of ARPE, in Section 2 and 3 we give conditions which can be verified in many applications and under which the ARPE can be calculated.

Section 4 contains the case of asymptotically normal or χ^2-distributed test statistics (including the case of different degrees of freedom) as well as some applications.

Throughout the paper, $\{P_\theta, \theta \in \Theta\}$ is a family of probability measures on a space $(\Omega, \mathfrak{A})$ where Θ is a topological space. Furthermore, for $\theta_0 \in \Theta$, $\{\phi_n\}$ is a sequence of level-α-tests ($\alpha > 0$) for $H:\theta = \theta_0$ against $K:\theta \in \Theta - \{\theta_0\} = \Theta'$ (say). In order to avoid complications we also assume that for every $\theta \neq \theta_0$

$$E_\theta(\phi_n) \geq \alpha \tag{1.1.a}$$

$$\lim_{n\to\infty} E_\theta(\phi_n) = 1. \tag{1.1.b}$$

$$\{\theta_0\} \neq C(\theta_0). \tag{1.2}$$

Here, $C(\theta_0)$ denotes the connected component of θ_0.

Usually, ϕ_n is a test based on n observations. Now the question arises how many observations are necessary to achieve a given power $\beta \in]\alpha, 1[$. Thus for $0 < \alpha < \beta < 1$, we define

Definition 1. A function $N:\Theta' \to \mathbb{N}$ is called a Pitman efficiency function for β (β-PEF), if

$$E_\theta(\phi_{N(\theta)}) \geq \beta \tag{1.3.a}$$

$$E_\theta(\phi_{N(\theta)-1}) < \beta \tag{1.3.b}$$

where $\phi_0 \equiv \alpha$.

Further, let

$$\underline{N_\beta}(\theta) = \inf\{n \in \mathbb{N}: E_\theta(\phi_n) \geq \beta\} \tag{1.4.a}$$

$$\overline{N_\beta}(\theta) = \inf\{n \in \mathbb{N}: E_\theta(\phi_m) \geq \beta \quad \text{for all} \quad m \geq n\}. \tag{1.4.b}$$

Remark. Clearly, $\underline{N_\beta}$, resp. $\overline{N_\beta}$, are the smallest, resp. the largest β-PEF. The existence

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of β-PEF's is guaranteed by (1.1). If $\{E_\theta(\phi_n)\}$ is an increasing sequence, the β-PEF is uniquely determined, but this property is frequently difficult to verify.

For our definition of Pitman efficiency, we modify the concept of Wieand (1976), using the notation Π for the set of all sequences $\{\theta_n\}$ satisfying $\theta_n \in \Theta'$, $\theta_n \to \theta_0$:

DEFINITION 2. Let $\{\phi_n^{(i)}\}_{n\in\mathbb{N}}$, $i = 1, 2$ be two sequences of level-α-tests with β-PEF $\underline{N_\beta}^{(i)}$, $\overline{N_\beta}^{(i)}$, respectively. Then

(1.5.a) $$e_{12}^- = \inf_\Pi \liminf_{n\to\infty} \frac{\underline{N_\beta}^{(2)}(\theta_n)}{\overline{N_\beta}^{(1)}(\theta_n)}$$

resp.

(1.5.b) $$e_{12}^+ = \sup_\Pi \limsup_{n\to\infty} \frac{\overline{N_\beta}^{(2)}(\theta_n)}{\underline{N_\beta}^{(1)}(\theta_n)}$$

are the lower (resp. upper) ARPE.

If $e_{12}^- = e_{12}^+ = e_{12}$ (say) then e_{12} is the ARPE of $\{\phi_n^{(1)}\}$ w.r.t. $\{\phi_n^{(2)}\}$.

Simple calculations show that under the conditions A, B, and C given below our definition of ARPE coincides with several somewhat different versions (e.g., those of Noether (1955), Fraser (1957), Olshen (1967), and Wieand (1976)).

2. Limiting behavior of efficiency functions. In this section we assume that the following condition is satisfied:

CONDITION A. There are functions $g{:}\Theta \to [0, \infty[$ and $H{:}[0, \infty[\to [\alpha, 1[$ such that

(2.1.a) g is continuous, and $g(\theta) = 0$ iff $\theta = \theta_0$

(2.1.b) H is strictly increasing and bijective

(2.1.c) For sequences $\{\theta_n\}$ in Θ satisfying $g(\theta_n)n \to \eta \geq 0$ as $n \to \infty$, we have $\lim_{n\to\infty} E_{\theta_n}(\phi_n) = H(\eta)$.

REMARKS. 1. By (2.1.b), H is continuous, $H(0) = \alpha$ and $\lim_{t\to\infty} H(t) = 1$

2. By (2.1.a) and (1.2), there is a $b > 0$ such that $[0, b] \subset \{g(\theta), \theta \in \Theta\}$.

3. Although Condition A is satisfied in many cases, its verification can become very tedious, generally uniformity or contiguity arguments are needed (cf., Section 4).

An easy consequence is

LEMMA 1. *For $k_n \in \mathbb{N}$, $k_n \to \infty$, $g(\theta_n)k_n \to \eta$, we have $E_{\theta_n}(\phi_{k_n}) \to H(\eta)$.*

PROOF. (a) If $\{k_n\}$ is strictly increasing, there is a sequence $\{\theta_m^*\}$ such that for $m > \eta/b$ (by remark 2 above)

(2.2.a) $$\theta_m^* = \theta_n \quad \text{if} \quad m = k_n$$

(2.2.b) $$mg(\theta_m^*) = \eta, \quad \text{otherwise.}$$

Then $g(\theta_m^*)m \to \eta$ and $E_{\theta_n}(\phi_{k_n})$ is a subsequence of $E_{\theta_m}(\phi_m)$ which tends to $H(\eta)$ by Condition A.

(b) If $\{k_n\}$ is not strictly increasing, each subsequence contains an increasing subsequence, hence each subsequence of $\{E_{\theta_n}(\phi_{k_n})\}$ contains a subsequence with limit $H(\eta)$ by part (a) of the proof and the result follows. □

The idea of the concept is to show that under simple conditions, for every sequence

$\{\theta_n\} \in \Pi$, $k_n = N(\theta_n)$ satisfies the conditions of the lemma with $\eta = H^{-1}(\beta)$. The conditions on $\{\theta_n\}$ we require are given by

DEFINITION 3. $\{\theta_n\} \in \Pi$ is called an essential sequence (ES) for the β-PEF N, if

$$N(\theta_n) \to \infty \tag{2.3.a}$$

$$\limsup_{n\to\infty} g(\theta_n)N(\theta_n) < \infty. \tag{2.3.b}$$

Then we have

THEOREM 1. *Let* $\{\theta_n\} \in \Pi$ *be an* ES *for the* β-PEF N, *then*

$$\lim_{n\to\infty} g(\theta_n)N(\theta_n) = H^{-1}(\beta). \tag{2.4}$$

PROOF. (a) Let $\{\theta_n'\}$ be a subsequence of $\{\theta_n\}$ such that $g(\theta_n')N(\theta_n') \to \eta^*$ (say). Then, by (2.3.a), $N(\theta_n') \to \infty$ and consequently by Lemma 1 $\beta \geq E_{\theta_n'}(\phi_{N(\theta_n')}) \to H(\eta^*)$ as well as $\beta \leq E_{\theta_n'}(\phi_{N(\theta_n')-1}) \to H(\eta^*)$, since $g(\theta_n')(N(\theta_n') - 1) \to \eta^*$. Hence $\beta = H(\eta^*)$ and $\eta^* = H^{-1}(\beta)$.

(b) By (2.1.b) each subsequence of $\{g(\theta_n)N(\theta_n)\}$ contains a convergent subsequence that must have the limit $H^{-1}(\beta)$ by part (a) of the proof. Thus the assertion follows. □

3. Essential sequences. Considering the definition of ARPE, it is useful to find conditions under which a sequence $\{\theta_n\} \in \Pi$ is essential for $\underline{N}_\beta$ and $\overline{N}_\beta$.

The goal of this section is to show that for every $\beta \in]\alpha, 1[$ each sequence of Π is an ES for $\underline{N}_\beta$ as well as for $\overline{N}_\beta$ (and hence for all PEF's), if the following two conditions are satisfied:

CONDITION B. For every $n \in \mathbb{N}$, the function $\psi_n:\theta \to E_\theta(\phi_n)$ is continuous at $\theta = 0$.

CONDITION C. For every sequence $\{\theta_n\} \in \Pi$ such that $g(\theta_n)n \to \infty$, we have $E_{\theta_n}(\phi_n) \to 1$.

REMARK. Note that C is an extension of A to the case $\eta = \infty$. A generalization similar to Lemma 1 is possible and will be used in the proof of Theorem 2.

However, yet we only assume A to be true. Then we have

LEMMA 2. *For every* $\beta \in]\alpha, 1[$ *and every sequence* $\{\theta_n\} \in \Pi$,
(a) (2.3.a) *holds for* $N = \overline{N}_\beta$
(b) (2.3.b) *holds for* $N = \underline{N}_\beta$.

PROOF. (a) For $\beta \in]\alpha, 1[$ and $\{\theta_n\} \in \Pi$, define $d = (\beta + \alpha)/2$, $\eta = H^{-1}(d)$ and $k_n = [\eta/g(\theta_n)]$ (where $[x] = \sup\{z \in \mathbb{Z}: z \leq x\}$). Then $k_n \to \infty$; $k_n g(\theta_n) \to \eta$ and thus $E_{\theta_n}(\phi_{k_n}) \to H(\eta) = d < \beta$. Hence, $k_n < \overline{N}_\beta(\theta_n)$ for sufficiently large n and the assertion follows.

(b) Define $d' = (\beta + 1)/2$, $\eta' = H^{-1}(d')$, $k_n = [\eta'/g(\theta_n)]$. Then by similar arguments $k_n \leq \overline{N}_\beta(\theta_n)$ and

$$\limsup g(\theta_n)\underline{N}_\beta(\theta_n) \leq \limsup g(\theta_n)k_n \leq \eta' < \infty. \qquad \square$$

Now we can show

THEOREM 2. *Assume Condition* A *is satisfied. Then*
(a) *Every sequence* $\{\theta_n\} \in \Pi$ *is an* ES *for* $\underline{N}_\beta$ *for all* $\beta \in]\alpha, 1[$ *if and only if Condition*

B *is satisfied.*

(b) *Every sequence* $\{\theta_n\} \in \Pi$ *is an* ES *for* $\overline{N_\beta}$ *for all* $\beta \in\,]\alpha, 1[$ *if and only if Condition* C *is satisfied.*

As a direct consequence of the theorem 1 and 2, we get

COROLLARY. *Under Conditions* A, B, *and* C, *for every sequence* $\{\theta_n\} \in \Pi$, *every* $\beta \in\,]\alpha, 1[$ *and every* β-PEF N, (2.4) *is satisfied.*

PROOF OF THEOREM 2. (a) Assume $0 < \alpha < \beta < 1$. Under B, for every $m \in \mathbb{N}$ there is a $\delta_m > 0$ such that for $|\theta| \leq \delta_m$ and for all $n \leq m$ we have $E_\theta(\phi_n) < \beta$. Thus $\underline{N_\beta}(\theta) > m$ for $|\theta| < \delta_m$. Hence every sequence $\{\theta_n\} \in \Pi$ satisfies (2.3.a) as well as (2.3.b) by L2(a). If B does not hold, there exists $k \in \mathbb{N}$, $\beta \in\,]\alpha, 1[$ and a sequence $\{\theta_n\} \in \Pi$ s.th. $E_{\theta_n}(\phi_k) > \beta$ for all $n \in \mathbb{N}$. Hence $\underline{N_\beta}(\theta_n) < k$ and (2.3.a) is not satisfied.

(b) Assume there exists $\beta \in\,]\alpha, 1[$ and $\{\theta_n\} \in \Pi$ such that $\{\theta_n\}$ is not an ES for $\overline{N_\beta}$. Then, by L2(b), w.l.o.g., $g(\theta_n)\overline{N_\beta}(\theta_n) \to \infty$ (and consequently $g(\theta_n)(\overline{N_\beta}(\theta_n) - 1) \to \infty$) can be assumed. Then, by C and the subsequent remarks, $E_{\theta_n}(\phi_{\overline{N_\beta}(\theta_n)-1}) \to 1$.

But this is a contradiction to $E_{\theta_n}(\phi_{\overline{N_\beta}(\theta_n)-1}) < \beta$. On the other hand, if C does not hold, there exists a $\delta > 0$ and a sequence $\{\theta_n\} \in \Pi$ such that $ng(\theta_n) \to \infty$ as well as $E_{\theta_n}(\phi_n) < 1 - \delta$. *Then, for* $\beta = 1 - \delta/2$, $\overline{N_\beta}(\theta_n) > n$ and $\overline{N_\beta}(\theta_n)g(\theta_n) \to \infty$, which is a contradiction to (2.3.b). □

Hence we obtain as a general result of the preceding arguments,

THEOREM 3. *Let* $\{\phi_n^{(i)}\}$, $i = 1, 2$ *be level-α-test sequences satisfying Conditions* A, B, *and* C *with function* g_i, H_i, *respectively. Further let*

$$g_{12}^{-} = \inf_\Pi \liminf_{n\to\infty} g_1(\theta_n)/g_2(\theta_n) \tag{3.1}$$

and define g_{12}^{+} *similarly* (cf., (1.5.b)). *Then*

$$e_{12}^{-}(\beta) = g_{12}^{-} H_2^{-1}(\beta)/H_1^{-1}(\beta) \tag{3.2.a}$$

$$e_{12}^{+}(\beta) = g_{12}^{+} H_2^{-1}(\beta)/H_1^{-1}(\beta). \tag{3.2.b}$$

PROOF. For $\{\theta_n\} \in \Pi$ and every β-PEF N, we have

$$\inf_\Pi \liminf_{n\to\infty} \frac{N^{(2)}(\theta_n)}{N^{(1)}(\theta_n)} = \lim_{n\to\infty} \frac{N^{(2)}(\theta_n)g_2(\theta_n)}{N^{(1)}(\theta_n)g_1(\theta_n)} \cdot \inf_\Pi \liminf \frac{g_1(\theta_n)}{g_2(\theta_n)}. \tag{3.3}$$ □

REMARK. Clearly, $g_{12}^{+} = 1/g_{21}^{-}$. If $g_{12}^{-} = g_{12}^{+}$, ARPE exists by Theorem 3, but generally depends on β.

4. Verification of condition A. By the arguments in the preceding section the calculation of ARPE mainly reduces to the problem of verifying Condition A and hence find suitable functions g and H. In this section we assume that ϕ_n is an upper level-α-test w.r.t. a test statistic T_n, i.e.,

$$\phi_n = \begin{cases} 1 & \text{if } T_n > t_n \\ \gamma_n & \phantom{\text{if }} = \\ 0 & \phantom{\text{if }} < \end{cases} \tag{4.1}$$

where t_n and γ_n are constants such that $E_{\theta_0}(\phi_n) = \alpha$. We shall consider the shape of H^{-1} if the distribution of T_n has one of the following asymptotic properties:

A_0. There is an $u > 0$ such that $g(\theta_n)n \to \eta$ implies $\mathscr{D}_{\theta_n}(T_n) \to N(\eta^u, 1)$ for every $\eta \geq 0$ and, for $K \in \mathbb{N}$.

A_K. There is an $u > 0$ such that $g(\theta_n)n \to \eta$ implies $\mathscr{D}_{\theta_n}(T_n) \to \chi^2(K, \eta^{2u})$, where $\chi^2(K, \delta^2)$ is a χ^2-distribution with K degrees of freedom and noncentrality parameter δ^2.

Then we obtain

THEOREM 4. *Assume $\{\phi_n\}$ is based on $\{T_n\}$ by (4.1). Then, for $0 < \alpha < \beta < 1$, we have: If T_n satisfies A_K for $K \geq 0$, Condition* A *holds with*

$$H^{-1}(\beta) = d^{1/u}(\alpha, \beta, K). \tag{4.2}$$

Here $d(\alpha, \beta, 0) = \Phi^{-1}(\beta) - \Phi^{-1}(\alpha)$, where Φ is the distribution function of the standard normal distribution, and, for $K \geq 1$, $d^2 = d^2(\alpha, \beta, K)$ is the (uniquely determined) noncentrality parameter s.th. *the β-fractile of $\chi^2(K, d^2)$ and the α-fractile of $\chi^2(K, 0)$ coincide.*

PROOF. For $K = 0$, $H(t) = 1 - \Phi(\Phi^{-1}(1 - \alpha) - t^u)$. But $H(t) = \beta$ iff $t = (\Phi^{-1}(\beta) - \Phi^{-1}(\alpha))^{1/u}$.

For $K > 0$, the assertion follows similarly using $H(t) = 1 - F_{\chi^2(K,t^{2u})}(F^{-1}_{\chi^2(K,0)}(1 - \alpha))$. Here F_μ denotes the distribution function of the distribution μ. □

REMARKS. 1. Assume that for $i = 1, 2$; $\{\phi_n^{(i)}\}$ are based on $\{T_n^i\}$ by (4.1) and let Conditions A_{K_i}, B, C be satisfied with functions g_i and constant K_i, u_i, respectively. For $g_{\overline{12}}$ as in (3.1), we have:

(a) If $K_1 = K_2$ and $u_1 = u_2$, then $e_{\overline{12}} = g_{\overline{12}}$ independent from α and β.

(b) If $K_1 \neq K_2$ or $u_1 \neq u_2$, we have

$$e_{\overline{12}}(\alpha, \beta) = g_{\overline{12}} d^{1/u_2}(\alpha, \beta, K_2)/d^{1/u_1}(\alpha, \beta, K_1), \tag{4.3}$$

which depends on α and β.

2. For $K > 0$, $d^2(\alpha, \beta, K)$ has been tabulated by Haynam et al. (1962) (cf. also Harter and Owen (1970)).

3. Often $\lim_{\theta\to\theta_0} g(\theta)/c(\theta) = 1$, where $c(\theta)$ is the Bahadur slope of the test statistic $\{T_n\}$ (cf., Bahadur (1960)). In these cases the Pitman efficiency factors into the product of the local Bahadur efficiency and a function of α and β which reflects only the analytic structure of the test statistic's limiting behavior.

4. Pitman's conditions in the modified version of Olshen (1967) imply our Condition A_0 with $u = \frac{1}{2}$, $g(\theta) = c^2(\theta - \theta_0)^2$. An analogous modification of the extensions due to Noether (1955) resp. Hannan (1956) lead to Condition A_0, resp. A_p, with $u = m\delta$, $g(\theta) = (\xi/m!)^{1/\delta m}(\theta - \theta_0)^{1/\delta}$, where $\xi = c$, or $\xi = (c'\Lambda^{-1}(\theta_0)c)^{1/2}$, in the notation of the respective authors.

5. Often contiguity arguments lead to Condition A. As an illustration, consider the rank statistic Q for the k-sample problem as defined by Hajek and Sidak (1967) in (VI. 3.1.2). In Chapter VI they show that suitable assumptions on the underlying model lead to (VI. 4.3.2), which in the case $n_j/n \to \lambda_j > 0$ for $1 \leq j \leq k$ is equivalent to our Condition A_{k-1} with $\Theta = \mathbb{R}^k$, $u = \frac{1}{2}$ and $g(\Delta) = \rho_1^2 I(f) \sum_{j=1}^k \lambda_j(\Delta_j - \bar{\Delta})^2$.

We close with two numerical examples:

1. Assume $T_n = (1/\sqrt{n}) \sum_{i=1}^n X_i$, where $X_i \sim N(\theta, 1)$ independent. For $H{:}\theta = 0$ against K: $\theta > 0$ we use $\{\phi_n^{(1)}\}$ based on $\{T_n\}$ by (4.1). How many observations are "lost", if we use the two-sided test although the problem is one-sided, i.e., if we use $\{\phi_n^{(2)}\}$ based on $\{T_n^2\}$? $\{T_n\}$ satisfies A_0, $\{T_n^2\}$ satisfies A_1, both with $g(\theta) = \theta$, $u = \frac{1}{2}$; Conditions B and C can be verified easily. Hence we have

$$e_{12}(\alpha, \beta) = \frac{d^2(\alpha, \beta, 1)}{(\Phi^{-1}(\alpha) - \Phi^{-1}(\beta))^2}.$$

Some values of this function are given in Table 1.

2. In a recent paper, Schach (1979), using the concept of Bahadur efficiency, compares a test proposed by Anderson (1959) with the method of n rankings using the optimal scores (cf., e.g., Puri and Sen (1971), Section 7). It can be shown that Conditions A_K, B and C with different K but the same g and u are satisfied for the two tests. Details are omitted and can be found in Rothe (1978). Hence the ARPE turns out to be

$$e_{\text{Anderson, opt. } n\text{-ranking}}(\alpha, \beta) = \frac{d^2(\alpha, \beta, p-1)}{d^2(\alpha, \beta, (p-1)^2)} \tag{4.4}$$

where p is the number of treatments in each block. Some values are given in Table 2, the values of $d^2(\alpha, \beta, K)$ have been taken from Harter and Owen (1970).

TABLE 1
ARPE *of two-sided against one-sided Gauss-test for one-sided alternatives.*

β \ α	0.1	0.050	0.010	0.005	0.001
0.2	0.332	0.519	0.732	0.778	0.842
0.4	0.548	0.665	0.795	0.826	0.871
0.6	0.655	0.736	0.831	0.855	0.890
0.7	0.693	0.762	0.845	0.866	0.897
0.8	0.736	0.788	0.859	0.878	0.906
0.9	0.768	0.815	0.873	0.890	0.914
0.99	0.825	0.858	0.901	0.912	0.930

TABLE 2
ARPE *of Anderson-test against method of* n*-rankings with optimal scores.*

α \ β	0.3	0.5	0.7	0.9	
0.1	0.727	0.758	0.780	0.812	
0.05	0.743	0.772	0.795	0.821	
0.01	0.777	0.800	0.819	0.840	p = 3
0.005	0.790	0.810	0.827	0.847	
0.001	0.812	0.829	0.844	0.860	
0.1	0.523	0.560	0.591	0.627	
0.05	0.541	0.575	0.604	0.639	
0.01	0.576	0.606	0.631	0.661	0 = 5
0.005	0.589	0.617	0.640	0.669	
0.001	0.614	0.639	0.660	0.685	
0.1	0.428	0.461	0.489	0.524	
0.05	0.443	0.474	0.501	0.535	
0.01	0.473	0.501	0.525	0.555	p = 7
0.005	0.484	0.510	0.533	0.562	
0.001	0.506	0.530	0.551	0.578	
0.1	0.330	0.356	0.379	0.408	
0.05	0.341	0.366	0.388	0.417	
0.01	0.363	0.386	0.406	0.433	p = 11
0.005	0.371	0.393	0.413	0.439	
0.001	0.388	0.408	0.427	0.451	

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