The Annals of Statistics
1981, Vol. 9, No. 2, 457–460

EFFICIENT NONPARAMETRIC ESTIMATORS OF PARAMETERS IN THE GENERAL LINEAR HYPOTHESIS[1]

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Asymptotically universally efficient estimators of the parameters in the general linear hypothesis are proposed. These estimators are based on ranks of residuals (or their absolute values); they are analogous to the linearized rank estimators proposed by Kraft and van Eeden in the sense that they are obtained by replacing, in these estimators, the function generating the scores by certain estimators of this function. Finally, it is shown that estimators of the score function, satisfying the conditions used, exist.

1. Introduction. In the one- and two-sample location models, among the authors who constructed uniformly asymptotically efficient estimators, are van Eeden (1970), Beran (1974) and Beran (1978) as the most recent reference concerning the problem; these estimators are asymptotically efficient in the sense that their asymptotic variance attains the Cramér-Rao lower bound for the distribution of the observations. Bhattacharya (1967) and Weiss and Wolfowitz (1970) and (1971) also obtained estimators which are nearly asymptotically efficient in the above sense.

In this paper, asymptotically universally efficient estimators of the parameters in the general linear hypothesis are proposed. These estimators are based on ranks of residuals; they are analogous to the linearized rank estimators proposed by Kraft and van Eeden (1972) in the sense that they are obtained by replacing certain estimates of the function $\phi(u, f), = -f'(F^{-1}(u))/f(F^{-1}(u))$ by certain estimators of this function.

The results are stated in Section 2 and an outline of the proofs is given in Section 3; in Section 4 it is shown that there exist estimators of $\phi(u, f)$ satisfying the conditions used in Section 2.

2. Linearized estimators based on ranks of residuals. Let $\mathscr{F}$ be the set of all distribution functions F with the following properties: F has an absolutely continuous density f and $\phi(u, f) = \phi_1(u) - \phi_2(u)$ where, for $s = 1, 2$, $\phi_s(u)$ is nondecreasing in u,

$$0 < u < 1, \quad \int_0^1 \phi_s(u)\, du = 0, \quad \text{and} \quad \int_0^1 \phi_s^2(u)\, du < \infty.$$

For each $\nu = 1, 2, \cdots$, let $Z_\nu = (Z_{\nu 1}, \cdots, Z_{\nu N_\nu})'$ be an $N_\nu \times 1$ vector of observations with $N_\nu \to \infty$ as $\nu \to \infty$, let X_ν be an $N_\nu \times p$ design matrix of known constants of rank p with $p \geq 1$, and let Δ be a $p \times 1$ vector of unknown parameters. The design matrix X_ν satisfies $\sum_{i=1}^{N_\nu} X_{ij} = 0$ for each $j = 1, \cdots, p$. The components of $Z_\nu - X_\nu \Delta$ are independently and identically distributed according to $F \in \mathscr{F}$. p will be fixed and limits will be as $\nu \to \infty$. The subscripts ν will be omitted wherever possible.

It is assumed that, for some integer $k \geq 1$, there exist k different rows $L_{\nu 1}, \cdots, L_{\nu k}$ in X, where, for $r = 1, \cdots, k$, $L_{\nu r}$ is M_r times repeated and $\min_{1 \leq r \leq k} M_r \to \infty$. Let for $r = 1, \cdots,$

Received August, 1976; revised December, 1979.

[1] This paper was written with the partial support of the National Research Council of Canada, Grant A-3986.

[2] The author wants to thank the referee for pointing out Weiss and Wolfowitz's (1971) article.

AMS 1970 *subject classifications.* Primary 62G05; secondary 62G20.

Keywords and phrases. Asymptotic efficiency, linearized estimators, Jurečková's Theorem.

k, N_r be a sequence of integers such that $N_r \le M_r$ and

(2.1) $$\min_{1\le r\le k} N_r \to \infty.$$

For $r = 1, \cdots, k$, the first N_r out of the M_r observations corresponding to the row that is M_r times repeated (and which are identically distributed as $F(z - \alpha_r)$ with $F \in \mathscr{F}$) will be denoted by $H_r = (H_{r1}, \cdots, H_{rN_r})'$. The vector $H = (H_1', \cdots, H_k')'$ will be used to estimate $\phi(u, f)$. Let $N^* = N - \sum_{r=1}^k N_r$ and suppose

(2.2) $$N^*/N \to 1.$$

For $\nu = 1, 2, \cdots$, let $\phi^*(u, H)$ be a sequence of estimators of $\phi(u, f)$ based on H. It will be supposed that the following conditions are satisfied:

A1. For each ν, $\phi^*(u, H)$ is constant on the N^* intervals $[(i-1)/N^*, i/N^*)$, $i = 1, \cdots, N^*$, almost surely;

A2. For each ν, $\phi^*(u, H) = \phi_1^*(u, H) - \phi_2^*(u, H)$, $0 \le u \le 1$, almost surely;

A3. For each ν and $s = 1, 2$, $\phi_s^*(u, H)$ and $\phi^*(u, H)$ are constant on the same intervals almost surely;

A4. For each ν and $s = 1, 2$, $\phi_s^*(u, H_1 - a_1, \cdots, H_k - a_k) = \phi_s^*(u, H)$ almost surely with $a_r = (a^{(r)}, \cdots, a^{(r)})' \in R^{N_r}$;

A5. $\phi_s^*(u, H)$ is consistent for $\phi_s(u)$, $s = 1, 2$, in the sense that for each $\epsilon > 0$ and $F \in \mathscr{F}$, $P_\nu(\int_0^1 (\phi_s^*(u, H) - \phi_s(u))^2\, du > \epsilon) \to 0$ as $\nu \to \infty$;

A6. For each ν and $s = 1, 2$, $\phi_s^*(u, H)$ is nondecreasing in u, $0 \le u \le 1$, almost surely;

A7. For each ν and $s = 1, 2$, $\int_0^1 \phi_s^*(u, H)\, du = 0$ almost surely.

REMARK The functions $\phi_s^*(u, H)$ satisfying A3–A7 do not need to be estimators of $\phi_s(u)$; they can depend upon F. The function $\phi^*(u, H)$, used below to construct an estimator of Δ, must be an estimator.

Let $Z^* = (Z_1^*, \cdots, Z_{N^*}^*)'$ be the vector of observations which are not used to estimate $\phi(u, f)$. It is assumed that, for the design matrix x corresponding to Z^*, $x^* = x - \bar{x}$ satisfies Kraft and van Eeden's condition (1972, page 44, Assumption B).

For $w = 1, \cdots, p$ define

(2.3) $$\hat{S}_{\xi_{\nu w}} = \sum_{i=1}^{N^*} x_{iw}^* \phi^*((R_{Z_i^* - \sum_{d=1}^p x_{id}^* \xi_{\nu d}})/(N^* + 1), H),$$

where $\xi_\nu = \xi/N^{*1/2}$, ξ is a $p \times 1$ vector and $R_{Z_i^* - \sum_{d=1}^p x_{id}^* \xi_{\nu d}} = R_{(Z^* - x^*\xi_\nu)_i}$ is the rank of the ith component of $Z^* - x^*\xi_\nu$ among all N^* components. Let $S_{\xi_\nu} = \hat{S}_{\xi_\nu 1}, \cdots \hat{S}_{\xi_\nu p})'$. Let

(2.4) $$\begin{aligned} \hat{\Delta} = \hat{\Delta}(Z) &= \hat{\Delta}_1 + (x^{*\prime}x^*)^{-1}\hat{K}^{-1}\hat{S}_{\hat{\Delta}_1} && \text{if } \hat{K} > 0 \\ &= \hat{\Delta}_1 && \text{if } \hat{K} = 0, \end{aligned}$$

be a linearized estimator of Δ, analogous to Kraft and van Eeden's (1972) estimator. Here, $\hat{K} = \int_0^1 \phi^{*2}(u, H)\, du$, and $\hat{\Delta}_1$ is an estimator of Δ based on Z and satisfying

(2.5) $$\begin{cases} \text{(a)} \ P_{\nu,\Delta}(N^{1/2}(\hat{\Delta}_1 - \Delta) \in A) \to P(A) \text{ for some fixed } p\text{-dimensional distribution } P, \\ \text{(b)} \ \hat{\Delta}_1(Z - Xa) = \hat{\Delta}_1(Z) - a \quad \text{for all } a \in R^p. \end{cases}$$

THEOREM 2.1. *If the conditions* (2.1) *and* (2.2) *are satisfied, if* $F \in \mathscr{F}$, *if* x^* *satisfies Kraft and van Eeden's* (1972) *Assumption* B, *if* $\phi^*(u, H)$ *is an estimator of* $\phi(u, f)$ *such that* A1–A7 *are satisfied, if there exists a* $\hat{\Delta}_1$ *satisfying* (2.5), *then, for* $\hat{\Delta}$ *defined by* (2.4), $N^{1/2}(\hat{\Delta} - \Delta)$ *has asymptotically a normal distribution with mean zero and covariance matrix* $\sum^{-1}K^{-1}$ *where* $K = \int_0^1 \phi^2(u, f)\, du$.

Analogous results hold for estimators of Δ based on the ranks of the absolute values of the residuals (see Dionne (1976)).

3. Outline of the proof of Theorem 2.1. For $w = 1, \cdots, p$ and $s = 1, 2$ define

$$S_{\xi_\nu ws} = \sum_{i=1}^{N^*} x_{iw}^* \phi_s((R_{Z_i^\cdot - \sum_{d=1}^p x_{id}^\cdot \xi_{\nu d}})/(N^* + 1)),$$

where ϕ_1 and ϕ_2 are the two parts of $\phi(u, f)$ and define

$$\hat{S}_{\xi_\nu ws} = \sum_{i=1}^{N^*} x_{iw}^* \phi_s^*((R_{Z_i^\cdot - \sum_{d=1}^p x_{id}^\cdot \xi_{\nu d}})/(N^* + 1), H),$$

where ϕ_1^* and ϕ_2^* are the two parts of $\phi^*(u, H)$. Denote

$$\hat{K}_s = \int_0^1 \phi_s^*(u, H)\phi^*(u, H)\, du, \qquad K_s = \int_0^1 \phi_s(u)\phi(u, f)\, du.$$

LEMMA 3.1. *If $F \in \mathscr{F}$, if the conditions* (2.1) *and* (2.2) *are satisfied, if $\phi^*(u, H)$ is such that* A1–A7 *are satisfied, if x^* satisfies Kraft and van Eeden's* (1972) *Assumption* B, *and if $\hat{S}_{\xi_\nu w}$ is defined by* (2.3) *then for each $w = 1, \cdots, p$,*

$$\lim_{\nu\to\infty} P_{\nu,0}\{\sup_{\|\xi\|\le C} N^{*-1/2} |\hat{S}_{\xi_\nu w} - \hat{S}_{0w} + \hat{K} N^{*-1/2} \textstyle\sum_{d=1}^p \xi_d \sum_{i=1}^{N^*} x_{iw}^* x_{id}^*| > \epsilon\} = 0$$

for each $C > 0$ and each $\epsilon > 0$, with $P_{\nu,0}$ corresponding to $p_{\nu,0}$ the joint density of $Z_1, \cdots, Z_n$ under the hypothesis $\Delta = 0$.

PROOF. It is sufficient to prove that for $s = 1, 2$ with $w = 1, \cdots, p$

$$\lim_{\nu\to\infty} P_{\nu,0}\{\sup_{\|\xi\|\le C} N^{*-1/2} |\hat{S}_{\xi_\nu ws} - \hat{S}_{0ws} + \hat{K}_s N^{*-1/2} \textstyle\sum_{d=1}^p \xi_d \sum_{i=1}^{N^*} x_{iw}^* x_{id}^*| > \epsilon\} = 0.$$

This follows (see Dionne (1976)) from Kraft and van Eeden's (1972) extension of Theorem 3.1 of Jurečková (1969), from the fact that

$$N^{*-1/2} |\hat{S}_{0ws} - S_{0ws}| \to_{P_{\nu,0}} 0,\ N^{*-1/2} |\hat{S}_{\xi_\nu ws} - S_{\xi_\nu ws}| \to_{P_{\nu,0}} 0, \qquad |K_s - \hat{K}_s| \to_{P_{\nu,0}} 0,$$

and from an argument analogous to the one used by Kraft and van Eeden (1972, page 54). □

PROOF OF THEOREM 2.1. Since $N^*/N \to 1$ when $\nu \to \infty$, it is sufficient to prove the result with N replaced by N^*. From A4 and (2.5.b) it follows that $\hat{\Delta}(Z - X\Delta) = \hat{\Delta}(Z) - \Delta$; so it can be supposed that $\Delta = 0$ in the study of the distribution of $\hat{\Delta}$. From the fact that $P_{\nu,0}(\hat{K} = 0) \to 0$ (see van Eeden (1970)), it follows that the asymptotic distribution of $N^{*1/2}\hat{\Delta}$ does not depend on the definition of $\hat{\Delta}$ when $\hat{K} = 0$.

It can be shown, by using Lemma 3.1., that for each $\epsilon > 0$, $\lim_{\nu\to\infty} P_{\nu,0}\{\| N^{*1/2}(\hat{K}\hat{\Delta} - (x^{*\prime}x^*)^{-1}\hat{S}_0 \| > \epsilon\} = 0$. (For the details, see Dionne (1976)). This result implies that $N^{*1/2}\hat{K}\hat{\Delta}$ and $N^{*1/2}(x^{*\prime}x^*)^{-1}\hat{S}_0$ have the same asymptotic distribution. That $N^{*1/2}(x^{*\prime}x^*)^{-1}\hat{S}_0$ and $N^{*1/2}(x^{*\prime}x^*)^{-1}S_0$ have the same asymptotic distribution follows from Kraft and van Eeden's (1972) Assumption B(ii) and from the fact that $N^{*-1/2} |\hat{S}_{0ws} - S_{0ws}| \to_{P_{\nu,0}} 0$, where $S_{\xi_\nu} = (S_{\xi_\nu 1}, \cdots, S_{\xi_\nu p})'$ with $S_{\xi_\nu w} = S_{\xi_\nu w1} - S_{\xi_\nu w2}$ $w = 1, \cdots, p$; from Hájek and Šidák (1967, page 163), (see also Kraft and van Eeden (1972), Theorem 2.1.), the asymptotic distribution of $N^{*1/2}(x^{*\prime}x^*)^{-1}S_0$ is $N(0, \sum^{-1}K)$. That $N^{*1/2}\hat{\Delta}$ is asymptotically $N(0, \sum^{-1}K^{-1})$ then follows from the fact that $|K_s - \hat{K}_s| \to_{P_{\nu,0}} 0$. □

4. Existence of a class of estimators of $\phi(u, f)$ satisfying the conditions A1–A7. In this section, a class of estimators of $\phi(u, f)$ based on k samples, and satisfying A1–A7 is given. Again the subscripts ν will be omitted wherever possible.

For one sample $Z = (Z_1, \cdots, Z_L)$ let Φ be the set of sequences of estimators $\hat{\phi}(u, Z)$ of $\phi(u, f)$ satisfying the following properties:

1. For each ν, $\hat{\phi}(u, Z)$ is constant almost surely on each of a finite number Q of intervals $I_j = [J_j, J_{j+1})$ of length l_j with $\min_{1\le j\le Q} l_j \ge 1/L$, and $Q/L \to 0$ when $\nu \to \infty$;

2. For each ν, $\hat{\phi}(u, Z - a) = \hat{\phi}(u, Z)$ almost surely with $a = (a, \cdots, a)' \in R^L$;

3. $\hat{\phi}(u, Z)$ is a consistent estimator of $\phi(u, f)$ in the sense that for each $\epsilon > 0$ and $F \in \mathscr{F}$

$$P_{\nu,0}\left(\int_0^1 (\hat{\phi}(u, Z) - \phi(u, f))^2 \, du > \epsilon\right) \rightarrow 0 \qquad \text{as} \quad \nu \rightarrow \infty.$$

The class Φ is not empty: the class of estimators of $\phi(u, f)$ proposed by Hájek and Šidák (1967, page 260) is included in Φ. Let M be an integer such that $M \geq L$ and let

$$\hat{\phi}'(u, Z) = \hat{\phi}(l/(M+1), Z) \qquad \text{for} \quad (l-1)/M \leq u < l/M, \tag{4.1}$$

$l = 1, \cdots, M$ and $\hat{\phi}(u, Z) \in \Phi$. The estimator defined in (4.1) satisfies A1 with $N^* = M$ and it still has the properties 2 and 3 of the class Φ. (For details see Hájek (1967, page 263) and Dionne (1976, pages 8–13).)

An estimator of $\phi(u, f)$ based on k samples of sizes $N_1, \cdots, N_k$, satisfying A1, A4, and the property 3 of Φ can be obtained by taking a weighted average of the estimators $\hat{\phi}_r'(u, Z_r)$, $r = 1, \cdots, k$ obtained from the separate samples with $M \geq \max_{1 \leq r \leq k} N_r$, where $Z_r = (Z_{r1}, \cdots, Z_{rN_r})'$.

In order to show that there exist estimators of $\phi(u, f)$ satisfying A1–A7 it is now sufficient to show that there exist estimators based on one sample and belonging to Φ, satisfying A2, A5, A6, and A7.

It can be shown (see Dionne (1976), Chapter 5) that a slightly modified version of Hájek and Šidák's estimator satisfies these conditions provided ϵ_L satisfies Hájek and Šidák's conditions (1967, page 260) and $L^{1/2}\epsilon_L^8 \rightarrow 0$. Here L is the size of the sample from which the estimator is computed and ϵ_L is defined as in Hájek and Šidák (1967, page 260). An example of such ϵ_L is $\epsilon_L = L^{-\alpha}$ with $1/16 < \alpha < 1/12$.

ACKNOWLEDGEMENTS. The author wishes to thank her thesis advisor, Professor Constance van Eeden from the Université de Montréal, for her helpful suggestions during both the elaboration of her thesis and the preparation of this paper based on it. Thanks are also due to the referee for his valuable comments on the manuscript.

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