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GENERALIZATION OF BLOCK-SAVITS' CONVOLUTION RESULT

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The convolution result of Block and Savits is generalized.

One of the main results, Theorem 2.8, of Block and Savits (1979) states:

THEOREM 1. *If the convolution of n* IFRA *distributions is exponential, then* $n - 1$ *of the distributions are degenerate at* 0 *and the other distribution is exponential.*

We obtain a more general result using elementary properties of the coefficient of variation (CV). For a nonnegative rv X we define $\mathrm{CV}(X)$ to be equal to $\sqrt{\mathrm{Var}(X)}/EX$ if $EX > 0$ and equal to 1 if $EX = 0$. It is shown in Barlow and Proschan (1975), page 118, that the CV for an IFRA distribution is ≥ 1, while the CV for an exponential $= 1$.

LEMMA 1. *Let X and Y be nonnegative* rv's (*possibly dependent*) *with* $EX \leq EY < \infty$, $\mathrm{CV}(X) \leq 1$, $\mathrm{CV}(Y) \leq 1$, *and* $\mathrm{CV}(X + Y) = 1$. *Then* $X = \alpha Y$ a.s., *where*

$$\alpha = EX/EY, \qquad EY > 0$$
$$0, \qquad EY = 0.$$

PROOF. $EX^2 + EY^2 + 2EXY = E(X + Y)^2 = 2E^2(X + Y) = 2E^2X + 2E^2Y + 4EXEY \geq EX^2 + EY^2 + 2(EX^2EY^2)^{1/2}$, so that $E^2XY \geq EX^2EY^2$. By the Cauchy-Schwartz inequality, equality must hold. This implies the desired conclusion. □

Note that the class of nonnegative rv's with $\mathrm{CV} \leq 1$ is closed under sums. Thus, to obtain Theorem 1 (Block and Savits), take $X_1, \cdots, X_n$ independent IFRA rv's with $EX_1 \leq \cdots \leq EX_n$ and assume $S_n \equiv X_1 + \cdots + X_n$ exponential. By Lemma 1, the rv's $S_n - X_n$ and X_n are linearly dependent as well as stochastically independent. Consequently, at least one of $S_n - X_n$ and X_n is degenerate (at 0), which in turn implies that $X_i \equiv 0$, $i = 1, \cdots, n - 1$.

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