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ON THE USE OF A STATISTIC BASED ON SEQUENTIAL RANKS TO PROVE LIMIT THEOREMS FOR SIMPLE LINEAR RANK STATISTICS

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A technique is introduced to prove limit theorems for simple linear rank statistics by means of an approximating statistic based on sequential ranks. This approximation is shown to be close enough to prove asymptotic normality of simple linear rank statistics under the null hypothesis and to obtain a bound on the rate of convergence to normality when the score function is unbounded. In addition, a law of the iterated logarithm and an invariance principle are given for simple linear rank statistics.

1. Introduction. Let $X_1, \cdots, X_n$ be independent random variables with common continuous distribution function F. Let R_{in} equal to the rank of X_i among $X_1, \cdots, X_n$. $R_{11}, \cdots, R_{nn}$ will be called the sequential ranks of $X_1, \cdots, X_n$.

Consider a simple linear rank statistic of the following form:

$$T_n = \sum_{i=1}^n c_{in} J_n(R_{in}/(n+1)), \tag{1.1}$$

where $c_{in}, \cdots, c_{nn}$ are known regression constants and $J_n(i/(n+1))$ for $i = 1, \cdots, n$ are scores generated in the following manner:

$$J_n(i/(n+1)) = EJ(U_{in}), \tag{1.2}$$

where U_{in} is the ith order statistic of n independent uniform (0, 1) random variables $U_1, \cdots, U_n$. We will assume

$$\int_0^1 J(u)du = 0, \tag{1.3}$$

$$0 < \int_0^1 J^2(u)du = A < \infty, \tag{1.4}$$

and

$$\sum_{i=1}^n c_{in} = 0. \tag{1.5}$$

Consider now the following statistic based on the sequential ranks:

$$M_n = \sum_{i=1}^n (c_{in} - \bar{c}_{i-1,n}) J_i(R_{ii}/(i+1)) \tag{1.6}$$

where $\bar{c}_{i-1,n} = \sum_{j=1}^{i-1} c_{jn}/(i-1)$, for $z \le i \le n$, and $\bar{c}_{on} = 0$.

Observe that M_n is a sum of independent random variables since $R_{11}, \cdots, R_{nn}$ are independent. See Theorem 1.1 of Barndorff-Nielsen (1963) for a proof of this fact.

If M_n can be shown to be sufficiently close to T_n, the asymptotic properties of T_n can be derived from the asymptotic properties of the sum of independent random variables M_n. Under various restrictions on J and on the regression constants, it will be shown that this

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approximation is close enough to prove the following: asymptotic normality of T_n under the same conditions as Theorem V.1.6a of Hájek and Šidák (1967); a law of the iterated logarithm and an invariance principle for T_n; and to obtain a bound on the rate at which T_n converges to normality. It is also shown that M_n is as efficient as T_n against regression alternatives under the same conditions as Theorem V.1.2.4 of Hájek and Šidák (1967).

The idea of using M_n to approximate T_n is latent in the proof of Theorem 2.1 of Sen and Ghosh (1972). Also more recently Lombard (1979b) has motivated M_n by observing that $M_n = \sum_{i=1}^n E(S_n | R_{ii})$, where $S_n = \sum_{i=1}^n c_{in} J(U_i)$.

The usual method of obtaining asymptotic properties of T_n is to approximate T_n by the projection $\hat{T}_n = \sum_{i=1}^n E(T_n | X_i)$. See Hájek (1968) for a discussion of this technique. Recently, Jurečková and Puri (1975), Bergström and Puri (1977), Hušková (1977) and Serfling (1977) have used the $\hat{T}_n$ approximation or variates of it to obtain bounds on the rate at which T_n converges to normality when J is bounded and in addition satisfies other regularity conditions. Hušková (1977) in particular has obtained the optimum rate of $O(n^{-1/2})$. To date, the projection technique has not succeeded in obtaining a rate for the case when J is unbounded. Using the M_n approximation we obtain close bounds on the rate at which T_n converges to normality when J is unbounded. See Theorem 5.1 below.

Some remarks must be made about the use of M_n as a test statistic. In sequential test procedures, M_n has an obvious advantage over T_n. With the advent of each new observation, only one new rank is computed in the recomputation of M_n, whereas if T_n is used, all the previous ranks must be recomputed. Lombard (1979a, 1979b) has recently investigated properties of sequential testing procedures based on the statistic M_n. For related work on statistics based on sequential ranks, see Parent (1965), Reynolds (1975), Lombard (1977), and Sen (1978).

2. A moment-inequality relating M_n to T_n. First we will introduce some additional notation and observations that will be used throughout the remainder of this paper. Let

$$C_{in}^2 = \sum_{j=1}^{i} (c_{jn} - \bar{c}_{in})^2 \quad \text{for} \quad i = 1, \cdots, n. \tag{2.1}$$

When $i = n$ we denote $C_n^2 = C_{nn}^2$.

$$A_n^2 = \sum_{i=1}^n J_n^2(i/(n+1))/n, \qquad \sigma_n^2 = \operatorname{Var} T_n \quad \text{and} \quad s_n^2 = \operatorname{Var} M_n. \tag{2.2}$$

Observe that $\sigma_n^2 = nC_n^2 A_n^2/(n-1)$, $s_n^2 = \sum_{j=2}^n (c_{jn} - \bar{c}_{j-1,n})^2 A_j^2$, and by Theorem V.1.3.a of Hájek and Šidák (1967) $A_n^2 \to_{n\to\infty} A$. For $1 \le j \le n$, let

$$T_{nj} = \sum_{i=1}^{j} (c_{in} - \bar{c}_{jn}) J_j(R_{ij}/(j+1)), \tag{2.3}$$

and

$$M_{nj} = \sum_{i=1}^{j} (c_{in} - \bar{c}_{i-1,n}) J_i(R_{ii}/(i+1)), \tag{2.4}$$

$$\phi_{jn} = T_{nj} - T_{n,j-1} - (M_{nj} - M_{n,j-1}) \tag{2.5}$$

and

$$W_{in} = J_n(R_{in}/(n+1)) - J_{n-1}(R_{i,n-1}/n). \tag{2.6}$$

Observe that for $2 \le j \le n$

$$\phi_{jn} = \sum_{i=1}^{j-1} (c_{in} - \bar{c}_{j-1,n}) W_{ij} \text{ a.s.} \tag{2.7}$$

For each $j \ge 1$, $\mathscr{F}_j$ will denote the σ-field generated by $R_{1j}, \cdots, R_{jj}$. By Lemma 2.1 of Sen and Ghosh (1972)

$$\{\langle T_{nj}, \mathscr{F}_j \rangle, 1 \le j \le n\} \text{ is a martingale.} \tag{2.8}$$

With the above notation we will prove the following moment inequality for $M_n - T_n$.

Theorem 2.1. *For each integer $k > 0$, there exists a constant $C(k) > 0$ dependent only on k such that for all $1 \leq j < n$*

$$E(T_n - M_n - (T_{nj} - M_{nj}))^{2k} \leq n^{k-1} C(k) \textstyle\sum_{i=j+1}^n C_{i-1,n}^{2k} E W_{1i}^{2k}.$$

Proof. Pick $1 \leq j < n$. Note

$$\textstyle\sum_{i=j+1}^n \phi_{in} = T_n - M_n - (T_{nj} - M_{nj}).$$

Now since $\{\langle T_{ni} - M_{ni}, \mathscr{F}_i\rangle;\ 1 \leq i \leq n\}$ is a martingale and $E\phi_{in} = 0$, the moment inequality for martingales of Dharmdikari et al. (1968) gives

$$E(T_n - M_n - (T_{nj} - M_{nj}))^{2k} \leq n^{k-1} A(k) \textstyle\sum_{i=j+1}^n E\phi_{in}^{2k},$$

where $A(k) > 0$ is a constant dependent only on k.

Lemma 2.1. *Let $W_1, \cdots, W_n$ be random variables such that for each set of positive integers $S = \{l_1, \cdots, l_m\}$ where $m \leq n$ $E(W_{i_1}^{l_1}, \cdots, W_{i_m}^{l_m}) = \eta_S$ for all permutations $i_1, \cdots, i_m$ of $1, \cdots, n$ taken m at a time.*

Also let $c_1, \cdots, c_n$ be constants such that $\sum_{i=1}^n c_i = 0$. Set $\phi_n = \sum_{i=1}^n c_i W_i$. For every integer $k > 0$, there exists a constant $B(k) > 0$ dependent only on k such that $E\phi_n^{2k} \leq B(k) C_n^{2k} E W_1^{2k}$ where $C_n^2 = \sum_{i=1}^n c_i^2$.

Proof. Pick $k > 0$. Let S be any set of integers $\{l_1, \cdots, l_m\}$ such that $1 \leq l_1 \leq \cdots \leq l_m$ and $\sum_{i=1}^m l_i = 2k$. Let $\mathscr{S} =$ the class of all such S where $1 \leq m \leq 2k$ and set $\mathscr{S}_n =$ be the subclass of $\mathscr{S}$ where $1 \leq m \leq 2k \wedge n$. Now

$$E\phi_n^{2k} = \textstyle\sum_{j_1+\cdots+j_n=2k} \binom{2k}{j_1, \cdots, j_n} c_1^{j_1}, \cdots, c_n^{j_n} E(W_1^{j_1}, \cdots, W_n^{j_n})$$

$$= \textstyle\sum_{S\in\mathscr{S}_n} \binom{2k}{l_1, \cdots, l_m} \sum_{i_1,\cdots,i_m \text{ distinct}} c_{i_1}^{l_1}, \cdots, c_{i_m}^{l_m} \eta_S.$$

Claim. For all $S = \{l_1, \cdots, l_m\} \in \mathscr{S}_n$

$$|\textstyle\sum_{i_1,\cdots,i_m \text{ distinct}} c_{i_1}^{l_1} \cdots c_{i_m}^{l_m}| \leq m! C_n^{2k}. \tag{2.9}$$

Proof. If $l_1 \geq 2$ then the left side of (2.9) is

$$\leq \textstyle\prod_{j=1}^m (\sum_{i=1}^n |c_i|^{l_j}) \leq \max_{1\leq j\leq n} |c_i|^{2k-2m} C_n^{2m} \leq C_n^{2k-2m}\, C_n^{2m} = C_n^{2k}.$$

If $l_1 = 1$ proceed as in Lemma 2.1 of Jurečková and Puri (1975), to show (2.9) $\leq m! C_n^{2k}$. □

Hence, $E\phi_n^{2k} \leq \sum_{S\in\mathscr{S}_n} \binom{2k}{l_1, \cdots, l_m} m! C_n^{2k} |\eta_S| \leq ((2k)!)^2 \text{ card } \mathscr{S} C_n^{2k} \max |\eta_S|$. Note that each $|\eta_S| \leq E W_1^{2k}$ and card $\mathscr{S}$ depends only on k. Let $B(k) = ((2k)!)^2$ card $\mathscr{S}$. □

To complete the proof of Theorem 2.1, we use observation (2.7) and note that $W_{1i}, \cdots, W_{i-1,i}$ satisfy the conditions of Lemma 2.1. Hence

$$E\phi_{in}^{2k} \leq B(k) C_{i-1,n}^{2k}\, E W_{1i}^{2k}.$$

Now set $C(k) = A(k)B(k)$. □

Remark 2.1. See Lemma 2.5 of Hušková (1977) for an inequality very similar to the inequality of Lemma 2.1. The inequality of Hušková (1977) is not sufficient for our purposes.

3. The asymptotic mean square equivalence of T_n and M_n. The moment inequality of Section 2 allows us now to give conditions under which T_n and M_n are asymptotically mean square equivalent.

THEOREM 3.1. *If*

$$\max_{1\le i\le n} c_{in}^2/C_n^2 = o(1) \tag{3.1}$$

then $E((T_n - M_n)/\sigma_n)^2 \to 0$ *as* $n \to \infty$.

PROOF. It is enough to show (A) $E((T_n - M_n)/C_n)^2 \to_{n\to\infty} 0$. First assume that J has a bounded first derivative, then it is easy to show that there exists a constant $C > 0$ such that $EW_{in}^2 \le Cn^{-2}$ for all $n \ge 1$.

Now pick any $\epsilon > 0$ and n_0 such that for all $n \ge n_0$

$$C \sum_{j=n_0+1}^n j^{-2} < \epsilon.$$

Application of Theorem 2.1 now gives for all $n > n_0$

$$E((T_n - M_n)/C_n)^2 \le \sum_{j=2}^{n_0} C_{j-1,n}^2 EW_{1j}^2/C_n^2 + \sum_{j=n_0+1}^n C_{j-1,n}^2 EW_{1j}^2/C_n^2.$$

Note that

$$C_{j-1,n}^2 \le \sum_{i=1}^{j-1} c_{in}^2 \le (j-1) \max_{1\le i\le n} c_{in}^2.$$

Hence, the above is

$$\le (\max_{1\le i\le n} c_{in}^2/C_n^2) \sum_{j=1}^{n_0} jEW_{1j}^2 + \epsilon.$$

Thus, by (3.1) $\limsup_{n\to\infty} E((T_n - M_n)/C_n)^2 \le \epsilon$ for all $\epsilon > 0$, which implies (A).

To prove the general case, we need only apply the following lemma in the obvious manner.

LEMMA 3.1. *For each* $\epsilon > 0$, *there exists a function* $a(u)$ *with bounded first derivative such that*

$$E((T_n - T_{an})/C_n)^2 \le \epsilon \tag{3.2}$$

and

$$E((M_n - M_{an})/C_n)^2 \le \epsilon \qquad \text{for all} \qquad n \ge 2 \tag{3.3}$$

where

$$T_{an} = \sum_{i=1}^n c_{in}(a_n(R_{in}/(n+1)) - \bar{a}),$$

$$M_{an} = \sum_{i=2}^n (c_{in} - \bar{c}_{i-1,n})(a_i(R_{ii}/(i+1)) - \bar{a}),$$

and

$$\bar{a} = \int_0^1 a(u)du.$$

PROOF. Since the polynomials are dense in $L^2(0, 1)$, for each $\epsilon > 0$, we can pick a polynomial $a(u)$ such that

$$\int_0^1 (J(u) - a(u))^2 du < \epsilon/4.$$

Now observe that for all $n \ge 2$

$$\sum_{i=1}^n (J_n(i/(n+1)) - (a_n(i/(n+1)) - \bar{a}))^2/(n-1)$$

$$\le \sum_{i=1}^n (J_n(i/(n+1)) - a_n(i/(n+1)))^2/(n-1)$$

$$\le \frac{n}{n-1} \int_0^1 (J(u) - a(u))^2 du < \epsilon/2.$$

Hence (3.2) $\leq \epsilon$ and (3.3) is

$$\leq \sum_{i=2}^{n} (c_{in} - \bar{c}_{i-1,n})^2 \epsilon/(2C_n^2) \leq \epsilon.$$

For the last inequality, we use the fact that

$$C_n^2 = \sum_{i=2}^{n} i^{-1}(i-1)(c_{in} - \bar{c}_{i-1,n})^2 \geq \tfrac{1}{2} \sum_{i=2}^{n} (c_{in} - \bar{c}_{i-1,n})^2. \qquad \square$$

REMARK 3.1. Under the conditions of Theorem 3.1, $s_n^2/\sigma_n^2 \to_{n\to\infty} 1$.
The asymptotic normality of T_n follows now as an easy corollary of Theorem 3.1.

COROLLARY 3.1. (Theorem V.1.6.a of Hájek and Šidák). *Under the conditions of Theorem* 3.1,

$$T_n/\sigma_n \to_d N(0, 1).$$

PROOF. Just apply the Lindeberg-Feller Theorem (Theorem 7.2.1 of Chung (1974)) to M_n/s_n and then Theorem 3.1 and Remark 3.1. $\square$

REMARK 3.2. It is not too hard to show that Corollary 3.1 remains true for scores given by

$$J_n(i/(n+1)) = J(i/(n+1)) \quad \text{for} \quad i = 1, \cdots, n, \tag{3.4}$$

when J is expressible as a finite sum of square integrable monotone functions, or

$$J_n(i/(n+1)) = n \int_{(i-1)/n}^{i/n} J(u)\,du \quad \text{for} \quad i = 1, \cdots, n \tag{3.5}$$

when J is square integrable. (See pages 164–165 of Hájek and Šidák (1967).)

COROLLARY 3.2. (Distribution of M_n under regression alternatives). *Under the same conditions as Theorem* V.1.2.4 *of Hájek and Šidák* (1967) *M_n is asymptotically normal with the same mean and variance as T_n.*

PROOF. Look at the proof of Theorem V.1.2.4 of Hájek and Šidák. Note that our T_n corresponds to Hájek and Šidák's S_c^ϕ. Now apply Theorem 3.1 to show that $(M_n - S_c^\phi)/\sigma_n \to_P 0$ as $n \to \infty$. The rest is exactly the same as in Hájek and Šidák.

REMARK 3.3. Under the assumptions of Theorem V.1.2.4 of Hájek and Šidák, M_n and T_n have the same Pitman efficiency.

COROLLARY 3.3. (An invariance principle for T_n). *For each $n \geq 1$, let V_n and W_n be random functions on* [0, 1] *defined as follows*:

$$V_n(t) = s_n^{-1}(M_{nk} + (M_{n,k+1} - M_{nk})(ts_n^2 - s_{nk}^2)/(s_{n,k+1}^2 - s_{nk}^2))$$

$$W_n(t) = s_n^{-1}(T_{nk} + (T_{n,k+1} - T_{nk})(ts_n^2 - s_{nk}^2)/(s_{n,k+1}^2 - s_{nk}^2))$$

whenever $s_{nk}^2 \leq ts_n^2 < s_{n,k+1}^2$ *for* $k = 0, \cdots, n-1$ *and*

$$V_n(t) = s_n^{-1}M_{nk} \quad \text{and} \quad W_n(t) = s_n^{-1}T_{nk}$$

if $s_{nk}^2 = ts_n^2 = s_{n,k+1}^2$; *where*

$$s_{nk}^2 = \operatorname{Var} M_{nk} \quad \text{and} \quad M_{n0} = T_{n0} \equiv 0.$$

Under the conditions of Theorem 3.1, *$V_n \Rightarrow W$ and $W_n \Rightarrow W$, where W is a standard Wiener process on* [0, 1].

PROOF. It is easy to show that under the conditions of Theorem 3.1 that the V_n process satisfies the conditions of Theorem 2.1 of Prokhorov (1956) to give $V_n \Rightarrow W$.

Observe that $\sup_{0\le t\le 1}|V_n(t) - W_n(t)| \le s_n^{-1}\max_{1\le k\le n}|M_{nk} - T_{nk}|$. Now since $M_{nk} - T_{nk}$, $k = 1, \cdots, n$ is a martingale, for any $\epsilon > 0$

$$P(\sup_{0\le t\le 1}|V_n(t) - W_n(t)| > \epsilon) \le \epsilon^{-2}s_n^{-2}E(T_n - M_n)^2. \tag{3.6}$$

Theorem 3.1 implies that (3.6) → 0, hence $W_n \Rightarrow W$. □

REMARK 3.4. See Theorem 1.2 of Sen and Ghosh (1972) for a related invariance principle proven under conditions similar to those of Theorem 5.1 below.

4. A strong moment inequality for $M_n - T_n$. In this section, we add an additional smoothness condition on J, which will allow the inequality of Section 2 to be strengthened so that a law of the iterated logarithm can be obtained for T_n. Later in Section 5, this inequality will be an essential tool in deriving a bound on the rate that T_n converges to normality.

THEOREM 4.1. *Suppose J is absolutely continuous inside* (0, 1) *and*

$$|J'(u)| \le K(u(1-u))^{-3/2+\delta} \tag{4.1}$$

for all $u \in (0, 1)$ where $K > 0$ and $0 < \delta < 1/2$, then for some constant $C(k) > 0$ dependent only on k, K and δ

$$E(T_n - M_n)^{2k} \le n^{k-1}C(k)\sum_{i=2}^{n} C_{i-1,n}^{2k}(i+1)^{k-2\delta k-2} \tag{4.2}$$

for all $n \ge 2$.

PROOF. We need only show $(A) EW_{1i}^{2k} = O((i+1)^{k-2\delta k-2})$ and then apply Theorem 2.1. The following two lemmas prove (A).

LEMMA 4.1. *Under the conditions on J in Theorem 4.1 there exists a constant $K' > 0$ such that for all $n \ge 2$ and $1 \le j \le n-1$*

$$|J_n((j+1)/(n+1)) - J_n(j/(n+1))| \le K'(n+1)^{-1}[(j+1)(n+1-j)/(n+1)^2]^{-3/2+\delta}.$$

PROOF. Pick any $1 \le j \le n-1$, $n \ge 2$. Note that there exists a $K_1 > 0$ such that

$$K(u(1-u))^{-3/2+\delta} < K_1(u^{-3/2+\delta} + (1-u)^{-3/2+\delta})$$

for all $u \in (0, 1)$. Hence,

$$\begin{aligned}&|J_n((j+1)/(n+1)) - J_n(j/(n+1))|\\ &\quad = |E\int_{U_{jn}}^{U_{j+1,n}} J'(u)\,du| < K_1E\int_{U_{jn}}^{U_{j+1,n}} (u^{-3/2+\delta} + (1-u)^{-3/2+\delta})\,du\\ &\quad = K_1(1/2-\delta)^{-1}E[(1-U_{j+1,n})^{-1/2+\delta} - (1-U_{jn})^{-1/2+\delta} - (U_{j+1,n})^{-1/2+\delta} + (U_{jn})^{-1/2+\delta}],\end{aligned}$$

which since

$$E(U_{in})^{-1/2+\delta} = \prod_{l=i}^{n} l/(l+\delta-1/2),$$

equals

$$\begin{aligned}&K_1(1/2-\delta)^{-1}[((n-j)/(n-j+\delta-1/2)-1)\\ &\quad\cdot\prod_{i=n+1-j}^{n} i/(i+\delta-1/2) + (j/(j+\delta-1/2)-1)\prod_{i=j+1}^{n} i/(i+\delta-1/2)].\end{aligned} \tag{4.3}$$

Now it is easy to show that there exists a $K_\delta > 0$ such that for all $1 \le k \le n - 1$ and $n \ge 2$

$$\prod_{i=n+1-k}^{n} i/(i + \delta - ½) \le K_\delta((n + 1 - k)/(n + 1))^{-1/2+\delta},$$

which implies that expression (4.3) is

$$\le K_1 K_\delta[(1/(n - j + \delta - ½))((n + 1 - j)/(n + 1))^{-1/2+\delta} + (1/(j + \delta - ½))(j/(n + 1))^{-1/2+\delta}]$$

$$= \frac{K_1 K_\delta}{n+1}[((n + 1 - j)/(n - j + \delta - ½))((n + 1 - j)/(n + 1))^{-3/2+\delta} + (j/(j + \delta - ½))(j/(n + 1))^{-3/2+\delta}]. \tag{4.4}$$

Note that $k/(k + \delta - ½) = 1/(1 + (\delta - ½)/k) \le (½ + \delta)^{-1}$ for $1 \le k \le n$. Hence, (4.4) is

$$\le K_1 K_\delta(½ + \delta)^{-1}(n + 1)^{-1}[(1 - j/(n + 1))^{-3/2+\delta} + (j/(n + 1))^{-3/2+\delta}].$$

It is easy to see that there exists a constant $K_2 > 0$ such that

$$(1 - u)^{-3/2+\delta} + u^{-3/2+\delta} \le K_2(u(1 - u))^{-3/2+\delta}$$

for all $u \in (0, 1)$. Now let $K' = K_1 K_2 K_\delta(½ + \delta)^{-1}$. □

LEMMA 4.2. *Under the conditions on J in Theorem 4.1 for every integer $k > 0$ there exists a constant $D(k) > 0$ dependent only on k and J such that for all $n \ge 2$*

$$EW_{1n}^{2k} \le D(k)(n + 1)^{k-2\delta k-2}.$$

PROOF. Pick $k \ge 1$. Note that

$$E(W_{1n}^{2k}|\mathscr{F}_{n-1}) = n^{-1}(n - R_{1,n-1})[J_n(R_{1,n-1}/(n + 1)) - J_{n-1}(R_{1,n-1}/n)]^{2k} + n^{-1}R_{1,n-1}[J_n((R_{1,n-1} + 1)/(n + 1)) - J_{n-1}(R_{1,n-1}/n)]^{2k}. \tag{4.5}$$

Now by application of the identity: for $1 \le i \le n - 1$

$$J_{n-1}(i/n) = n^{-1}(n - i)J_n(i/(n + 1)) + n^{-1}iJ_n((i + 1)/(n + 1)),$$

we get expression (4.5)

$$= (1 - R_{1,n-1}/n)(R_{1,n-1}/n)[(R_{1,n-1}/n)^{2k-1} + (1 - R_{1,n-1}/n)^{2k-1}].$$
$$[J_n((R_{1,n-1} + 1)/(n + 1)) - J_n(R_{1,n-1}/(n + 1))]^{2k}$$
$$\le (1 - R_{1,n-1}/n)(R_{1,n-1}/n)[J_n((R_{1,n-1} + 1)/(n + 1)) - J_n(R_{1,n-1}/(n + 1))]^{2k}.$$

Thus,

$$EW_{1n}^{2k} \le (n - 1)^{-1} \sum_{i=1}^{n-1} (1 - i/n)(i/n)[J_n((i + 1)/(n + 1)) - J_n(i/(n + 1))]^{2k}. \tag{4.6}$$

which by Lemma 4.1 is

$$\le K''(n + 1)^{-2k} \sum_{i=1}^{n} [(1 - i/(n + 1))(i/(n + 1))]^{-3k+2\delta k+1}/(n - 1) \tag{4.7}$$

for some $K' > 0$.
By an integral approximation expression (4.7)

$$\le D(k)(n + 1)^{k-2\delta k-2}$$

for some $D(k) > 0$ for all $n \ge 2$. □

COROLLARY 4.1. *Suppose J satisfies the conditions on J in Theorem 4.1 and if*

$$\max_{1\le i\le n} c_{in}^2/C_n^2 = O(1/n), \tag{4.8}$$

then $E((T_n - M_n)/\sigma_n)^2 = O(n^{-2\delta})$.

PROOF. It is sufficient to show $E((T_n - M_n)/C_n)^2 = O(n^{-2\delta})$. By Theorem 4.1

$$E((T_n - M_n)/C_n)^2 \le C(1) \sum_{i=2}^n C_{i-1,n}^2 (i+1)^{-2\delta-1}/C_n^2,$$

which since

$$C_{i-1,n}^2 \le \sum_{j=1}^i c_{jn}^2$$

and by condition (4.8) is

$$\le C \sum_{i=2}^n (i+1)^{-2\delta}/n$$

for some constant $C > 0$ dependent only on J and condition (4.8). A simple integral approximation completes the proof. □

REMARK 4.1. Under the conditions of Corollary 4.1, $|\sigma_n/s_n - 1| = O(n^{-\delta})$.

COROLLARY 4.2. (A law of the iterated logarithm for T_n). *Under the conditions of Corollary 4.1, when the c_{in} are of the form $c_{in} = c_i - \bar{c}$ for $i = 1, \cdots, n$ and $\bar{c} = \sum_{i=1}^n c_i/n$,*

$$\limsup_{n\to\infty} T_n/\sqrt{2\sigma_n^2 \ln\ln \sigma_n^2} = 1 \quad \text{a.s.}$$

PROOF. Using the maximal inequality for martingales over appropriately chosen blocks of $T_n - M_n$, it can be demonstrated using standard techniques that

$$T_n/\sigma_n = M_n/\sigma_n + 0(1) \quad \text{a.s.}$$

Now Theorem 6 on page 115 of Petrov (1975) can be shown to imply $\limsup_{n\to\infty} M_n/\sqrt{2\sigma_n^2 \ln\ln \sigma_n^2} = 1$ a.s. □

REMARK 4.2. Sen and Ghosh (1972) prove essentially this same result, though they add one more condition on the regression constants. Their proof consists of a lengthy verification of the conditions of a martingale law of the iterated logarithm of Strassen (1967). For laws of the iterated logarithms for T_n under other conditions, see Mason (1978).

5. A bound on the rate that T_n converges to normality. In this section, we will use various martingale inequalities along with the moment inequality of Section 4 to obtain a bound on the rate that T_n converges to normality when J is unbounded. We remark here that Theorem 4.1 alone cannot prove Theorem 5.1, since the moment bound increases along with k, rather than decreases.

THEOREM 5.1. *If J is absolutely continuous inside* (0, 1) *such that*

$$|J'(u)| \le K(u(1-u))^{-3/2+\delta} \tag{5.1}$$

for some $0 < \delta < 1/2$ and $K > 0$ and all $u \in (0, 1)$, and

$$\max_{1\le i\le n} c_{in}^2/C_n^2 < c/n \tag{5.2}$$

for some $c > 0$ and all $n \ge 1$, then for all $0 < \delta^* < \delta$

$$D_n \equiv \sup_{-\infty<x<\infty} |P(T_n \le x\sigma_n) - \Phi(x)| = O(n^{-\delta^*}).$$

PROOF. Pick any $0 < \delta^* < \delta$. We will first show that

$$\sup_{-\infty<x<\infty} |P(T_n \le xs_n) - \Phi(x)| = O(lnn\, n^{-\delta^*}). \tag{5.3}$$

For any $0 \le a \le 1$, let

$$\Delta_n(a) = T_{n,[n^a]} - M_{n,[n^a]}, \qquad \text{where } [x] = \text{greatest integer} \le x.$$

Pick $0 < a < 1$ then the left side of expression (5.3) is

$$\le \sup_{-\infty<x<\infty} |P(T_n \le xs_n) - P(M_n + \Delta_n(a) \le xs_n)| \tag{5.4}$$

$$+ \sup_{-\infty<x<\infty} |P(M_n + \Delta_n(a) \le xs_n) - \Phi(x)|. \tag{5.5}$$

A trivial argument shows that for all $h > 0$ expression (5.4) is

$$\le \sup_{-\infty<x<\infty} |P(M_n + \Delta_n(a) \le (x+h)s_n) - P(M_n + \Delta_n(a) \le xs_n)| \tag{5.6}$$

$$+ P(|\Delta_n(1) - \Delta_n(a)|/s_n > h). \tag{5.7}$$

It is easy to show that (5.6) $\le 2(5.5) + h\sqrt{2/\pi}$. Hence (5.3) $\le 3(5.5) + (5.7) + h\sqrt{2/\pi}$.

The proof will be completed by showing (I) there exists a $c > 0$ and an $0 < a < 1$ such that for $h = c\, lnn\, n^{-\delta^*}$ (5.7) $= O(n^{-\delta})$, and (II) for all $0 < a < 1$ (5.5) $= O(n^{-\delta})$.

PROOF OF (I). We will show that for every $\lambda > 0$, there exists a $c > 0$ and $0 < a < 1$ such that for all n sufficiently large

$$P(|\Delta_n(1) - \Delta_n(a)|/s_{1\le m\le n} > c\, lnn\, n^{-\delta^*}) < n^{-\lambda}. \tag{5.8}$$

Pick $\lambda > 0$. Let $t_m^2 = \sum_{i=1}^m E(\phi_{in}^2 | \mathscr{F}_{i-1})/s_n^2$ for $\le m \le n$. Pick $0 < a < 1$ and let $\tau_a = \min\{m: t_{m+1}^2 - t_{[n^a]}^2 \ge n^{-2\delta^*}, n \ge m \ge [n^a]\}$ and equal to ∞ if the set is empty. Let

$$X_i = ((T_{n,[n^a]+i} - T_{n,[n^a]} - (M_{n,[n^a]+i} - M_{n,[n^a]}))/s_n$$

for $i = 0, \cdots, n-[n^a], \mathscr{F}_i^* = \mathscr{F}_{i+[n^a]}$ $i = 0, \cdots, n-[n^a]$ and

$$\mathscr{F}_\infty^* = \vee_{0\le i\le n-[n^a]} \mathscr{F}_i^*.$$

It is easy to see that $\{\langle X_i, \mathscr{F}_i^* \rangle : i = 0, \cdots, n-[n^a]\}$ is a martingale. Set$\tau_a^* = \tau_a - [n^a]$. Also it is not difficult to show that τ_a^* is optional relative to $\{\mathscr{F}_i^* : i = 0, \cdots, n-[n^a], \infty\}$.

Now by page 324 of Chung (1974), $\{\langle X_{i\wedge\tau_a^*}, \mathscr{F}_{i\wedge\tau_a^*}^* \rangle : i = 0, \cdots, n-[n^a]\}$ is a martingale. Observe that the left side of (5.8) is

$$= P(|\dot{X}_{\tau_a^* \wedge;1nmi[n^a])}| > c\, lnn\, n^{-\delta^*}, \tau_a^* \ge n - [n^a]) \tag{5.9}$$

$$+ P(|X_{n-[n^a]}| > c\, lnn\, n^{-\delta^*}, \tau_a^* < n - [n^a]). \tag{5.10}$$

Note that expression (5.9)

$$\le P(\sup\{|n^{\delta^*} X_{\tau_a^* \wedge i}| : 0 \le i \le n - [n^a]\} \ge c\, lnn). \tag{5.11}$$

At this point, we need a lemma.

LEMMA 5.1. *Under the conditions of Theorem* 4.1, *there exists constant* $C > 0$ *such that for all* $n \ge 2$ *and* $2 \le m \le n$

$$|T_{nm} - T_{n,m-1} - (M_{nm} - M_{n,m-1})| \le C\, C_n m^{-\delta} \quad \text{a.s.} \tag{5.12}$$

PROOF. Note that by (2.7), the left side of (5.12)

$$= |\textstyle\sum_{i=1}^{m-1} (c_{in} - \bar{c}_{m-1,n})(J_m(R_{im}/(m+1)) - J_{m-1}(R_{i,m-1}/m))|.$$

Now (5.2) implies $|c_{in} - \bar{c}_{m-1,n}| \le B\, n^{-1/2}\, C_n$ for some constant $B > 0$ independent of i, m and n. Hence, the left side of (5.12) is

$$\le n^{-1/2}\, B\, C_n \textstyle\sum_{i=1}^{m-1} |J_m(R_{im}/(m+1)) - J_{m-1}(R_{i,m-1}/m)|. \tag{5.13}$$

Application of the same identity used in Lemma 4.2 gives (5.13)

$$\le n^{-1/2}\, B\, C_n \textstyle\sum_{i=1}^{m-1} |J_m((i+1)/(m+1)) - J_m(i/(m+1))| \quad \text{a.s.}$$

Lemma 4.1, along with an integral approximation, completes the proof. □

Now by picking $0 < a < 1$ sufficiently close to 1, it is not hard to show, using Lemma 5.1, that for all n sufficiently large

$$\sup\{|n^{\delta^*}(X_{\tau_a^*\wedge i} - X_{\tau_a^*\wedge(i-1)})|; 1 \le i \le n-[n^a]\} \le 1 \quad \text{a.s.}$$

Also it is a routine matter to verify that

$$\textstyle\sum_{i=1}^{n-[n^a]} E(n^{2\delta^*}(X_{\tau_a^*\wedge i} - X_{\tau_a^*\wedge(i-1)})^2 \mid \mathscr{F}^*_{\tau_a^*\wedge(i-1)}) \le n^{2\delta^*}(t_{\tau_a}^2 - t_{[n^a]}^2) \le 1.$$

All the conditions are now satisfied for the application of the exponential inequality for martingales on page 69 of Meyer (1972). Thus, for all n sufficiently large, (5.9) $\le \exp(c\, lnn)/(c\, lnn + 1)^{c\, lnn+1}$, which for $c > 0$ sufficiently large is $< n^{-\lambda}$.

Now observe that (5.10) $\le P(\tau_a^* < n - [n^a]) = P(\tau_a < n)$, which since $t_m^2 - t_{[n^a]}^2$ is nondecreasing, equals

$$P(t_n^2 - t_{[n^a]}^2 > n^{-2\delta^*}). \tag{5.14}$$

LEMMA 5.2. *Under the conditions of Theorem* 5.1 *for every* $k > 0$, *there exists a* $C(k) > 0$ *such that*

$$E(E(\phi_{mn}^2 \mid \mathscr{F}_{m-1})/C_n^2)^k \le C(k)\, m^{-2k\delta}\, n^{-k}$$

for all $n \ge 2$, $2 \le m \le n$.

PROOF. The steps to prove this inequality are briefly sketched on page 349 of Sen and Ghosh (1972). For a more detailed proof, see Mason (1978b). □

Observe that

$$E(t_n^2 - t^2_{[n^a]})^k \le n^{k-1} \textstyle\sum_{i=1+[n^a]}^{n} E(E(\phi_{in}^2 \mid \mathscr{F}_{i-1}))^k / s_n^{2k}. \tag{5.15}$$

It is not difficult to show, using Lemma 5.2, that (5.15) $\le n^{-1} D(k) \sum_{i=[n^a]+1}^{n} i^{-2k\delta}$ for some constant $D(k) > 0$ independent of n. An integral approximation gives (5.15) $\le B(k)\, n^{-2k\delta a+a-1}$ for some constant $B(k) > 0$. So, by Markov's inequality, expression (5.14) is

$$\le B(k)\, n^{2k(\delta^* - \delta a)+a-1}. \tag{5.16}$$

Now pick $0 < a < 1$ sufficiently close to 1 so that $\delta^* - \delta a < 0$ and k sufficiently large so that (5.16) $< n^{-\lambda}$. This completes the proof of (I).

PROOF OF (II). Let $f_{in}(t) = E \exp(it(c_{in} - \bar{c}_{i-1,n})J_i(R_{ii}/(i+1))/s_n)$ for $i = 2, \cdots, n$. By the Esseen lemma (see Lemma 2 on page 227 of Chung (1974)) for each $b > 0$, (5.5) is

$$\le \frac{2}{\pi}\int_0^{bn^\delta} |E \exp(it(M_n + \Delta_n(a))/s_n) - \exp(-t^2/2)|\, t^{-1} dt + 24(bn^\delta)^{-1}/(2\pi^3)^{1/2}. \tag{5.17}$$

Observe that the first term of (5.17) is

$$\le \frac{2}{\pi}\int_0^{bn^\delta} |E \exp(it(M_n + \Delta_n(a))/s_n) - E \exp(it\, M_n/s_n)|\, t^{-1} dt \tag{5.18}$$

(5.19) $$+\frac{2}{\pi}\int_0^{bn^\delta} |E\exp(it\,M_n/s_n)-\exp(-t^2/2)|\,t^{-1}dt.$$

Since $R_{ii}i=[n^a]+1,\cdots,n$ are independent of $R_{i[n^a]},\ldots,R_{[n^a],[n^a]}$, (5.18) is

$$=\frac{2}{\pi}\int_0^{bn^\delta} |\textstyle\prod_{j=1+[n^a]}^n f_{jn}(t)(E\exp(it\,T_{n[n^a]}/s_n)-E\exp(it\,M_{n[n^a]}/s_n)|\,t^{-1}dt$$

(5.20) $$\le\frac{2}{\pi}\int_0^{bn^\delta} |\textstyle\prod_{j=1+[n^a]}^n f_{jn}(t)|(E((T_{n]n^a]}-M_{n[n^a]})/s_n)^2)^{1/2}dt.$$

Now by Corollary 4.1, for some $C>0$ independent of n, (5.20) $\le\frac{2}{\pi}\int_0^{bn^\delta}|\prod_{j=1+[n^a]}^n f_{jn}(t)|Cn^{-\delta a}\cdot n^{(a-1)/2}dt$, which is

(5.21) $$\le\frac{2cn^{-\delta}}{\pi}\int_0^{bn^\delta} |\textstyle\prod_{j=1+[n^a]}^n f_{jn}(t)|\,dt.$$

To complete the proof, we need the following lemma.

LEMMA 5.3. *Let* $y_{jn}=(c_{jn}-\bar c_{j-1,n})J_j(R_{jj}/(j+1))/s_n$ *for* $2\le j\le n$. *For each* $0\le a\le1$, *let*

$$L_{an}(\delta)=\textstyle\sum_{j=1+[n^a]}^n E|y_{jn}|^{2+2\delta},\qquad L_{on}(\delta)=L_n(\delta),$$

and

$$d_{an}^2=\textstyle\sum_{j=1+[n^a]}^n E|y_{jn}|^2.$$

Under the conditions of Theorem 5.1, *there exist constants* $a_0>0$, $c_0>0$ *and* $d_0>0$ *independent of n such that*

(5.22) $$|\textstyle\prod_{j=1+[n^a]}^n f_{jn}(t)|<\exp(-c_0d_{an}^2t^2)$$

when $|t|\le(d_0L_{an}(\delta))^{-1/2\delta}$ *and*

(5.23) $$|\textstyle\prod_{j=2}^n f_{jn}(t)-\exp(-t^2/2)|\le a_0L_n(\delta)|t|^{2+2\delta}\exp(-c_0t^2)$$

when $|t|\le(d_0L_n(\delta))^{-1/2\delta}$.

PROOF. Note that $\left|\exp(it)-1-it+\frac{t^2}{2}\right|\le|t|^{2+2\delta}B_\delta$ for all t where $B_\delta=2^{1-2\delta}/((1+2\delta)(2+2\delta))$. To prove (5.22), we note that the above inequality implies $\left|\cos t-1+\frac{t^2}{2}\right|\le B_\delta|t|^{2+2\delta}$. The proof is thus a straightforward modification of the proof of Lemma 4 on page 229 of Chung (1974).

To prove (5.23), note that the above inequality implies $|f_{nj}(t)-1+E|y_{jn}|^2t^2/2|\le B_\delta|t|^{2+2\delta}E|y_{jn}|^{2+2\delta}$, which implies that for each t there exists a complex number θ where $|\theta|\le1$ such that

$$f_{jn}(t)=1-E|y_{jn}|^2t^2/2+\theta B_\delta|t|^{2+2\delta}E|y_{jn}|^{2+2\delta}.$$

The proof now proceeds by steps analogous to the proofs of Lemmas 3 and 5 on pages 228–229 of Chung (1974). □

To finish the proof of (II), note that the conditions on J and the regression constants imply that there exists $b>0$ such that $d_0L_{an}(\delta)<b^{-1}n^{-\delta}$ for all $0\le a<1$. Hence, by

Lemma 5.3 and the fact that $d_{an}^2 \to 1$, expression (5.21) is

$$\leq \frac{2cn^{-\delta}}{\pi} \int_0^{bn^\delta} \exp(-c_0 d_{an}^2 t^2)\, dt = O(n^{-\delta}).$$

Also by Lemma 5.3 expression (5.19)

$$= \frac{2}{\pi} \int_0^{bn^\delta} \left| \prod_{j=2}^{n} f_{jn}(t) - \exp(-t^2/2) \right| t^{-1} dt$$

$$\leq \frac{2a_0}{\pi} L_n(\delta) \int_0^{bn^\delta} |t|^{1+2\delta} \exp(-c_0 t^2)\, dt = O(n^{-\delta}) \qquad \square$$

To complete the proof of Theorem 5.1, we use Remark 4.1 along with the inequalities on page 114 of Petrov (1975) to show $D_n = O(\ln n\, n^{-\delta^*})$, which implies that $D_n = O(n^{-\delta^*})$ for all $0 < \delta^* < \delta$. $\square$

REMARK 5.1. Observe that when $\delta > 1/6$, $\sum_{i=2}^{n} E|y_{in}|^3 = O(n^{-1/2})$. Theorem 5.1 gives, in this case, $D_n = O(n^{-\delta^*})$ for all $1/6 < \delta^* < \delta$. So if the rate for T_n is the same as the corresponding rate for sums of independent random variables, our rate misses the optimum Berry-Esseén rate of $O(n^{-1/2})$. If J is the inverse of the standard normal distribution or has a bounded derivative, then condition (5.1) holds for all $0 < \delta < 1/2$. Theorem 5.1 then gives $D_n = O(n^{-\delta})$ for all $0 < \delta < 1/2$. This rate is comparable to the result of Jurečková and Puri (1975) for bounded J with bounded first derivative though not to Hušková (1977), who obtains the optimum rate $O(n^{-1/2})$ under the additional assumption that $\int_0^1 (J''(u))^2 du < \infty$.

REMARK 5.2. Theorem 5.1 remains true if the scores are replaced by (3.4) or (3.5).

6. Some remarks on the rate in the non-i.i.d. Case. Nathan (1975) has shown that $D_n = O(\ln n/n^\delta)$ where $0 < \delta < 1/4$, when T_n is the Chernoff-Savage (1958) two-sample statistic, if it is assumed that $|J'(u)| \leq K(u(1-u))^{-5/4+\delta}$ and some slightly modified assumptions of Pyke and Shorack (1968). Essential to his proof in a strong embedding of the empirical process due to Müller (1970). Presumably, using the more recent embedding due to Komlós, et al., (1975), his rate can be improved to give the rate of Theorem 5.1 under the bounding condition (5.1). See page 104 of Nathan (1975) for further discussion of this point. He considers the two-sample problem, so thus does not require the X_i's to be identically distributed. He does need some added smoothness assumptions on the distributions.

Jurečková and Puri (1975) and Hušková (1977) were able to find rates at which T_n converges to normality under local regression alternatives when J is bounded. The M_n technique does not appear amenable to finding a rate in this case. To use the Jurečková and Puri technique would require a weak convergence result for the simple linear rank statistic process under the assumptions of Theorem 5.1. See Jurečková (1973) for the definition of this process. A result of this kind does not exist at present under the assumptions of Theorem 5.1. Even when J is bounded the proof of such a weak convergence result is a long and involved affair. On the other hand, Hušková's method relies heavily on the assumption that J has a bounded derivative and on the fact that the projection is still a sum of independent random variables when $X_1, \cdots, X_n$ are not i.i.d. M_n is not necessarily a sum of independent random variables in the non-i.i.d. case. Finally, we mention that if $|J'(u)| \leq K(u(1-u))^{-1}$, it is possible to obtain a $O(n^{-\delta})$ rate for all $0 < \delta < 1/4$ using an a.s. linearity result of Ghosh and Sen (1972). See their Theorem 3.1.

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