

DISTRIBUTIONS OF SOME MATRIX VARIATES AND LATENT ROOTS IN MULTIVARIATE BEHRENS-FISHER DISCRIMINANT ANALYSIS¹

BY YASUKO CHIKUSE

Kagawa University, Japan and University of Pittsburgh

In this paper it is shown that some distributions of the matrix variates and latent roots arising in the multivariate Behrens-Fisher discriminant problem can be explicitly expressed in terms of the invariant polynomials with two matrix arguments, due to A. W. Davis, extending the zonal polynomials of matrix argument.

1. Introduction. We are interested in the discrimination problem of two multivariate normal populations under the heterogeneity of population covariance matrices. Distributional problems for the univariate case have been investigated by many authors (e.g., McCullough, Gurland and Rosenberg (1960), Ray and Pitman (1961) and Welch (1947)), and Gurland and McCullough (1962) and Wehrhahn and Ogawa (1975) took into consideration the preliminary testing procedures for the null hypothesis of the equality of the variances.

Yao (1965) considered approximate distributions of a Hotelling's T^2 -type statistic in the multivariate case. However, to the author's knowledge very little work has been published on the multivariate Behrens-Fisher discriminant problem. Our Behrens-Fisher matrix statistic for the multivariate case may be in the form

$$(1.1) \quad D = (S_1 + S_2)^{-1/2} S_0 (S_1 + S_2)^{-1/2}$$

where the $m \times m$ matrices S_i are independently distributed as, in general, noncentral Wishart $W_m(n_i, \Sigma_i, \Omega_i)$, $i = 0, 1, 2$. In practical terms, S_1 and S_2 are sample error sums of products matrices, proportional to estimates of the distinct population covariance matrices, and S_0 is an effects sum of products matrix, estimating the difference of the population mean matrices. In the case when $\Sigma_1 = \Sigma_2 = \sigma^2 \Sigma_0$ ($\sigma^2 > 0$) and $\Omega_1 = \Omega_2 = 0$, D is the well-known MANOVA matrix (central if $\Omega_0 = 0$ and noncentral if $\Omega_0 \neq 0$). Hence, D is an extension of the MANOVA matrix in discriminant analysis.

The matrix $A = (S_1 + S_2)^{-1/2} S_1 (S_1 + S_2)^{-1/2}$ is a suitable preliminary test matrix for the hypothesis $\Sigma_1 = \Sigma_2$, where in this paper we assume $\Omega_1 = \Omega_2 = 0$ whenever the preliminary testing procedures are considered. When $\Sigma_1 \neq \Sigma_2$, D and A are dependent, and we may be interested in the conditional or unconditional distributions of the roots, of D , or suitable functions of them (conditional on the roots of A).

In this paper, it is shown that some distributions of the matrix variates and latent roots arising in the multivariate Behrens-Fisher discriminant problem, for the null case, $\Omega_0 = 0$, can be explicitly expressed in terms of invariant polynomials with two matrix arguments. The invariant polynomials with two matrix arguments have been defined by Davis (1980a), (1980b), in the need of extending the zonal polynomials due to Constantine (1963) and James (1964). Properties, relationships between them and applications in multivariate distribution theory are discussed in the same papers of Davis. It is noted here that explicit

Received May 1979; revised February 1980.

¹ The work of this author was partially supported by the Air Force Office of Scientific Research, Air Force Systems Command under Contract F49620-79-C-0161.

AMS 1970 subject classifications. Primary 62H10; secondary 62H30.

Key words and phrases. Behrens-Fisher discriminant analysis, invariant polynomials with two matrix arguments, Wishart distributions, preliminary test of equality of covariance matrices.

forms for those distributions for the nonnull case, $\Omega_0 \neq 0$, i.e., when population mean matrices are different, can be obtained by introducing invariant polynomials with three matrix arguments, extending the work of Davis. The investigation of invariant polynomials with larger number of matrix arguments will be presented in subsequent papers (see, e.g., Chikuse (1980)).

The invariant polynomials with two matrix arguments are briefly discussed in Section 2. The joint distributions of the roots of D and A are derived in Section 3, and the marginal (and conditional) distributions of the roots of D (conditional on those of A) are investigated in Section 4, for the null case, $\Omega_0 = 0$.

2. Invariant polynomials with two matrix arguments. Davis (1980a), (1980b) has defined a class of homogeneous polynomials $C_\phi^{\kappa,\lambda}(X, Y)$ of degrees k and l in the elements of the $m \times m$ symmetric complex matrices X and Y , invariant under the simultaneous transformations

$$(2.1) \quad X \rightarrow H'XH, \quad Y \rightarrow H'YH, \quad H \in O(m),$$

where $O(m)$ is the group of $m \times m$ orthogonal matrices. These satisfy the basic relationship

$$(2.2) \quad \int_{O(m)} C_\kappa(AH'XH)C_\lambda(BH'YH) dH = \sum_{\phi \in \kappa \cdot \lambda} C_\phi^{\kappa,\lambda}(A, B)C_\phi^{\kappa,\lambda}(X, Y)/C_\phi(I),$$

where $C_\kappa, C_\lambda, C_\phi$ are zonal polynomials, indexed by the ordered partitions κ, λ, ϕ of the nonnegative integers $k, l, f = k + l$ respectively into not more than m parts (for the zonal polynomials, see, e.g., Constantine (1963) and James (1964)). Letting $Gl(m, R)$ denote the group of $m \times m$ real nonsingular matrices, $\phi \in \kappa \cdot \lambda$ signifies that the irreducible representation of $Gl(m, R)$ indexed by 2ϕ occurs in the decomposition of the Kronecker product $2\kappa \otimes 2\lambda$ of the irreducible representations indexed by $2\kappa, 2\lambda$. The properties and relationships satisfied by the $C_\phi^{\kappa,\lambda}$, which are especially utilized in the later sections, are summarized in the following (see Davis (1980a), (1980b) for proofs and details):

$$(2.3) \quad C_\phi^{\kappa,\lambda}(X, X) = \theta_\phi^{\kappa,\lambda} C_\phi(X), \quad \text{where } \theta_\phi^{\kappa,\lambda} = C_\phi^{\kappa,\lambda}(I, I)/C_\phi(I),$$

$$(2.4) \quad C_\phi^{\kappa,\lambda}(X, I) = [\theta_\phi^{\kappa,\lambda} C_\phi(I)/C_\kappa(I)]C_\kappa(X), \quad \text{and similarly for } C_\phi^{\kappa,\lambda}(I, X),$$

$$(2.5) \quad C_\kappa^{\kappa,0}(X, Y) \stackrel{\text{def}}{=} C_\kappa(X), \quad C_\lambda^{0,\lambda}(X, Y) \stackrel{\text{def}}{=} C_\lambda(Y),$$

$$(2.6) \quad C_\phi^{\kappa,\lambda}(\alpha X, \beta Y) = \alpha^k \beta^l C_\phi^{\kappa,\lambda}(X, Y) \quad (\alpha, \beta \text{ complex constants}),$$

$$(2.7) \quad C_\kappa(X)C_\lambda(X) = \sum_{\phi \in \kappa \cdot \lambda} (\theta_\phi^{\kappa,\lambda})^2 C_\phi(X) = \sum_{\phi \in \kappa \cdot \lambda} g_{\kappa,\lambda}^\phi C_\phi(X),$$

where $g_{\kappa,\lambda}^\phi = \sum_{\phi' = \phi} (\theta_{\phi'}^{\kappa,\lambda})^2$, which is defined in Constantine (1966, Equation (27)),

$$(2.8) \quad \int_{O(m)} C_\phi^{\kappa,\lambda}(AH'XH, AH'YH) dH = C_\phi^{\kappa,\lambda}(X, Y)C_\phi(A)/C_\phi(I),$$

$$(2.9) \quad \int_{R>0} \text{etr}(-WR) |R|^{-a-p} C_\phi^{\kappa,\lambda}(XR, YR) dR = \Gamma_m(a, \phi) |W|^{-a} C_\phi^{\kappa,\lambda}(XW^{-1}, YW^{-1}),$$

where $p = (m + 1)/2$,

$$(2.10) \quad \int_{O(m)} C_\phi^{\kappa,\lambda}(A'H'XHA, B) dH = C_\phi^{\kappa,\lambda}(A'A, B)C_\kappa(X)/C_\kappa(I),$$

$$(2.11) \quad \int_0^X |R|^{t-p} |I - R|^{u-p} C_\lambda(AR) dR \\ = \Gamma_m(p) |X|^t \sum_{k=0}^\infty \sum_{\kappa; \phi \in \kappa \cdot \lambda} \Gamma_m(t, \phi) (-u + p)_{\kappa} \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(X, AX) / k! \Gamma_m(t + p, \phi),$$

$$(2.12) \quad \int_0^I |R|^{t-p} |I - R|^{u-p} C_\phi^{\kappa,\lambda}(R, I - R) dR = \Gamma_m(t, \kappa) \Gamma_m(u, \lambda) \theta_\phi^{\kappa,\lambda} C_\phi(I) / \Gamma_m(t + u, \phi).$$

A multivariate generalization $L_\phi^t(X, A)$ of the Laguerre polynomial due to Khatri (1977) has an expansion

$$(2.13) \quad L_\phi^t(X, A) = (t + p)_\phi \sum_{\kappa,\lambda(\phi \in \kappa \cdot \lambda)} (t)_\kappa \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(A, -XA) / (t + p)_\lambda.$$

The following lemma is also useful.

LEMMA 2.1.

$$(2.14) \quad \begin{aligned} (i) \quad & \int_0^I \int_0^I |I - R - T|^{a-p} |R|^{b-p} |T|^{c-p} C_\phi^{\kappa,\lambda}(R, T) dR dT \\ & = \Gamma_m(a) \Gamma_m(b, \kappa) \Gamma_m(c, \lambda) \theta_\phi^{\kappa,\lambda} C_\phi(I) / \Gamma_m(a + b + c, \phi), \quad 0 < R + T < I \end{aligned}$$

$$(2.15) \quad \begin{aligned} (ii) \quad & \int_0^X |R|^{a-p} C_\phi^{\kappa,\lambda}(ARA', B) dR \\ & = \Gamma_m(p) \Gamma_m(a, \kappa) |X|^\alpha C_\phi^{\kappa,\lambda}(AXA', B) / \Gamma_m(a + p, \kappa). \end{aligned}$$

PROOF. (i) We evaluate the integral form

$$\Gamma = \int_{X>0} \int_{Y>0} \int_{Z>0} \text{etr} - (X + Y + Z) |X|^{a-p} |Y|^{b-p} |Z|^{c-p} C_\phi^{\kappa,\lambda}(Y, Z) dX dY dZ.$$

Making the transformations $U = X + Y + Z, R = U^{-1/2} Y U^{-1/2}, T = U^{-1/2} Z U^{-1/2}$ and using (2.9) gives

$$\Gamma = \Gamma_m(a + b + c, \phi) \Lambda,$$

where Λ is the left-hand side of (2.14). On the other hand, it may be shown that

$$\Gamma = \Gamma_m(a) \Gamma_m(b, \kappa) \Gamma_m(c, \lambda) \theta_\phi^{\kappa,\lambda} C_\phi(I).$$

Hence (2.14) follows.

(ii) The integral is given by the coefficient of $C_\phi^{\kappa,\lambda}(U, V) / k!!! C_\phi(I)$ in

$$\begin{aligned} & \int_{O(m)} \int_0^X |R|^{a-p} \text{etr}(ARA'H'UH) dR \text{etr}(BH'VH) dH \\ & = [\Gamma_m(p) \Gamma_m(a) / \Gamma_m(a + p)] |X|^\alpha \sum_{k,l=0}^\infty \sum_{\kappa,\lambda} (a)_\kappa \int_{O(m)} C_\kappa(AXA'H'UH) \\ & \quad \cdot C_\lambda(BH'VH) dH / k!!! (a + p)_\kappa, \quad \text{because of Constantine (1963, Equation (60)),} \end{aligned}$$

leading to the desired result (2.15) on using (2.2).

3. Joint distributions of the roots of D and A. The latent roots of $A = (S_1 + S_2)^{-1/2} S_1 (S_1 + S_2)^{-1/2}$ may be utilized as suitable preliminary test statistics for the hypothesis $\Sigma_1 = \Sigma_2$, where the S_i are independently distributed as Wishart $W_m(n_i, \Sigma_i), i = 1, 2$. The distributions of the roots of $S_2^{-1/2} S_1 S_2^{-1/2}$ have been considered for the case $\Omega_2 = 0$ by Davis (1980b) who expressed them in terms of the $C_\phi^{\kappa,\lambda}$. We consider the matrix $D = (S_1 + S_2)^{-1/2} S_0 (S_1 + S_2)^{-1/2}$ in connection with the discriminant problem under the heterogeneity of covariance matrices, where S_0 is distributed as noncentral Wishart $W_m(n_0, \Sigma_0, \Omega_0)$ independently of S_1 and S_2 . When $\Sigma_1 = \Sigma_2, D$ and A are independent and D is the well-

known test matrix for the null hypothesis $\Omega_0 = 0$ in MANOVA. When $\Sigma_1 \neq \Sigma_2$, they are dependent and we may be interested in the conditional distributions of the roots of D or suitable functions of them (conditional on the roots of A).

Let d_1, d_2, \dots, d_m and a_1, a_2, \dots, a_m denote the latent roots of D and A respectively ($d_1 > d_2 > \dots > d_m > 0, 1 > a_1 > a_2 > \dots > a_m > 0$). In this section, we shall derive the joint density function of d_1, \dots, d_m and a_1, \dots, a_m and the joint distribution functions of d_1 and a_1 . The ‘marginal’ and conditional distributions of d_1, \dots, d_m will be considered in the next section. It is shown that these distributions can be expressed in terms of the $C_\phi^{\kappa,\lambda}$ for the null case, $\Omega_0 = 0$.

Make the transformations

$$(3.1) \quad D = S^{-1/2}S_0S^{-1/2}, \quad A = S^{-1/2}S_1S^{-1/2}, \quad S = S_1 + S_2$$

in the joint density function of S_0, S_1 and S_2 ; then the Jacobian of the above transformation is $|S|^{2p}$. D and A have the same roots as $\bar{D} = H'DH$ and $\bar{A} = H'AH, H \in O(m)$, respectively; averaging over $O(m)$, we have the joint density function of \bar{D}, \bar{A} and S in the form, using (2.2),

$$(3.2) \quad f(\bar{D}, \bar{A}, S) = \left[\prod_{i=0}^2 \Gamma_m(\frac{1}{2}n_i) |2\Sigma_i|^{1/2n_i} \right]^{-1} \cdot |\bar{D}|^{1/2n_0-p} |\bar{A}|^{1/2n_1-p} |I - \bar{A}|^{1/2n_2-p} \text{etr}(-\frac{1}{2}\Sigma_2^{-1}S) |S|^{1/2\Sigma_{i=0}^2 n_i - p} \cdot \sum_{\kappa,\lambda;\phi}^{\infty} (-1)^f C_\phi^{\kappa,\lambda}(\Sigma_0^{-1}S, (\Sigma_1^{-1} - \Sigma_2^{-1})S) C_\phi^{\kappa,\lambda}(\bar{D}, \bar{A}) / k! l! 2^f C_\phi(I).$$

Integrating (3.2) over $S > 0$, using (2.9), yields the joint density function of \bar{D} and \bar{A} in the form

$$(3.3) \quad f(\bar{D}, \bar{A}) = C_{41} |\bar{D}|^{1/2n_0-p} |\bar{A}|^{1/2n_1-p} |I - \bar{A}|^{1/2n_2-p} \sum_{\kappa,\lambda;\phi}^{\infty} (-1)^f (\frac{1}{2}\Sigma_{i=0}^2 n_i)_\phi \cdot C_\phi^{\kappa,\lambda}(\Sigma_0^{-1}\Sigma_2, \Sigma_1^{-1}\Sigma_2 - I) C_\phi^{\kappa,\lambda}(\bar{D}, \bar{A}) / k! l! C_\phi(I),$$

where

$$(3.4) \quad C_{41} = \Gamma_m(\frac{1}{2}\Sigma_{i=0}^2 n_i) \left[\prod_{i=0}^2 \Gamma_m(\frac{1}{2}n_i) \prod_{j=0}^1 |\Sigma_j \Sigma_2^{-1}|^{1/2n_j} \right]^{-1}.$$

Here we should acknowledge the fact that (3.3) will not be convergent for all \bar{D} , which is a multivariate generalization of F .

The joint density function of d_1, \dots, d_m and a_1, \dots, a_m . By the usual method, the joint density function of d_1, \dots, d_m and a_1, \dots, a_m is derived from (3.3) in the form

$$(3.5) \quad f(d_1, \dots, d_m, a_1, \dots, a_m) = C_{42} |D|^{1/2n_0-p} |A|^{1/2n_1-p} |I - A|^{1/2n_2-p} \prod_{i < j}^m (d_i - d_j) \prod_{i < j}^m (a_i - a_j) \cdot \sum_{\kappa,\lambda;\phi}^{\infty} (-1)^f (\frac{1}{2}\Sigma_{i=1}^2 n_i)_\phi \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(\Sigma_0^{-1}\Sigma_2, \Sigma_1^{-1}\Sigma_2 - I) C_\kappa(D) C_\lambda(A) / k! l! C_\kappa(I) C_\lambda(I),$$

where $C_{42} = C_{41} [\pi^{1/2m^2} / \Gamma_m(\frac{1}{2}m)]^2$ with C_{41} given by (3.4).

The joint distribution function of d_1 and a_1 . We shall obtain the joint distribution of the largest roots d_1 and a_1 of D and A respectively, $P(0 < d_1 < \delta, 0 < a_1 < \alpha) \equiv P(0 < \bar{D} < \delta I, 0 < \bar{A} < \alpha I)$. It may be shown that

$$(3.6) \quad \int_0^{\delta I} |\bar{D}|^{1/2n_0-p} C_\phi^{\kappa,\lambda}(\bar{D}, \bar{A}) d\bar{D} = \Gamma_m(p) \Gamma_m(\frac{1}{2}n_0, \kappa) \theta_\phi^{\kappa,\lambda} C_\phi(I) \delta^{1/2mn_0+k} C_\lambda(\bar{A}) / \Gamma_m(\frac{1}{2}n_0 + p, \kappa) C_\lambda(I),$$

because of (2.15), and then (2.4),

$$\int_0^{\alpha I} |\bar{A}|^{1/2n_1-p} |I - \bar{A}|^{1/2n_2-p} C_\lambda(\bar{A}) d\bar{A}$$

$$(3.7) \quad = \Gamma_m(p) \sum_{s=0}^{\infty} \sum_{\sigma, \tau \in \sigma, \lambda} \Gamma_m(\frac{1}{2}n_1, \tau)(-\frac{1}{2}n_2 + p)_{\sigma}(\theta_{\tau}^{\sigma, \lambda})^2 C_{\tau}(I) \alpha^{1/2mn_1+t} / \Gamma_m(\frac{1}{2}n_1 + p, \tau) s!,$$

because of (2.11).

Thus, (3.6) and (3.7) with (3.3) establish the joint distribution function of d_1 and a_1 in the form

$$(3.8) \quad P(0 < d_1 < \delta, 0 < a_1 < \alpha) = C_{43} \delta^{1/2mn_0} \alpha^{1/2mn_1} \sum_{\kappa, \lambda; \phi}^{\infty} (-1)^f (\frac{1}{2} \sum_{i=0}^2 n_i)_{\phi} (\frac{1}{2} n_0)_{\kappa} \theta_{\phi}^{\kappa, \lambda} \\ C_{\phi}^{\kappa, \lambda} (\Sigma_0^{-1} \Sigma_2, \Sigma_1^{-1} \Sigma_2 - I) \delta^k \sum_{s=0}^{\infty} \sum_{\sigma, \tau \in \sigma, \lambda} (\frac{1}{2} n_1)_{\tau} (-\frac{1}{2} n_2 + p)_{\sigma} g_{\sigma, \lambda}^{\tau} C_{\tau}(I) \alpha^t / k! l! s! (\frac{1}{2} n_0 + p)_{\kappa} (\frac{1}{2} n_1 + p)_{\tau} C_{\lambda}(I),$$

where

$$C_{43} = [\Gamma_m(p)]^2 \Gamma_m(\frac{1}{2} \sum_{i=0}^2 n_i) [\Gamma_m(\frac{1}{2} n_2) \prod_{i=0}^1 \Gamma_m(\frac{1}{2} n_i + p) |\Sigma_i \Sigma_2^{-1}|^{1/2n_i}]^{-1}.$$

4. Marginal and conditional distributions of the roots of D . Following the notation in Section 3, we shall derive the ‘marginal’ joint density function of d_1, \dots, d_m , the distribution function of d_1 and the density function of $\text{tr } D$, in this section. The conditional joint density function of d_1, \dots, d_m given a_1, \dots, a_m and the conditional distribution function $P(0 < d_1 < \delta | 0 < a_1 < \alpha)$ are obtained from these ‘marginal’ distributions and the results of Section 3.

The joint density function of \tilde{D} and \tilde{A} , (3.3), is integrated over $0 < \tilde{A} < I$ to obtain the density function of \tilde{D} ; making the transformation $\tilde{A} \rightarrow H' \tilde{A} H, H \in O(m)$, and averaging over $O(m)$ with the use of (2.10) and (2.4), and then averaging over $0 < \tilde{A} < I$ with the use of Constantine (1963, Equation (22)), we can establish the density function of \tilde{D} in the form

$$(4.1) \quad f(\tilde{D}) = C_{51} |\tilde{D}|^{1/2n_0-p} \sum_{\kappa, \lambda; \phi}^{\infty} (-1)^f (\frac{1}{2} n_1)_{\lambda} (\frac{1}{2} \sum_{i=0}^2 n_i)_{\phi} \theta_{\phi}^{\kappa, \lambda} \\ C_{\phi}^{\kappa, \lambda} (\Sigma_0^{-1} \Sigma_2, \Sigma_1^{-1} \Sigma_2 - I) C_{\kappa}(\tilde{D}) / k! l! (\frac{1}{2} (n_1 + n_2))_{\lambda} C_{\kappa}(I),$$

where $C_{51} = \Gamma_m(\frac{1}{2} \sum_{i=0}^2 n_i) [\Gamma_m(\frac{1}{2} n_0) \Gamma_m(\frac{1}{2} (n_1 + n_2)) \prod_{i=0}^1 |\Sigma_i \Sigma_2^{-1}|^{1/2n_i}]^{-1}$.

The joint density function of d_1, \dots, d_m . By the usual method, the joint density function of d_1, \dots, d_m is readily derived from (4.1) in the form

$$(4.2) \quad f(d_1, \dots, d_m) = C_{52} |D|^{1/2n_0-p} \prod_{i < j}^m (d_i - d_j) \sum_{\kappa, \lambda; \phi}^{\infty} (-1)^f (\frac{1}{2} n_1)_{\lambda} \\ \cdot (\frac{1}{2} \sum_{i=0}^2 n_i)_{\phi} \theta_{\phi}^{\kappa, \lambda} C_{\phi}^{\kappa, \lambda} (\Sigma_0^{-1} \Sigma_2, \Sigma_1^{-1} \Sigma_2 - I) C_{\kappa}(D) / k! l! (\frac{1}{2} (n_1 + n_2))_{\lambda} C_{\kappa}(I),$$

where

$$(4.3) \quad C_{52} = C_{51} \pi^{1/2m^2} / \Gamma_m(\frac{1}{2} m).$$

The distribution function of d_1 . The distribution function of $d_1, P(d_1 < \delta) \equiv P(\tilde{D} < \delta I)$, is obtained from (4.1) in the form, using Constantine (1963, Equation (22)),

$$(4.4) \quad P(d_1 < \delta) = C_{53} \delta^{1/2mn_0} \sum_{\kappa, \lambda; \phi}^{\infty} (-1)^f (\frac{1}{2} n_0)_{\kappa} (\frac{1}{2} n_1)_{\lambda} (\frac{1}{2} \sum_{i=0}^2 n_i)_{\phi} \theta_{\phi}^{\kappa, \lambda} \\ C_{\phi}^{\kappa, \lambda} (\Sigma_0^{-1} \Sigma_2, \Sigma_1^{-1} \Sigma_2 - I) \delta^k / k! l! (\frac{1}{2} n_0 + p)_{\kappa} (\frac{1}{2} (n_1 + n_2))_{\lambda},$$

where

$$C_{53} = \Gamma_m(p) \Gamma_m(\frac{1}{2} \sum_{i=0}^2 n_i) [\Gamma_m(\frac{1}{2} n_0 + p) \Gamma_m(\frac{1}{2} (n_1 + n_2)) \prod_{i=0}^1 |\Sigma_i \Sigma_2^{-1}|^{1/2n_i}]^{-1}.$$

The density function of $\text{tr } D$. By the method similar to that used for deriving the Hotelling trace in Davis (1980b, Section 8), the density function of $d = \text{tr } D$ equal to $\text{tr } \tilde{D}$,

is obtained from (4.1) in the form

$$(4.6) \quad f(d) = C_{54} d^{1/2mn_0-1} \sum_{\kappa, \lambda; \phi}^{\infty} (-1)^f (\frac{1}{2}n_0)_{\kappa} (\frac{1}{2}n_1)_{\lambda} (\frac{1}{2}\Sigma_{i=0}^2 n_i)_{\phi} \cdot \theta_{\phi}^{\kappa, \lambda} C_{\phi}^{\kappa, \lambda} (\Sigma_0^{-1}\Sigma_2, \Sigma_1^{-1}\Sigma_2 - I) d^k / k! l! (\frac{1}{2}mn_0)_{\kappa} (\frac{1}{2}(n_1 + n_2))_{\lambda},$$

where $C_{54} = \Gamma_m(\frac{1}{2}\Sigma_{i=0}^2 n_i) [\Gamma(\frac{1}{2}mn_0) \Gamma_m(\frac{1}{2}(n_1 + n_2))] \prod_{i=0}^1 |\Sigma_i \Sigma_2^{-1}|^{1/2n_i}]^{-1}$. (4.6) is a generalization of Constantine's series for the Hotelling trace (see Constantine (1966)). Presumably it may converge only for $|d| < 1$.

The conditional distributions of the roots of D given those of A . It is easily shown that the joint density function of the roots a_1, \dots, a_m of A is given by

$$(4.7) \quad f(a_1, \dots, a_m) = C_{55} |A|^{1/2n_1-p} |I - A|^{1/2n_2-p} \prod_{i < j}^m 1(a_i - a_j) {}_1F_0(\frac{1}{2}(n_1 + n_2); \Sigma_1^{-1}\Sigma_2 - I, -A),$$

where $C_{55} = \Gamma_m(\frac{1}{2}(n_1 + n_2)) \pi^{1/2m^2} [\Gamma_m(\frac{1}{2}m) \prod_{i=1}^2 \Gamma_m(\frac{1}{2}n_i) \cdot |\Sigma_1 \Sigma_2^{-1}|^{1/2n_1}]^{-1}$, and that the distribution function of a_1 is given by

$$(4.8) \quad P(a_1 < \alpha) = C_{56} [\alpha(1 - \alpha)^{-1}]^{1/2mn_1} {}_2F_1(\frac{1}{2}n_1, \frac{1}{2}(n_1 + n_2); \frac{1}{2}n_1 + p; -\alpha(1 - \alpha)^{-1}\Sigma_1^{-1}\Sigma_2),$$

where $C_{56} = \Gamma_m(p) \Gamma_m(\frac{1}{2}(n_1 + n_2)) [\Gamma_m(\frac{1}{2}n_2) \Gamma_m(\frac{1}{2}n_1 + p) |\Sigma_1 \Sigma_2^{-1}|^{1/2n_1}]^{-1}$, and the ${}_2F_1$ is the Gaussian hypergeometric function of matrix argument (see, e.g., James (1964)).

Hence, the conditional joint density function of d_1, \dots, d_m given a_1, \dots, a_m is obtained as

$$(4.9) \quad f(d_1, \dots, d_m | a_1, \dots, a_m) = f(d_1, \dots, d_m, a_1, \dots, a_m) / f(a_1, \dots, a_m),$$

where $f(d_1, \dots, d_m, a_1, \dots, a_m)$ and $f(a_1, \dots, a_m)$ are given by (3.5) and (4.7) respectively. When $\Sigma_1 = \Sigma_2$, (4.9) becomes independent of A in the form of the density function of the roots of the F matrix, expressed in terms of the ${}_1F_0$ hypergeometric function with two matrix arguments (see, e.g., James (1964, Equation (65))).

The conditional distribution function of d_1 given a_1 is obtained as

$$(4.10) \quad P(d_1 < \delta | a_1 < \alpha) = P(d_1 < \delta, a_1 < \alpha) / P(a_1 < \alpha),$$

where $P(d_1 < \delta, a_1 < \alpha)$ and $P(a_1 < \alpha)$ are given by (3.8) and (4.8) respectively. When $\Sigma_1 = \Sigma_2$, (4.10) reduces to the distribution function of the largest root of the F matrix, expressed in terms of the ${}_2F_1$ hypergeometric function of matrix arguments (see, e.g., Chikuse (1977, Equation (2.4))).

Before closing this paper it is noted that we can extend some of these results by introducing invariant polynomials with larger numbers of argument matrices, extending the work of Davis (1980a), (1980b). These will be discussed in subsequent papers (see, e.g., Chikuse (1980)).

Acknowledgment. The author would like to express her sincere thanks to Dr. A. W. Davis at the Division of Mathematics and Statistics, C.S.I.R.O., Australia for many valuable discussions during the course of the preparation of this paper. Also, thanks are due to the referee and the associate editor for very helpful comments.

REFERENCES

CHIKUSE, Y. (1977). Asymptotic expansions for the joint and marginal distributions of the latent roots of $S_1 S_2^{-1}$. *Ann. Inst. Statist. Math.* **21** 309-320.
 CHIKUSE, Y. (1980). Invariant polynomials with matrix arguments and their applications. In *Multivariate Analysis*. 53-68. (R.P. Gupta, ed.) North Holland, Amsterdam.

- CONSTANTINE, A. G. (1963). Some non-central distribution problems in multivariate analysis. *Ann. Math. Statist.* **34** 1270-1285.
- CONSTANTINE, A. G. (1966). The distribution of Hotelling's generalized T_0^2 . *Ann. Math. Statist.* **37** 215-225.
- DAVIS, A. W. (1980a). Invariant polynomials with two matrix arguments, extending the zonal polynomials. In *Multivariate Analysis V* (P. R. Krishnaiah, ed.). North-Holland, Amsterdam.
- DAVIS, A. W. (1980b). Invariant polynomials with two matrix arguments extending the zonal polynomials: applications to multivariate distribution theory. *Ann. Inst. Statist. Math.* To appear.
- GURLAND, J. and McCULLOUGH, R. S. (1962). Testing equality of means after a preliminary test of equality of variances. *Biometrika* **49** 403-417.
- HERZ, C. S. (1955). Bessel functions of matrix argument. *Ann. Math.* **61** 474-523.
- JAMES, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475-501.
- KHATRI, C. G. (1977). Distribution of a quadratic form in noncentral normal vectors using generalized Laguerre polynomials. *South African Statist. J.* **11** 167-179.
- MCCULLOUGH, R. S., GURLAND, J. and ROSENBERG, L. (1960). Small sample behavior of certain tests of the hypothesis of equal means under variance heterogeneity. *Biometrika* **47** 345-353.
- RAY, W. D. and PITMAN, E. N. T. (1961). An exact distribution of the Fisher-Behrens-Welch Statistic. *J. Roy. Statist. Soc. Ser. B* **23** 377-384.
- WEHRHAHN, K. and OGAWA, J. (1975). On the bias of the t -test used in the two means problem under conditions of heterogeneity of variances. Research Paper No. 29494, Depart. Math. and Statist., Univ. Calgary.
- WELCH, B. L. (1947). The generalization of Student's problem when several different population variances are unequal. *Biometrika* **34** 28-35.
- YAO, Y. (1965). An approximate degrees of freedom solution to the multivariate Behrens-Fisher problem. *Biometrika* **52** 139-147.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA 93106