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ROBUSTNESS OF ONE- AND TWO-SAMPLE RANK TESTS AGAINST GROSS ERRORS[1]

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The classical nonparametric hypotheses of symmetry and equality of distribution functions are extended to hypotheses of approximate symmetry and approximate equality, by allowing for gross errors, which is indispensable for practical applications. It is shown that for these hypotheses one- and two-sample rank statistics maintain their distribution freeness, which now refers to their stochastically extreme laws. These laws are evaluated asymptotically, also under similarly extended alternatives, in the author's previous local framework, which has not yet covered rank statistics due to a subtle asymptotic fine structure of infinitesimal neighborhoods. Consequences on asymptotic maximum size, minimum power and relative efficiency of rank tests are drawn. In particular, it is shown that if the scores are unbounded, then rank tests fail completely; and by suitable truncation of the classically optimal scores, an asymptotic maximin rank test is obtained.

1. Introduction. Because symmetry about zero of a distribution function and equality of two distribution functions are rather stringent properties that can be destroyed by the slightest amount of contamination, the classical nonparametric hypotheses in the one- and two-sample problems do not appear to be compatible with the inevitable occurrence of gross errors or other indeterminacies in practice, although distribution freeness of rank statistics for these hypotheses has sometimes been regarded as a kind of robustness; cf., e.g., Puri and Sen (1971), page 2. Further somewhat contradictory statements about the robustness of rank tests have been made by Hodges and Lehmann (1963), page 598, who use the unproven robustness of rank tests against gross errors as motivation for their rank estimators, discarding their earlier observation that the normal scores rank test is so sensitive to heavier tails that it loses its superiority to the Wilcoxon when there are only about 2.5% (rather innocent looking) outliers; cf. Hodges and Lehmann (1961), page 317.

The reason why a subject like this appears to be controversial may be that there is no unique answer; the robustness or not of rank tests may depend on the method by which we assess robustness. Unlike in robust estimation, where about three or four different approaches are known to describe the various aspects of robustness (cf. Huber (1977), Chapter II), such alternative ways have not yet been developed for robust testing, or only unconsciously or tentatively so. The topics of this paper shall be distribution freeness and infinitesimal robustness of rank tests, with the further concepts of qualitative robustness and breakdown point to follow in a subsequent paper ([26]).

To formally account for gross errors or other indeterminacies of distribution functions, we first enlarge the classical nonparametric null hypotheses to the nonparametric hypothesis of approximate symmetry in the one-sample case, and to the nonparametric hypothesis of approximate equality in the two-sample case. Thus we formulate a nonparametric and robust

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framework which, apparently, has also been demanded by Bickel and Lehmann (1975), Section 4 (neighborhoods of nonparametric models with natural parameters). It extends the previous robustness models that have been tied to parametric families, and simultaneously the classical nonparametric models which have used the above-mentioned stringent null hypotheses and, as for alternatives, have equally employed parametric families (cf., e.g., Hájek and Sidak (1967)). It is shown that for these hypotheses rank statistics maintain their distribution freeness, which now refers to the stochastically extreme laws of test statistics (Section 3).

These extreme laws are quantitatively evaluated in the asymptotic, infinitesimal framework of Rieder (1978), which seems to be the most canonical approach in view of the fact that test statistics are commonly standardized so as to achieve local power rather than just consistency of tests. The local alternatives, which we employ, similarly relax the classical requirement of stochastic ordering to ordering restricted to some compact, leaving the unreliable tails of distribution functions unspecified. By tedious calculations it is mathematically justified (Section 4) that rank statistics can indeed be formally subsumed under the result of Rieder (1978), which is necessary due to a subtle asymptotic fine structure of infinitesimal neighborhoods. At first glance, they may appear sufficiently small so that, despite the occurrence of asymptotically orthogonal sequences, a restriction to contiguous sequences won't (and actually does not) essentially affect the results. However, it is shown that they are rather sufficiently large so that the extreme laws can be approximated under contiguous sequences, which would not be true automatically for smaller neighborhoods (Section 6.1).

Section 5 states the consequences on asymptotic maximum size and minimum power, as well as on asymptotic relative efficiency of rank tests, which, aside from their invariance properties that are maintained in this framework and, for example, yield a robust version of the t-test, do not reveal extra robustness properties. Thus, on one hand, unbounded scores cause a total failure; on the other hand, by a suitable truncation of the classically optimal scores, an asymptotic maximin rank test can be obtained. Also the classical ARE-results are invalidated; for example, under sufficiently contaminated normal shift alternatives, the sign-test surpasses the Wilcoxon, which even may turn biased.

In order not to overload the presentation, the exposition concentrates on the one-sample case, the two-sample analog being briefly treated in Section 6.2. In Section 6.3 the possibility of other asymptotic approaches is mentioned.

2. Preliminaries. This section introduces the notation for the one-sample case and presents two auxiliary results which will be used throughout.

The sample space in this paper is the extended real line $[-\infty, +\infty]$, endowed with its Borel σ-field $\mathscr{B}$. We denote by $\mathscr{M}$ the set of all probability measures (pm's) on $\mathscr{B}$, by $\mathscr{M}_c$ the subset of $\mathscr{M}$ that corresponds to continuous distribution functions (df's) which assign probability zero to $-\infty$ and $+\infty$; and by $\mathscr{M}_{cs}$ the subset of $\mathscr{M}_c$ that corresponds to df's which are symmetric about zero. A pm G shall be identified with its df and with its expectation operator; hence the identities $G(t) = G([-\infty, t])$ for $t \in [-\infty, +\infty]$, $G(\{t\}) = G(t) - G(t-0)$ for $t \in (-\infty, +\infty]$, and $G(B) = G(I_B) = \int_B dG$ for $B \in \mathscr{B}$. The left-continuous pseudoinverse of G is given by $G^{-1}(s) = \inf\{t \in [-\infty, +\infty] \mid G(t) \geqq s\}$, $s \in (0, 1)$. Note that, if the random variable (rv) u is distributed uniformly on (0, 1) (to be abbreviated subsequently as $u \sim \lambda$), then $G^{-1}(u) \sim G$, even if $G(\{-\infty, +\infty\}) > 0$. For n, a positive integer, and $G_1, \cdots, G_n \in \mathscr{M}$ the symbol $\otimes_{i=1}^n G_i$ stands for the stochastic product of $G_1, \cdots G_n$, and if $\mathscr{P} \subset \mathscr{M}$, then $\mathscr{P}^{(n)} = \{W_n \mid W_n = \otimes_{i=1}^n G_i, G_i \in \mathscr{P}, i = 1, \cdots, n\}$ and $\mathscr{P}^n = \{G^n \mid G^n = \otimes_{i=1}^n G, G \in \mathscr{P}\}$; these classes represent all possible joint laws of the observations $x_1, \cdots, x_n$ which are always assumed to be independent and, sometimes, identically distributed, when the law of the single observation varies over the class $\mathscr{P}$. This $\mathscr{P}$ is usually a neighborhood (nbd) of the following kind. Given $F \in \mathscr{M}$ and $\epsilon, \delta \in [0, 1]$, $0 < \epsilon + \delta < 1$, the gross error nbd $\mathscr{P}$ of F with radii ϵ, δ is defined as $\mathscr{P} = \mathscr{P}(F; \epsilon, \delta) = \{G \in \mathscr{M} \mid \forall B \in \mathscr{B}, G(B) \geqq (1-\epsilon)F(B) - \delta\}$. These nbd's conveniently generalize ϵ-contamination and total variation nbd's by means of a composition, cf. [23], Remark 5, page 1082, and at least in these more special forms, they have been used in robust statistics from the start. While ϵ-contamination nbd's are not defined by a metric total variation nbd's can be generated by the total variation distance, which is given by $\|G - F\| = \sup\{|G(B) - F(B)| \mid B \in \mathscr{B}\}$.

The rank tests considered in this paper are obtained by inserting a rank statistic into a test of the form $\varphi_{\gamma,k}(t) = (1-\gamma)I(t > k) + \gamma I(t \geqq k)$, $t \in [-\infty, +\infty]$, where γ is the randomization constant and k the critical value. Because discontinuous df's ought not to be ruled out by artificial assumptions, our definition of rank statistics must respect ties among the absolute values of the observations $x_1, \cdots, x_n$, and also zero observations, cf. Hájek and Sidak (1967), page 118. Although one can think of other combinations of treating the two types of ties, we restrict ourselves to either averaging or randomization, for lucidity's sake; the particular treatment of ties does not affect the results anyway. Now, for every sample size n, let scores $a_n(1), \cdots, a_n(n) \in (-\infty, +\infty)$ be given. We arrange the ordered absolute values of $x_1, \cdots, x_n$ into $g+1$ different ties (tie 0 possibly empty) so that $0 = |x|^{(1)} = \cdots = |x|^{(\tau_0)} < |x|^{(\tau_0+1)} = \cdots = |x|^{(\tau_1)} < \cdots < |x|^{(\tau_g)}$; tie k has length $\mu_k = \tau_k - \tau_{k-1}$, $k = 0, \cdots, g$; $\tau_{-1} = 0$, $\tau_g = n$. Then averaged scores $\bar{a}_n(\cdot)$ are defined by

$$\bar{a}_n(i) = \mu_k^{-1} \sum \{a_n(j) \mid \tau_{k-1} < j \leqq \tau_k\}, \tau_{k-1} < i \leqq \tau_k, \quad k = 0, \cdots, g, \tag{2.1}$$

and the average scores rank statistic $\bar{R}_n$ is defined to be of the form

$$\bar{R}_n = n^{-1/2} \sum_{i=1}^n (\operatorname{sign}(x_i) + \tfrac{1}{2} I(x_i = 0))\bar{a}_n(\overset{+}{r}_i); \tag{2.2}$$

here and subsequently, $\operatorname{sign}(t) = -1, 0, 1$ according to $t < 0, = 0, > 0$, and the $\overset{+}{r}_i$'s denote the absolute ranks,

$$\overset{+}{r}_i = \sum_{j=1}^n I(|x_j| \leqq |x_i|), \qquad i = 1, \cdots, n. \tag{2.3}$$

For the randomization method, additional random variables are required: $u_1, \cdots, u_n$ i.i.d. $\sim \lambda$, and $b_1, \cdots, b_n$ i.i.d. $\sim \mathscr{B}(1, \tfrac{1}{2})$ (binomial), such that the u-, b-vectors and the x-vector are independent. Then the randomized absolute ranks $\overset{+}{r}{}_i^*$ can be defined as

$$\overset{+}{r}{}_i^* = \sum_{j=1}^n I(\overset{+}{r}_j + u_j \leqq \overset{+}{r}_i + u_i), \qquad i = 1, \cdots, n; \tag{2.4}$$

thus, the observations in tie k inherit their ranks from the corresponding u_j's. The randomized scores rank statistic R_n^* is by definition of the form

$$R_n^* = n^{-1/2} \sum_{i=1}^n \operatorname{sign}^*(x_i) a_n(\overset{+}{r}{}_i^*), \tag{2.5}$$

where $\operatorname{sign}^*(x_i) = -1$ if $x_i < 0$, or $x_i = 0$ and $b_i = 0$, and $\operatorname{sign}^*(x_i) = 1$ if $x_i > 0$, or $x_i = 0$ and $b_i = 1$. The common symbol for $\bar{R}_n$ and R_n^* will be R_n.

Although it does not seem to be essential for the results of this paper to hold, we assume for convenience that the scores $a_n(\cdot)$ are nonnegative and increasing,

$$0 \leqq a_n(1) \leqq \cdots \leqq a_n(n). \tag{2.6}$$

Then the rank statistics R_n are stochastically increasing.

PROPOSITION 2.1. *Let the rank statistic R_n be given by* (2.2) *or* (2.5), *and assume* (2.6). *Then the law $(\otimes_{i=1}^n G_i) \circ R_n^{-1}$ is increasing in each argument G_i, $i = i, \cdots, n$, with respect to stochastic ordering.*

PROOF. We shall consider $\bar{R}_n$ first. By the argument of Lehmann (1959), Lemma 2, page 74, it suffices to show that, as a point function, $\bar{R}_n(x_1, \cdots, x_n)$ is increasing in each argument. In showing this, we may without restriction keep $x_2, \cdots, x_n$ fixed, while x_1 is moved from the left to the right. Several cases occur, which can be settled by similar arguments; to illustrate the idea, we sketch the following cases: x_1 leaving zero, $x_1 = |x_1|$ entering, and then leaving, tie k.

The only part of $n^{1/2}\bar{R}_n$ that is affected when x_1 leaves zero is $\tfrac{1}{2} \sum\{\bar{a}_n(\overset{+}{r}_i) \mid \overset{+}{r}_i = \tau_0\} = \tfrac{1}{2} \sum_{j=1}^{\tau_0} a_n(j)$. This expression turns to $\tfrac{1}{2} \sum_{j=1}^{\tau_0 - 1} a_n(j) + a_n(\tau_0)$, and thus increases by the amount of $\tfrac{1}{2} a_n(\tau_0)$.

When $x_1 = |x_1| = |x|^{(\tau_{k-1})}$ enters tie k from the left, the part of $n^{1/2}\bar{R}_n$ that corresponds to $\overset{+}{r}_i < \tau_{k-1}$ or $\overset{+}{r}_i > \tau_k$ is not affected. The remaining part is $a_n(\tau_{k-1}) + \sum\{\text{sign}(x_i)\bar{a}_n(\overset{+}{r}_i) \mid \overset{+}{r}_i = \tau_k\}$. The score of x_1 increases from $a_n(\tau_{k-1})$ to $(\mu_k + 1)^{-1}\sum\{a_n(j) \mid \tau_{k-1} \leqq j \leqq \tau_k\}$, while the common score of the x_i's in tie k decreases from $\mu_k^{-1} \sum\{a_n(j) \mid \tau_{k-1} < j \leqq \tau_k\}$ to $(\mu_k + 1)^{-1}\sum\{a_n(j) \mid \tau_{k-1} \leqq j \leqq \tau_k\}$. Therefore, the amount of increase is minimized if all x_i's in tie k are positive, and then it is zero.

When $x_1 = |x_1| = |x|^{(\tau_k)}$ leaves tie k, the only part of $n^{1/2}\bar{R}_n$ that is affected is $\sum\{\text{sign}(x_i)\bar{a}_n(\overset{+}{r}_i) \mid \overset{+}{r}_i = \tau_k\}$, where $\bar{a}_n(\tau_k) = \mu_k^{-1}\sum\{a_n(j) \mid \tau_{k-1} < j \leqq \tau_k\}$. The score of x_1 increases from this value to $a_n(\tau_k)$, the common score of the other x_i's in tie k decreases to $(\mu_k - 1)^{-1}\sum\{a_n(j) \mid \tau_{k-1} < j < \tau_k\}$. Therefore, the amount of increase is minimized if, again, all x_i's in tie k are positive, and then again, it is zero.

To treat R_n^*, we may again invoke Lehmann's lemma, in view of the independence of the u-, k-vectors and the x-vector, and must show that, as a point function, $R_n^*(x_1, \cdots, x_n)$ is increasing in each of its arguments, when the values of the randomization variables u_i, b_i, $i = 1, \cdots, n$, are fixed. Without restriction, keep $x_2, \cdots, x_n$ fixed and vary x_1. We sketch the argument for the following cases: x_1 increases while remaining negative, and x_1 entering zero from the left. The remaining cases can be treated similarly.

When x_1 increases and remains negative, then its absolute value decreases, and so does $\overset{+}{r}_1$, whereas $\overset{+}{r}_i$ increases for $i > 1$. Because the value of the randomization vector u is fixed, the same can be shown, by going back to definition (2.4), for the randomized absolute ranks: $\overset{+}{r}{}_1^*$ decreases, $\overset{+}{r}{}_i^*$ increases for $i > 1$. Then, because also the b-vector is fixed, $n^{1/2}R_n^* = 2 \sum \{a_n(\overset{+}{r}{}_i^*) \mid x_i > 0 \text{ or } x_i = 0, b_i = 1\} - \sum_{i=1}^n a_n(i)$ increases.

When x_1 approaches the tie at zero from the left, then $\overset{+}{r}{}_1^* = \overset{+}{r}_1 = \tau_0 + 1$, and the part of $n^{1/2}R_n^*$ that corresponds to $\overset{+}{r}{}_i^* > \tau_0 + 1$ won't be affected. The remaining part is given by the expression $\sum_{i \in I} \text{sign}^*(x_i)a_n(\overset{+}{r}{}_i^*) - a_n(\tau_0 + 1)$, where $I = \{i \mid x_i = 0\}$. Let $\hat{r}_i$, $i \in I$, be the ranks of u_i, $i \in I$. By definition, $\overset{+}{r}{}_i^* = \hat{r}_i$, $i \in I$. Let $\tilde{r}_i$, $i \in I \cup \{1\}$, denote the ranks of u_i, $i \in I \cup \{1\}$; put $I' = \{i \in I \mid u_i > u_1\}$. Then the above expression's increase amounts at least to

$$a_n(\tau_0 + 1) - a_n(\tilde{r}_1) - \textstyle\sum_{i \in I'}(a_n(\hat{r}_i + 1) - a_n(\hat{r}_i)),$$

which is nonnegative, since $\tilde{r}_1 \leqq \hat{r}_i \leqq \tau_0$ for $i \in I'$. □

The preceding proposition tells us in particular that a rank statistic R_n attains its stochastically extreme laws with respect to some $\mathscr{P}^{(n)}$, $\mathscr{P} \subset \mathscr{M}$, at the stochastically extreme df's in $\mathscr{P}$ itself (provided these exist). To be precise, the notion of stochastically extreme law or df is used in the following sense: a set $\mathscr{Q} \subset \mathscr{M}$ has a *stochastic supremum* iff there is a $G_0 \in \mathscr{M}$ such that, first, $G(t) \geqq G_0(t)$ for all $G \in \mathscr{Q}$, $t \in [-\infty, +\infty]$, and second, for every ζ, $\kappa \in (0, +\infty)$ there is a $G \in \mathscr{Q}$ such that $\sup_{|t|<\kappa} |G(t) - G_0(t)| < \zeta$; then we write $G_0 = \sup \mathscr{Q}$. The definition of *stochastic infimum* ($\inf \mathscr{Q}$) is entirely analogous. The usefulness of these notions, which will be extended to an asymptotic setting in Section 4, rests on the following applications: if $\varphi_{\gamma,k} \circ T_n$ denotes a test of the previously introduced form, based on a statistic $T_n : [-\infty, +\infty]^n \to [-\infty, +\infty]$, and if for some $\mathscr{P} \subset \mathscr{M}$, $G_0 = \sup\{W_n \circ T_n^{-1} \mid W_n \in \mathscr{P}^{(n)}\}$, $G_1 = \inf\{W_n \circ T_n^{-1} \mid W_n \in \mathscr{P}^{(n)}\}$, then $\sup\{W_n(\varphi_{\gamma,k} \circ T_n) \mid W_n \in \mathscr{P}^{(n)}\} = G_0(\varphi_{\gamma,k})$ and $\inf\{W_n(\varphi_{\gamma,k} \circ T_n) \mid W_n \in \mathscr{P}^{(n)}\} = G_1(\varphi_{\gamma,k})$.

The next proposition provides a further reduction, to the continuous and i.i.d. case. It also shows that for gross error nbd's of centers $F \in \mathscr{M}_c$ the particular treatment of ties has no influence on the extreme laws of rank statistics. Despite its technical nature, the proof also gives some insight into a peculiar robustness property of rank statistics that might be called "bring-in effect of ranks".

PROPOSITION 2.2. *If R_n is a rank statistic of form* (2.2) *or* (2.5) *and* (2.6) *holds, then for $F \in \mathscr{M}_c$ and $\mathscr{P} = \mathscr{P}(F; \epsilon, \delta)$ we have*

$$\sup\{W_n \circ R_n^{-1} \mid W_n \in \mathscr{P}^{(n)}\} = \sup\{G^n \circ R_n^{-1} \mid G \in \mathscr{P} \cap \mathscr{M}_c\}$$

$$\inf\{W_n \circ R_n^{-1} \mid W_n \in \mathscr{P}^{(n)}\} = \inf\{G^n \circ R_n^{-1} \mid G \in \mathscr{P} \cap \mathscr{M}_c\}.$$

PROOF. The nbd $\mathscr{P} = \mathscr{P}(F; \epsilon, \delta)$ has a stochastically largest element G_0, which is given by $G_0(t) = ((1-\epsilon)F(t) - \delta)^+$, $t < +\infty$. It puts mass $\epsilon + \delta$ to $+\infty$, and so it is not in $\mathscr{M}_c$. We would like to bring in this outlying mass smoothly without changing the law $G_0^n \circ R_n^{-1}$ much.

Observe the following phenomenon (bring-in effect of ranks). Let $k \in (0, +\infty)$ be fixed. If $t_1, \cdots t_n \in (-k, +\infty]$ and $\tilde{t}_1, \cdots, \tilde{t}_n \in (-k, +\infty]$ such that $|t_i| < k \Rightarrow \tilde{t}_i = t_i$ and $t_i \geqq k \Rightarrow \tilde{t}_i \geqq k$, for $i = 1, \cdots, n$, then $R_n(\tilde{t}_1, \cdots, \tilde{t}_n) = R_n(t_1, \cdots, t_n)$. (In the case $R_n = R_n^*$ we assume fixed values of the randomization rv's.) Thus, outliers to the right, and similarly to the left, are automatically brought in (e.g., $t_i = +\infty$ may be exchanged for $\tilde{t}_i = k$).

Now choose a $H_b \in \mathscr{M}_c$ such that $H_b(b) = 0$ and define $\tilde{G}_0 \in \mathscr{M}_c$ by $\tilde{G}_0(t) = G_0(t) + (\epsilon + \delta)H_b(t)$, $t < +\infty$. Then $\tilde{G}_0 \in \mathscr{P}$, because $\tilde{G}_0(B) \leqq G_0(B \cap (-\infty, +\infty)) + \epsilon + \delta \leqq (1-\epsilon)F(B) + \epsilon + \delta$ for $B \in \mathscr{B}$. Furthermore $\| \tilde{G}_0^n \circ R_n^{-1} - G_0^n \circ R_n^{-1} \| \leqq \tilde{G}_0^n(\exists i, \tilde{t}_i \leqq -b) + G_0^n(\exists i, t_i \leqq -b) \leqq 2nG_0(-b)$, and this bound tends to zero as $b \to +\infty$. (Here, the approximation of the stochastic supremum even holds in total variation distance.)

The argument for the second assertion runs similarly. □

REMARKS (1). This proposition remains true if $\mathscr{P}$ is a Kolmogorov or Lévy or Prokhorov nbd.

(2) If $\delta > 0$, we can generally achieve equality of $\tilde{G}_0^n \circ R_n^{-1}$ and $G_0^n \circ R_n^{-1}$. However, note that b may necessarily have to tend to infinity if $\delta \to 0$. Therefore, the bring-in effect of ranks may (and actually does) disappear for infinitesimal nbd's; cf. Section 5.

3. The nonparametric hypothesis of approximate symmetry, and distribution freeness of rank statistics. The classical nonparametric null hypothesis in the one-sample case is the set $\mathscr{M}_{cs}$, i.e., the underlying df F, which for convenience is assumed continuous, shall be tested for symmetry about zero, without any further consideration of its particular shape. For this problem rank statistics R_n of form (2.2) or (2.5) bring along a very attractive property: the law $F^n \circ R_n^{-1}$ does not depend on the particular $F \in \mathscr{M}_{cs}$; more generally, if π denotes a probability density with respect to $\lambda_{(-1/2, 1/2)}$ (Lebesgue measure restricted to $(-\frac{1}{2}, \frac{1}{2})$), then for $F_\pi(dt) = \pi(F(t) - \frac{1}{2})\, F(dt)$ the law $F_\pi^n \circ R_n^{-1}$ does not depend on the particular $F \in \mathscr{M}_{cs}$. This distribution freeness is useful to guarantee uniform level α for $F \in \mathscr{M}_{cs}$, and it has also been used to obtain lucid, uniform power results under nonparametric alternatives (cf. Lehmann (1953) and Behnen (1972)).

However, symmetry about zero is a rather stringent requirement which, artificial and trivial cases excepted, will hardly ever be met in practice; already the slightest asymmetric contamination suffices to destroy the relation $F(-t) = 1 - F(t)$, $t \in [-\infty, +\infty]$. Consequently, the danger is imminent that an unmodified rank test becomes significant in the presence of only minor deviations from a symmetric df that we would readily neglect.

The situation concerning classical alternatives may be argued similarly: the optimality of a rank test, and the relative efficiency of two rank tests, are usually determined under the assumption of positively asymmetric df's, in the sense that,

$$\forall t \in [-\infty, +\infty], \qquad F(-t) \leqq 1 - F(t), \qquad \exists t_0 \in [-\infty, +\infty], \qquad F(-t_0) < 1 - F(t_0).$$

Also these relations can be destroyed by the slightest contamination (when positive mass is moved towards $-\infty$, for example). Thus the superiority of one rank test to another may depend on practically too restrictive alternatives. It will be shown quantitatively in Sections 4 and 5 that gross error deviations have indeed nonnegligible, and possibly disastrous effects. Therefore, a robustification of the classical nonparametric approach is needed.

The basic idea that also prevails in other areas of distributional robustness will be to replace single pm's by nbd's. On the type of nbd's and their size, one has to agree on a priori grounds. In our case we are lead to the following notion of *approximate symmetry*: a df G is approximately symmetric about zero iff there is a $F \in \mathscr{M}_{cs}$ such that $G \in \mathscr{P}(F; \epsilon, \delta)$. The nonparametric hypothesis of approximate symmetry is $\mathscr{H}_{\epsilon,\delta}^s = \cup \{ \mathscr{P}(F; \epsilon, \delta) \mid F \in \mathscr{M}_{cs} \}$. In the case of ϵ-contamination nbd's, $G \in \mathscr{H}_{\epsilon,0}^s$ means that we observe a continuous and symmetric rv with probability $1 - \epsilon$, and, with probability ϵ, a completely unspecified rv. In the case of total variation nbd's and $G \in \mathcal{M}_c$, $G \in \mathcal{H}^s_{0,\delta}$ is equivalent to $\sup\{|G(B) - G(-B)\| B \in \mathcal{B}\} \leqq 2\delta$. When the classical alternatives of positive asymmetry are enlarged in the same manner to collections of alternatives, one could equally strive for an interpretation (of approximate positive asymmetry, e.g., as positive asymmetry of df's on some compact). However, one should not care too much because the main purpose of alternatives, as formulated by Lehmann (1953), is to provide a technically feasible basis for power comparisons between the various tests. Furthermore, we choose our alternatives in generalization of the classical ones and in accordance with the type of deviations that occur under the null hypothesis.

Leaving quantitative aspects to other sections, our interest here concentrates on the question whether rank statistics preserve their distribution freeness in the following generalized sense.

DEFINITION. A statistic $T_n : [-\infty, +\infty]^n \to [-\infty, +\infty]$ is distribution free for a class $\mathcal{M}_0 \subset \mathcal{M}$ with respect to nbd's $\mathcal{P}(\cdot; \epsilon, \delta)$ iff for every $F_0 \in \mathcal{M}_0$ the stochastically extreme laws, $\sup\{W_n \circ T_n^{-1} | W_n \in \mathcal{P}(F_0; \epsilon, \delta)^{(n)}\}$ and $\inf\{W_n \circ T_n^{-1} | W_n \in \mathcal{P}(F_0; \epsilon, \delta)^{(n)}\}$, exist and do not depend on the particular $F_0 \in \mathcal{M}_0$.

REMARK. In view of tests of the form $\varphi_{\gamma,k} \circ T_n$, it would be sufficient to require that, separately, the supremum is independent of F_0, $F_0 \in$ null hypothesis and the infimum is independent of F_1, $F_1 \in$ alternative hypothesis, in order to get a "distribution free" test. But subsequently, the stronger condition is simultaneously fulfilled, or not. Moreover, it appears to lead to a more canonical notion.

THEOREM 3.1. *Let π be a probability density with respect to $\lambda_{(-1/2,1/2)}$, and let $\mathcal{M}_{cs}(\pi) = \{F_\pi \in \mathcal{M}_c | \exists F \in \mathcal{M}_{cs}, F_\pi(dt) = \pi(F(t) - ½)F(dt)\}$. Then rank statistics R_n of form* (2.2) *or* (2.5) *are distribution free for $\mathcal{M}_{cs}(\pi)$ with respect to* nbd's $\mathcal{P}(\cdot; \epsilon, \delta)$.

REMARK. While the theorem remains valid for Kolmogorov nbd's, rank statistics are not distribution free for $\mathcal{M}_{cs}$ with respect to Lévy and Prokhorov nbd's. These latter nbd's would lead to a notion of approximate symmetry that is not scale invariant (as follows from the subsequent proof); more severely, a look at the extreme case of one-point measures I_t, $|t| \leqq \delta$, reveals that such a notion would collide with the presumed unrestricted accuracy by which we can evaluate the sign-function.

PROOF. Without restriction, the argument concerning the stochastic supremum is given. For nbd's $\mathcal{P}(F_\pi; \epsilon, \delta)$, $F \in \mathcal{M}_{cs}$, this is attained, in view of Proposition 2.1, under G^n where $G(t) = ((1 - \epsilon)F_\pi(t) - \delta)^+$, $t < +\infty$. (If $\epsilon = 0$, then the same G is the stochastic supremum of the Kolmogorov nbd of radius δ centered at F_π, so these nbd's are covered automatically.)

Let $x \sim F_\pi$, and $c \sim \mathcal{B}(1, \epsilon)$, independent of x (and the randomization rv's). Define the rv y by

$$y = (xI(c = 0) + \infty I(c = 1))I(x > a) + \infty I(x \leqq a),$$

where $a = F_\pi^{-1}(\delta/(1 - \epsilon))$. Then y has df G and, without changing its law, it can be represented as

$$y = (F^{-1}(½ + v)I(c = 0) + \infty I(c = 1))I(v > a_0) + \infty I(v \leqq a_0),$$

where $v \sim F_{0\pi}$, $F_0 = \lambda_{(-1/2,1/2)}$, and $a_0 = F_{0\pi}^{-1}(\delta/(1 - \epsilon))$. Indeed, by the definitions, we have $F^{-1}(½ + v) \sim F_\pi$ and $F^{-1}(½ + s) \leqq a \Leftrightarrow s \leqq a_0$, for $|s| < ½$. Let the transformation $f: (-½, ½) \cup \{+\infty\} \to [-\infty, +\infty]$ be defined by $f(s) = F^{-1}(½ + s)$, $|s| < ½$, and $f(+\infty) = +\infty$. Then we have $y = f(w)$, where w is the rv $(vI(c = 0) + \infty I(c = 1))I(v > a_0) + \infty I(v \leqq a_0)$, whose df is the stochastic supremum of $\mathcal{P}(F_{0\pi}; \epsilon, \delta)$. Note that the function f is strictly increasing, and that $f(-s) = -f(s + 0)$, $|s| < ½$.

Therefore, if w_i, $i = 1, \cdots, n$, are n independent copies of w, and $y_i = f(w_i)$, then the vector of signs and absolute ranks based on the w_i's coincides almost surely with that computed from the y_i's (the only possible exceptional set is included in $\{\exists i,\ w_i = 0\}$). This implies $R_n(y_1, \cdots, y_n) = R_n(w_1, \cdots, w_n)$ almost surely, and the theorem is proved.

The remark is based on the following observations. Take a $F_1 \in \mathscr{M}_{cs}$ with compact support. Note that the stochastic supremum of the Lévy nbd, and also of the Prokhorov nbd, which is centered at F_1 and has radius $\delta \in (0, 1)$, is given by $G_1(t) = (F_1(t - \delta) - \delta)^+$, $t < +\infty$. This is the df of the rv $y_1 = (x_1 + \delta)I(x_1 > a_1) + \infty I(x_1 \leqq a_1)$, where $x_1 \sim F_1$ and $a_1 = F_1^{-1}(\delta)$. Now make scale changes, $x_\sigma = \sigma^{-1}x_1$, $\sigma \in (0, +\infty)$. This gives $F_\sigma(t) = F_1(\sigma t)$, $t \in [-\infty, +\infty]$, and $a_\sigma = F_\sigma^{-1}(\delta) = \sigma^{-1}a_1$; moreover, with self-explanatory notation, $y_\sigma = (\sigma^{-1}x_1 + \delta)I(x_1 > a_1) + \infty I(x_1 \leqq a_1)$. Because R_n is invariant under scale changes, we may equivalently consider $\sigma y_\sigma = (x_1 + \sigma\delta)I(x_1 > a_1) + \infty I(x_1 \leqq a_1)$. If σ is large enough, then σy_σ is positive with probability 1, and so $R_n = n^{-1/2} \sum_{i=1}^n a_n(i)$ almost surely. For sufficiently small σ however, $x_1 + \sigma\delta < 0$ holds with positive probability, hence $R_n < 0$ with positive probability. □

The quantitative evaluation of the extreme laws of a rank statistic with respect to gross error nbd's is certainly not easier than in the classical situation where one already resorts to asymptotic (as.) approximations. For ϵ-contamination, the extreme laws can at least be reduced explicitly to the classical case: for $F \in \mathscr{M}_c$, the stochastically largest law $G_0 = \sup\{W_n \circ R_n^{-1} \mid W_n \in \mathscr{P}(F;\epsilon, 0)^{(n)}\}$, for example, is given by

$$G_0(t) = \sum_{v=1}^n (1-\epsilon)^v \epsilon^{n-v} \sum_{d \in D_v} F^v(v^{1/2}R_v^d \leqq n^{1/2}t - \textstyle\sum_{i \notin d} a_n(i)) \\ + \epsilon^n I(n^{-1/2} \textstyle\sum_{i=1}^n a_n(i) \leqq t), \quad t < +\infty$$

where $D_v = \{d \subset \{1, \cdots, n\} \mid \#d = v\}$, and for $d \in D_v$, R_v^d is the rank statistic at sample size v, with scores $a_n(i)$, $i \in d$; $v = 1, \cdots, n$. In particular, distribution freeness is directly visible.

4. Extreme limit laws of rank statistics. In this section, an as. investigation of one-sample rank statistics shall be carried out in the model of [23]. This model employs local alternatives in the same manner as, e.g., Le Cam (1960), Hájek and Sidak (1967), Roussas (1972). As test statistics are commonly standardized so as to achieve local power rather than just consistency of tests, such a differential approach appears to be the most canonical way to assess the influence of outliers quantitatively. A peculiar feature of this model is that the nbd's have to shrink at an appropriate rate; otherwise, the maximin test based on least favorable pairs, at level α, would either degenerate to the test identically α (in case the nbd's shrink too slowly so that they overlap eventually) or it would be as. equivalent to the classically best test between the nbd-centers (in case the nbd's shrink too quickly).

Let $\{F_\theta \mid \theta \in \Theta\} \subset \mathscr{M}$ be a parametric family, whose parameter space Θ is a subset of $[-\infty, +\infty]$ and contains 0 in its interior, and which satisfies the following conditions (4.1)–(4.3).

(4.1) $$F_0 \in \mathscr{M}_{cs}.$$

(4.2) $$F_\theta \ll F_0, \qquad \theta \in \Theta.$$

There exists a function $\Lambda \in L_2(dF_0)$ such that, if f_θ denotes the F_0-density of F_θ, then

(4.3) $$\frac{f_\theta^{1/2} - 1}{\theta} \to \frac{1}{2}\Lambda \quad \text{in } L_2(dF_0) \text{ as } \theta \to 0.$$

Let parameters $\epsilon, \delta \in [0, +\infty)$ be given (they determine the size of the nbd's to follow) and some $\tau \in (0, +\infty)$ (which determines the distance between the classical null hypothesis and alternative). The following condition is imposed,

(4.4) $$0 < \eta < F_0(\Lambda^+); \qquad \eta = \frac{\epsilon + 2\delta}{\tau}.$$

The right-hand inequality is a disjointness condition (cf. Remark 4 of [23], page 1082); equivalently, it ensures strict as. unbiasedness of the as. maximin test (cf. (5.14), (5.15) below).

For all positive integers n, the numbers τ_n, ϵ_n and δ_n are defined by

(4.5) $$\xi_n = n^{-1/2}\xi, \qquad \xi = \tau, \epsilon, \delta.$$

Then the null hypothesis H_{0n} and the alternative H_{1n} at sample size n (sufficiently large) are of the form

$$(4.6) \qquad H_{0n} = \mathscr{P}(F_0; \epsilon_n, \delta_n)^{(n)}, \qquad H_{1n} = \mathscr{P}(F_{\tau_n}; \epsilon_n, \delta_n)^{(n)},$$

and the as. null hypothesis H_0 and the as. alternative H_1 are defined by

$$(4.7) \qquad H_j = \{(W_n) \mid \forall n,\ W_n \in H_{jn}\}, \qquad j = 0, 1.$$

If (T_n) is a sequence of statistics $T_n : [-\infty, +\infty]^n \to [-\infty, +\infty]$ that defines an as. test (φ_n) in the usual way, $\varphi_n = \varphi_{\gamma,k} \circ T_n$, where $\gamma \in [0, 1]$, $k \in (-\infty, +\infty)$, then as. approximations of the maximum size $\alpha_n(\varphi_n)$ and the minimum power $\beta_n(\varphi_n)$,

$$(4.8) \qquad \alpha_n(\varphi_n) = \sup\{W_n(\varphi_n) \mid W_n \in H_{0n}\}, \qquad \beta_n(\varphi_n) = \inf\{W_n(\varphi_n) \mid W_n \in H_{1n}\},$$

can be obtained from the extreme limit laws of (T_n). This latter notion generalizes the notion of stochastic extremum of a set $\mathscr{Q} \subset \mathscr{M}$, as introduced in Section 2, to the present asymptotic setting, where the class $\mathscr{Q}$ now consists of sequences of pm's, $\mathscr{Q} = \{(G_n)\}$. We shall say that $\mathscr{Q}$ has a *stochastic limit superior* iff there exists a $G_0 \in \mathscr{M}$ such that, first $G_0(t) \leqq \liminf_n G_n(t)$ for all $(G_n) \in \mathscr{Q}$, $t \in [-\infty, +\infty]$, and second, for every $\zeta, \kappa \in (0, +\infty)$ there is a sequence $(G_n) \in \mathscr{Q}$ such that $\limsup_n \sup_{|t|<\kappa} |G_n(t) - G_0(t)| < \zeta$; then we write $G_0 = \limsup \mathscr{Q}$. The definition of *stochastic limit inferior* ($\liminf \mathscr{Q}$) is entirely analogous. The usefulness of these notions rests on the following application to tests of the above form: if $G_0 = \limsup\{(W_n \circ T_n^{-1}) \mid (W_n) \in H_0\}$ and $G_1 = \liminf\{(W_n \circ T_n^{-1}) \mid (W_n) \in H_1\}$, then $\limsup_n \alpha_n(\varphi_n) = \sup\{\limsup_n W_n(\varphi_n) \mid (W_n) \in H_0\} = G_0(\varphi_{\gamma,k})$ and $\liminf_n \beta_n(\varphi_n) = \inf\{\liminf_n W_n(\varphi_n) \mid (W_n) \in H_1\} = G_1(\varphi_{\gamma,k})$.

For the determination of the extreme limit laws of one-sample rank statistics R_n of form (2.2) or (2.5) we make the following assumptions. The scores $a_n(i)$ are generated by a function $a : [0, 1) \to [0, +\infty)$ in either one of the two ways,

$$(4.9) \qquad \begin{aligned} a_n(i) &= a\left(\frac{i}{n+1}\right), && \text{for } i = 1, \cdots, n, \\ a_n(i) &= Ea(v_n^{(i)}), && \text{for } i = 1, \cdots, n, \end{aligned}$$

where $v_n^{(i)}$ denotes to the ith order statistics in a random sample of size n from the pm λ (as previously, Lebesgue measure on (0, 1)). The scores generating function a shall satisfy the following conditions.

$$(4.10) \qquad a \text{ is increasing, nonconstant, } a(0) = 0.$$

$$(4.11) \qquad \begin{gathered} a \text{ is absolutely continuous on } [0, r] \text{ for every } r \in (0, 1), \\ \text{and } \dot{a}(s) < K(1-s)^{-3/2+\vartheta} \text{ a.e. } \lambda, \text{ for some } K, \vartheta \in (0, +\infty). \end{gathered}$$

$$(4.12) \qquad \begin{gathered} a \text{ is Lipschitz bounded of order 1 on } [0, s_0) \text{ and convex on } (s_0, 1), \\ \text{for some } s_0 \in [0, 1]. \end{gathered}$$

To some extent, these conditions are dictated by the subsequent use of the Chernoff-Savage type theorem of Sen (1970). They are fulfilled, for example, by the functions $a(s) = s$ and $a(s) = \Phi^{-1}(\tfrac{1}{2} + \tfrac{1}{2}s)$, $s \in [0, 1)$ (Φ denoting the standard normal df), which correspond to the Wilcoxon and van der Waerden, Fisher-Yates one-sample rank tests; the sign-test ($a \equiv 1$) is already covered by the results of [23].

By virtue of Proposition 2.2, under these assumptions, the extreme limit laws of rank statistics over H_0 and H_1 may be calculated in the restricted model H_0', H_1' (continuous i.i.d. case),

$$(4.13) \qquad \begin{aligned} H_0' &= \{(G_n^n) \mid \forall n,\ G_n \in \mathscr{P}(F_0; \epsilon_n, \delta_n) \cap \mathscr{M}_c\}, \\ H_1' &= \{(G_n^n) \mid \forall n,\ G_n \in \mathscr{P}(F_{\tau_n}; \epsilon_n, \delta_n) \cap \mathscr{M}_c\}. \end{aligned}$$

Let the functional $R: \mathscr{M}_c \to [0, +\infty)$ be given by

$$R(G) = \int_{(0,+\infty)} a(G(t) - G(-t))G(dt), \qquad G \in \mathscr{M}_c. \tag{4.14}$$

The following uniform asymptotic normality is basic.

PROPOSITION 4.1. *Let (R_n) be a sequence of rank statistics of form* (2.2) *or* (2.5) *with the scores $a_n(i)$ and the scores generating function a satisfying* (4.9)–(4.12). *Then*

$$\mathscr{L}(R_n - n^{1/2}(2R(G_n) - \lambda(a)) \mid G_n^n) \Rightarrow \mathscr{N}(0, \lambda(a^2)) \quad \text{as} \quad n \to +\infty$$

for all $(G_n^n) \in H_0' \cup H_1'$.

PROOF. For $G \in \mathscr{M}_c$ let $G_+ \in \mathscr{M}_c$ be given by $G_+(t) = G(t) - G(-t)$, $t \in [0, +\infty]$, and define the (substochastic) measure ν_G on (0, 1) by

$$\nu_G(ds) = \frac{dG}{dG_+}(G_+^{-1}(s))\lambda(ds). \tag{4.15}$$

Moreover, let the function $\sigma^2: \mathscr{M}_c \to [0, +\infty)$ be defined by

$$\begin{aligned}\sigma^2(G) = \int\int \{&s \wedge t(1 - s \vee t)\dot{a}(s)\dot{a}(t) + (1 - s \vee t)a(s \wedge t)\dot{a}(s \vee t) \\ &- s \wedge t\dot{a}(s \wedge t)a(s \vee t)\}\nu_G(ds)\nu_G(dt) + \nu_G(a^2) - (\nu_G(a))^2.\end{aligned} \tag{4.16}$$

This functional defines the scaling constants that are given by formulae (2.13), (2.14) of Sen (1970), as can be seen upon a transformation to (0, 1) × (0, 1), which is possible in view of the continuity of G and the relation $G^{-1} \circ G = id_{[-\infty,+\infty]}$ a.e. G. As will be proved below, σ^2 is continuous at F_0 in the weak topology. This entails that $\lim_n \sigma^2(G_n) = ¼\lambda(a^2)$ for all $(G_n^n) \in H_0' \cup H_1'$, because $\sigma^2(F_0) = ¼\lambda(a^2)$ (cf., e.g., (3.39) of [29]) and even $\lim_n \| G_n - F_0 \| = 0$. Then we invoke Sen's (1970) Theorem 2.1 and Theorem 5.1 (apppealing to the considerations on pages 61 and 62 of [29] as for the choices of scores) and conclude that for all $(G_n^n) \in H_0' \cup H_1'$,

$$\mathscr{L}\left(R_n - n^{1/2}\left(2R(G_n) - n^{-1}\sum_{i=1}^n a\left(\frac{i}{n+1}\right)\right) \mid G_n^n\right) \Rightarrow \mathscr{N}(0, \lambda(a^2)) \quad \text{as} \quad n \to +\infty. \tag{4.17}$$

Then it remains to show that $\lim_n n^{1/2}(n^{-1}\sum_{i=1}^n a(i/(n+1)) - \lambda(a)) = 0$. To see this most easily, we insert the sequence $G_n = F_0$ into (4.17) and make use of the equality $R(F_0) = ½\lambda(a)$ and the asymptotic normality $\mathscr{L}(R_n \mid F_0^n) \Rightarrow \mathscr{N}(0, \lambda(a^2))$; this latter convergence is obtained from the asymptotic expansion

$$R_n - n^{-1/2}\sum_{i=1}^n \text{sign}(x_i)a(2F_0(|x_i|) - 1) \to_{F_0^n} 0 \quad \text{as} \quad n \to +\infty, \tag{4.18}$$

cf. Hájek and Sidak (1967).

To prove continuity of σ^2 at F_0, let the sequence (G_n), $G_n \in \mathscr{M}_c$, tend to F_0 weakly. Then $\sup_{t\in[-\infty,+\infty]} |G_n(t) - F_0(t)| \to 0$, $\sup_{t\in[0,+\infty]} |G_{n+}(t) - F_{0+}(t)| \to 0$, and hence $G_{n+}^{-1}(s) \to F_{0+}^{-1}(s)$ as $n \to +\infty$ at all continuity points of F_{0+}^{-1}. Because the measures ν_{G_n} have df's $\nu_{G_n}(s) = G_n(G_{n+}^{-1}(s)) - G_n(0)$ and $\nu_{F_0}(s) = ½s$, $s \in (0, 1)$, we conclude that $\nu_{G_n}(s) \to \nu_{F_0}(s)$ a.e. λ, and therefore $\nu_{G_n}(h) \to \nu_{F_0}(h)$ for all continuous $h: (0, 1) \to (-\infty, +\infty)$ with compact support. The set $C_c(0, 1)$ of such functions is dense in $L_1(d\lambda)$. Furthermore, the densities of the measures ν_{G_n} are uniformly bounded a.e. λ, because generally $dG/dG_+ \leqq 1$ a.e. G, for $G \in \mathscr{M}_c$. This convergence, therefore, extends to every $h \in L_1(d\lambda)$, in particular to the functions $h = a$ and $h = a^2$. In order to show convergence of the double integral in the defining expression (4.16) we recall that the set of all finite sums of functions $(s, t) \to h_1(s)h_2(t)$, $h_1, h_2 \in C_c(0, 1)$, is dense in $L_1(d\lambda \otimes d\lambda)$; so the same argument applies because by the Chernoff-Savage condition

(4.11), cf. also (3.17) of [29], the integrand is indeed in $L_1(d\lambda \otimes d\lambda)$. □

The determination of the extreme limit laws is thus reduced to the extreme limits of the centering constants. Their maximization and minimization is carried out for every n in Lemma 4.2. Lemma 4.3 contains the asymptotic evaluation. The assumptions are the same as for Proposition 4.1 (although weaker assumptions would suffice for Lemma 4.2). For the statement of the lemmas, let the functionals $\tilde{v}_n(a)$, $\tilde{u}_n(a)$ be defined by

$$\tilde{v}_n(a) = \int_{(1-\epsilon_n)F_0(0)-\delta_n}^{1-(\epsilon_n+\delta_n)} a\left(s - \left((1-\epsilon_n)F_0\left(-F_0^{-1}\left(\frac{s+\delta_n}{1-\epsilon_n}\right)\right) - \delta_n\right)^+\right)\lambda(ds) + \int_{1-(\epsilon_n+\delta_n)}^{1} a\,d\lambda, \tag{4.19}$$

$$\tilde{u}_n(a) = \int_{(1-\epsilon_n)F_{\tau_n}(0)+\epsilon_n+\delta_n}^{1} a\left(s - \left((1-\epsilon_n)F_{\tau_n}\left(-F_{\tau_n}^{-1}\left(\frac{s-(\epsilon_n+\delta_n)}{1-\epsilon_n}\right)\right) + \epsilon_n + \delta_n\right)\right)\lambda(ds).$$

(That these expressions are well defined for sufficiently large n follows from the subsequent proofs.)

LEMMA 4.2. *It holds that*

$$\sup\{R(G_n) \mid G_n \in \mathscr{P}(F_0; \epsilon_n, \delta_n) \cap \mathscr{M}_c\} = \tilde{v}_n(a),$$
$$\inf\{R(G_n) \mid G_n \in \mathscr{P}(F_{\tau_n}; \epsilon_n, \delta_n) \cap \mathscr{M}_c\} = \tilde{u}_n(a).$$

PROOF. Independently of the asymptotic setting we shall find the extrema of R with respect to $\mathscr{P} = \mathscr{P}(F; \epsilon, \delta) \cap \mathscr{M}_c$, where $F \in \mathscr{M}_c$ and $\epsilon, \delta \in [0, 1)$, $0 < \epsilon + \delta < 1$. Without restriction, we shall give the argument concerning the supremum. Since only continuous df's G are considered, for which $G^{-1} \circ G = id_{[-\infty,+\infty]}$ a.e. G generally, we may make a transformation to the unit interval so as to obtain that $R(G) = \int_{G(0)}^1 a(s - G(-G^{-1}(s)))\lambda(ds)$. Because a is increasing, nonnegative, it follows from this representation that R is increasing with respect to stochastic ordering. Moreover, for all $G \in \mathscr{P}$, we have $G(t) \geqq ((1-\epsilon)F(t) - \delta)^+$, $t \in [-\infty, +\infty]$, and $G^{-1}(s) \geqq F^{-1}((s+\delta)/(1-\epsilon))$, $s \in (0, 1-(\epsilon+\delta))$. Therefore,

$$R(G) \leqq \int_{((1-\epsilon)F(0)-\delta)^+}^{1-(\epsilon+\delta)} a\left(s - \left((1-\epsilon)F\left(-F^{-1}\left(\frac{s+\delta}{1-\epsilon}\right)\right) - \delta\right)^+\right)\lambda(ds) + \int_{1-(\epsilon+\delta)}^{1} a\,d\lambda, \qquad G \in \mathscr{P}.$$

It remains to show that this bound can be approximated arbitrarily closely. Consider the particular elements of $\mathscr{P}$ that are given by $G(t) = ((1-\epsilon)F(t) - \delta)^+ + (\epsilon + \delta)\lambda_{(k,k+1)}(t)$, $t \in [-\infty, +\infty]$, where $\lambda_{(k,k+1)}$ denotes Lebesgue measure restricted to $(k, k+1)$; $k \in (-\infty, +\infty)$. Assume $k > 0$ so that $G(0) = ((1-\epsilon)F(0) - \delta)^+$, and k even so large that $(1-\epsilon)F(k) - \delta > 0$; then $G^{-1}(s) = F^{-1}((s+\delta)/(1-\epsilon))$ for $s \leqq G(k)$. Because $s > ((1-\epsilon)F(0) - \delta)^+$ implies that $-F^{-1}((s+\delta)/(1-\epsilon)) < 0$, we have altogether that

$$R(G) = \int_{((1-\epsilon)F(0)-\delta)^+}^{G(k)} a\left(s - \left((1-\epsilon)F\left(-F^{-1}\left(\frac{s+\delta}{1-\epsilon}\right)\right) - \delta\right)^+\right)\lambda(ds) + \int_{G(k)}^{1} a(s - G(-G^{-1}(s)))\lambda(ds).$$

First, the second integral shall be shown to tend towards $\int_{1-(\epsilon+\delta)}^1 a\,d\lambda$ as $k \to +\infty$. Indeed, because $G(k) \to 1 - (\epsilon + \delta)$ and $a \in L_1(d\lambda)$, it suffices to bound the difference $\int_{G(k)}^1 \{a(s) - a(s - G(-G^{-1}(s)))\}\lambda(ds)$. Since a is increasing and $0 \leqq G(-G^{-1}(s)) \leqq G(-k)$ for $s > G(k)$, this difference can be bounded from above by $\int_{G(k)}^1 \{a(s) - a(s - G(-k))\}\lambda(ds) = \int_{1-G(-k)}^1 a d\lambda - \int_{G(k)-G(-k)}^{G(k)} a d\lambda$. By dominated convergence, this bound tends to zero as $k \to +\infty$. As for the first integral it suffices to show that $\int_{G(k)}^{1-(\epsilon+\delta)} a d\lambda \to 0$ as $k \to +\infty$, because of the monotonicity and nonnegativity of a. However, this is again obvious as $G(k) \to 1 - (\epsilon + \delta)$ and $a \in L_1(d\lambda)$. □

Subsequently, we set $a(1) = a(1 - 0)$, and let the function $IC: [-\infty, +\infty] \to [-\infty, +\infty]$ be defined by

$$IC(t) = \operatorname{sign}(t) a(2F_0(|t|) - 1), \qquad t \in [-\infty, +\infty]. \tag{4.20}$$

LEMMA 4.3. *It holds that*

$$\lim_n n^{1/2}(2\bar{v}_n(a) - \lambda(a)) = (\epsilon + 2\delta)a(1),$$
$$\lim_n n^{1/2}(2\tilde{u}_n(a) - \lambda(a)) = \tau F_0(IC\cdot\Lambda) - (\epsilon + 2\delta)a(1).$$

PROOF. In view of the continuity and symmetry of F_0 the expression for $\bar{v}_n(a)$ simplifies considerably so that

$$n^{1/2}(2\bar{v}_n(a) - \lambda(a)) = n^{1/2} \int_{1-(\epsilon_n + 2\delta_n)}^1 a d\lambda; \tag{4.21}$$

the latter integral tends to $(\epsilon + 2\delta)a(1)$, as a is increasing.

As for $\tilde{u}_n(a)$ note that, by virtue of (4.18) and Le Cam's third lemma, $\mathscr{L}(R_n \mid F_{\tau_n}^n) \Rightarrow \mathscr{N}(\tau F_0(IC\cdot\Lambda); \lambda(a^2))$ and consequently, in view of Proposition 4.1, $n^{1/2}(2R(F_{\tau_n}) - \lambda(a)) \to \tau F_0(IC\cdot\Lambda)$. Thus it remains to investigate the differences

$$\begin{aligned} &-2n^{1/2}(\tilde{u}_n(a) - R(F_{\tau_n})) \\ &\quad = n^{1/2} \int_{c_n'}^{c_n''} a(s - C_n'(s))\lambda(ds) + n^{1/2} \int_{c_n''}^1 \{a(s - C_n'(s)) - a(s - C_n''(s))\}\lambda(ds), \end{aligned} \tag{4.22}$$

where the following notation has been employed:

$$\begin{aligned} &c_n' = 2F_{\tau_n}(0) - 1, \qquad c_n'' = 2((1 - \epsilon_n)F_{\tau_n}(0) + \epsilon_n + \delta_n) - 1; \\ &C_n'(2s - 1) = F_{\tau_n}(-F_{\tau_n}^{-1}(s)) - (1 - s), \qquad s \in (F_{\tau_n}(0), 1); \\ &C_n''(2s - 1) = (1 - \epsilon_n)F_{\tau_n}\left(-F_{\tau_n}^{-1}\left(\frac{s - \epsilon_n - \delta_n}{1 - \epsilon_n}\right)\right) + (\epsilon_n + \delta_n) - (1 - s), \\ &\qquad s \in \left(\frac{1}{2}(1 + c_n''), 1\right). \end{aligned} \tag{4.23}$$

Because (4.3) entails that

$$\frac{f_\theta - 1}{\theta} \to \Lambda \quad \text{in } L_1(d\lambda) \text{ as } \theta \to 0, \tag{4.24}$$

it holds that $\| F_{\tau_n} - F_0 \| = O(n^{-1/2})$; hence $c_n', c_n'' = O(n^{-1/2})$ and $n^{1/2}(c_n'' - c_n') \to \epsilon + 2\delta$ as $n \to +\infty$. It will follow from (4.26), (4.27) below that also the maximum c_n of the sup-norms of the functions C_n' and C_n'' is of the order $O(n^{-1/2})$. Moreover, it will be shown that

$$\lim_n \sup_{s \in (1/2(1+c_n''), 1)} | n^{1/2}(C_n''(2s - 1) - C_n'(2s - 1)) - (\epsilon + 2\delta)| = 0. \tag{4.25}$$

Then the proof can be concluded in the following way. By a monotony argument, the first integral in (4.22) tends to $(\epsilon + 2\delta)a(0) = 0$. We split up the second integral according to assumption (4.12). By dominated convergence, the integral

$$n^{1/2}\int_{c''_n}^{s_0-c_n}\{a(s-C'_n(s))-a(s-C''_n(s))\}\,\lambda(ds)$$

$$=\int_{c''_n}^{s_0-c_n} n^{1/2}\,(C''_n(s)-C'_n(s))\cdot\frac{\{a(s-C'_n(s))-a(s-C''_n(s))\}}{C''_n(s)-C'_n(s)}\,\lambda(ds)$$

converges to $(\epsilon+2\delta)a(s_0)$, in view of (4.25), the Lipschitz boundedness of a, and its absolute continuity that ensures differentiability a.e. λ. The corresponding integral over $(s_0-c_n,\ s_0+c_n)$ obviously tends to zero. Because $\dot a$ is increasing on $(s_0,\ 1)$, outside some λ-nullset, the remaining integral over $(s_0+c_n,\ 1)$ is bounded from below by

$$\int_{s_o+c_n}^{1} n^{1/2}\,(C''_n(s)-C'_n(s))\dot a(s-c_n)\lambda(ds)$$

$$\geqq(\epsilon+2\delta)+o(1))(a(1-c_n)-a(s_o))\rightarrow(\epsilon+2\delta)(a(1)-a(s_o)).$$

It is bounded from above by

$$\int_{s_o+c_n}^{1-c_n} n^{1/2}\,(C''_n(s)-C'_n(s))\dot a(s+c_n)\,\lambda(ds)+n^{1/2}c_n(a(1)-a(1-c_n))$$

$$\leqq(\epsilon+2\delta+o(1))(a(1)-a(s_0+2c_n))+n^{1/2}c_n(a(1)-a(1-c_n))$$

$$\rightarrow(\epsilon+2\delta)(a(1)-a(s_0)).$$

Therefore, $2n^{1/2}(\tilde u_n(a)-R(F_{\tau_n}))\rightarrow-(\epsilon+2\delta)a(1)$, as desired.

In order to prove the outstanding formula (4.25), let a sequence (γ_n) be given such that $\gamma_n>0$, $\gamma_n\rightarrow 0$, and for every $s\in(0,\ 1)$ a sequence $(s_n)\subset(0,\ 1)$ such that $|s_n-s|<\gamma_n$. In view of (4.24) we have that $\|F_{\tau_n}-(F_0+\tau_n\Lambda\ dF_0)\|=o(n^{-1/2})$ and so

$$\text{(4.26)}\qquad \sup\nolimits_{t\in[-\infty,\,+\infty]}\left|F_{\tau_n}(-t)-\left\{1-F_{\tau_n}(t)+2\tau_n\int_{-\infty}^{t}\tilde\Lambda\ dF_0\right\}\right|=o(n^{-1/2}),$$

where $\tilde\Lambda(t)=\frac{1}{2}(\Lambda(t)-\Lambda(-t))$, $t\in[-\infty,+\infty]$. Furthermore,

$$F_0^{-1}(s-\gamma_n-\|F_{\tau_n}-F_0\|)\leqq F_{\tau_n}^{-1}(s_n)\leqq F_0^{-1}(s+\gamma_n+\|F_{\tau_n}-F_0\|),$$

where we may set $F_0^{-1}(s)=-\infty$ for $s\leqq 0, F_0^{-1}(s)=+\infty$, for $s\geqq 1$, and so we obtain that

$$\text{(4.27)}\qquad \left|\int_{-\infty}^{F_{\tau_n}^{-1}(s_n)}\tilde\Lambda\ dF_0-\int_{-\infty}^{F_0^{-1}(s)}\tilde\Lambda\ dF_0\right|^2\leqq 2(\gamma_n+\|F_{\tau_n}-F_0\|)F_0(\Lambda^2).$$

Now insert $s_n=(s-(\epsilon_n+\delta_n))/(1-\epsilon_n)$ for $s\in(1/2(1+c''_n),\ 1)$ to conclude (4.25) directly from (4.26), (4.27). □

By means of Proposition 2.2, Proposition 4.1, Lemma 4.2 and Lemma 4.3, we have thus derived the following result about the extreme limit laws of one-sample rank statistics in the model $H_0\cup H_1$.

THEOREM 4.4. *Let* (R_n) *be a sequence of rank statistics of form* (2.2) *or* (2.5) *with the scores* $a_n(i)$ *and the scores generating function a satisfying conditions* (4.9)–(4.12). *Then*

$$\limsup\{(W_n\circ R_n^{-1})\,|\,(W_n)\in H_0\}=\mathcal{N}((\epsilon+2\delta)a(1);\lambda(a^2)),$$

$$\liminf\{(W_n\circ R_n^{-1})\,|\,(W_n)\in H_1\}=\mathcal{N}(\tau F_0(IC\cdot\Lambda)-(\epsilon+2\delta)a(1);\lambda(a^2)),$$

where IC is given by (4.20).

Note that for this IC, in addition to $\lambda(a^2) = F_0(IC^2)$, also $a(1) = \sup IC = \sup_{[F_0]} IC = -\inf_{[F_0]} IC = -\inf IC$. So if we formally insert this IC into [23], then the preceding theorem agrees with the results obtained in this paper about the extreme limit laws of sequences of statistics T_n,

$$T_n = n^{-1/2} \sum_{i=1}^n IC(x_i), \tag{4.28}$$

where $IC \in L_2(dF_0)$, $F_0(IC) = 0$.

REMARK. As mentioned in Remark 6, page 1082, of [23], this paper can be modified so that the parameter sequence $(-\tau_n)$ is exhanged for the sequence identically zero. Moreover, it should be noted that all arguments go through under the present weaker differentiability assumptions about the densities. Condition (4.24) is sufficient for most conclusions (apart from the log likelihood expansion in Section 4 of [23], for which (4.3) is required). Where the proof of Lemma 4.5 of [23] employs pointwise convergence, a subsequence argument can be used instead.

Statistics of form (4.28) were considered because many other test statistics prove to be as. equivalent (in a suitable sense) to statistics of this kind; as for rank statistics, such an as. equivalence, in the sense of equality of the extreme limit laws, is demonstrated by the preceding theorem for the present context. It should be noted that the Hájek and Sidak (1968) approximation (4.18) is not sufficient for this model, although it extends, of course, to all sequences (W_n) which are contiguous to (F_o^n) (to be denoted by $(W_n) \ll (F_o^n)$). Unfortunately, the as. hypotheses H_0 and H_1 (as well as H_0' and H_1') contain sequences which are not contiguous, and even asymptotically orthogonal, to (F_o^n); cf. Proposition 6.1 of [23]. Therefore, the stronger Chernoff-Savage approximation of Sen (1970) was used. (After a simplification of the centering constants in the manner of Hoeffding (1973), the results of Huskova (1970), which are based on Hájek's (1968) projection method, would be equally suited.) It is true that also by these other techniques rank statistics R_n are approximated (in probability) by sums T_n of form (4.18), but with $IC = IC_n$ depending on n and, moreover, on the particular sequence (W_n) considered; cf. (3.4)–(3.6) of [29] and (12), (13) of [15]. Rather than approximating these various IC_n by some unique IC so that the results of [23] would become directly applicable, it turned out to be easier to straightforwardly evaluate the normalizing constants.

5. Local robustness of rank tests. In this section we collect and state the consequences of Theorem 4.4 on the as. maximum size and as. minimum power of one-sample rank tests, as well as on the as. relative efficiency. Moreover, the established correspondence via IC is utilized to derive an as. maximin rank test from the results of [23]. Finally, examples are given. Assumptions and notation are those of the preceding section.

To illustrate the influence of gross errors on size and power of rank tests, let $(\varphi_n) = (\varphi_{\gamma_n, k_n} \circ R_n)$ be the classical sequence of one-sample rank tests for (F_o^n) vs. $(F_{\tau_n}^n)$ at as. level α, where $(\gamma_n) \subset [0, 1]$ and $(k_n) \subset (-\infty, +\infty)$ such that $\lim_n k_n = k_\alpha(a)$,

$$k_\alpha(a) = u_\alpha(\lambda(a^2))^{1/2}, \qquad u_\alpha = \Phi^{-1}(1-\alpha); \tag{5.1}$$

thus

$$\begin{aligned} &\lim\nolimits_n F_0^n(\varphi_n) = \alpha, \\ &\lim\nolimits_n F_{\tau_n}^n(\varphi_n) = 1 - \Phi\left(u_\alpha - \frac{\tau F_0(IC \cdot \Lambda)}{(\lambda(a^2))^{1/2}}\right) \end{aligned} \tag{5.2}$$

where IC is given by (4.20). We have the following corollary to Theorem 4.4.

COROLLARY 5.1. *The* as. *maximum size and* as. *minimum power of the original* as. *rank test* (φ_n) *are given by*

$$\lim_n \alpha_n(\varphi_n) = 1 - \Phi\left(u_\alpha - \frac{(\epsilon + 2\delta)a(1)}{(\lambda(a^2))^{1/2}}\right) \tag{5.3}$$
$$\lim_n \beta_n(\varphi_n) = 1 - \Phi\left(u_\alpha - \frac{(\tau F_0(IC\cdot\Lambda) - (\epsilon + 2\delta)a(1))}{(\lambda(a^2))^{1/2}}\right).$$

Thus, if the scores generating function is unbounded, a drastic breakdown occurs. If $a(1) < +\infty$, then it makes sense to adjust (i.e., increase) the as. critical value so as to maintain as. level α. Let in this case $l_\alpha(a)$ be defined by

$$l_\alpha(a) = k_\alpha(a) + (\epsilon + 2\delta)a(1), \tag{5.4}$$

and denote by (ψ_n) the sequence of modified rank tests, $\psi_n = \varphi_{\gamma_n, l_n} \circ R_n$, where $\lim_n l_n = l_\alpha(a)$.

COROLLARY 5.2. *If* $a(1) < +\infty$, *then the* as. *maximum size and* as. *minimum power of the adjusted* as. *rank test* (ψ_n) *are given by*

$$\lim_n \alpha_n(\psi_n) = \alpha, \tag{5.5}$$
$$\lim_n \beta_n(\psi_n) = 1 - \Phi\left(u_\alpha - \tau\frac{(F_0(IC\cdot\Lambda) - 2\eta a(1))}{(\lambda(a^2))^{1/2}}\right).$$

Towards the derivation of an as. maximin rank test, let us first recall that in the classical situation the sequence of statistics $T_n = n^{-1/2} \sum_{i=1}^n \Lambda(x_i)$ defines an as. maximin test for (F_o^n) vs. $F_{\tau_n}^n)$ at some as. level α. By virtue of (4.18) this test is as. equivalent (in the sense of equal as. size, as. power under (F_o^n), $(F_{\tau_n}^n)$) to an as. rank test based on the scores generating function $\hat{a}$ (the scores $a_n(i)$ satisfying $\lim_n \int (a_n([1 + ns]) - \hat{a}(s))^2\lambda\,(ds) = 0$ would be sufficient at this point) provided that the following relation can be achieved,

$$\Lambda(t) = \operatorname{sign}(t)\hat{a}(2F_0(|t|) - 1), \qquad t \in [-\infty, +\infty]; \tag{5.6}$$

and then the as. rank test based on $\hat{a}$ is as. maximin for H_0 vs. H_1 among all as. tests at a prescribed as. level. This is outlined here in detail because the correspondence (5.6) makes the following restriction on the kind of alternatives, for which one-sample rank tests can be optimal, necessary, which has not yet been spelled out explicitly although it is inherent in, e.g., Hájek and Sidak (1967) and Behnen (1972), namely that

$$\Lambda(-t) = -\Lambda(t), \qquad t \in [-\infty, +\infty]. \tag{5.7}$$

This condition entails that the law of the absolute value $|x|$ is the same (up to as. negligible remainder terms of order $o(n^{-1/2})$ in total variation) under F_0 as under F_{τ_n}. Then the classically optimal choice of $\hat{a}$ is

$$\hat{a}(s) = (\Lambda \circ F_0^{-1})\,(½ + ½\, s), \qquad s \in [0, 1). \tag{5.8}$$

Second, in deriving an as. maximin rank test for H_0 vs. H_1 at some as. level α, recall Theorem 3.7 and Theorem 4.4 of [23] (slightly modified to account for the substitution of $(F_{-\tau_n}^n)$ by (F_o^n)), which tell us that the appropriate choice of the IC-function is

$$IC_*(t) = d_0 \vee \Lambda(t) \wedge d_1, \qquad t \in [-\infty, +\infty], \tag{5.9}$$

where the truncation points d_0 and d_1 are defined by

$$F_0((d_0 - \Lambda)^+) = \eta = F_0((\Lambda - d_1)^+). \tag{5.10}$$

To make the fulfillment of (5.9) by the IC-function of a rank test possible, we retain assumption

(5.7); thus in particular $-d_0 = d_1$, and then the optimal choice of the scores generating function is

$$a_*(s) = \hat{a}(s) \wedge d_1, \qquad s \in [0, 1), \tag{5.11}$$

where $\hat{a}$ is given by (5.8). Moreover, to guarantee the fulfillment of (4.10)–(4.12) by this a_*, we assume that

(5.12) Λ is increasing, and continuously differentiable on $(-\infty, +\infty)$,

(5.13) F_0 is absolutely continuous, with a density that is continuous and strictly positive on $\{0 < F_0 < 1\}$.

So we can state the following minimax result.

COROLLARY 5.3. *Let (R_n) be a sequence of one-sample rank statistics of form* (2.2) *or* (2.5) *based on the scores generating function a_** ((5.11)) *so that* (4.9) *is satisfied. Let $(\psi_n^*) = (\varphi_{\gamma_n, l_n} \circ R_n)$ be the corresponding* as. *rank test, where $(\gamma_n) \subset [0, 1]$, $(l_n) \subset (-\infty, +\infty)$ and $\lim_n l_n = l_\alpha(a_*)$* ((5.4)). *Assume that in addition to* (4.1)–(4.4) *also* (5.7) *and* (5.12), (5.13) *hold. Then (ψ_n^*) is* as. *maximin for H_0* vs. *H_1 at* as. *level α; its* as. *minimum power is $1 - \Phi(u_\alpha - \tau(\lambda(a_*^2))^{1/2})$.*

In Definition 5.2 of [23] the classical asymptotic relative efficiency was generalized to a minimax setting by the introduction of the as. minimax relative efficiency ($\mathrm{ARE}_{\mathrm{mx}}$), which is based on the minimum standardized shifts of the limiting normals of test statistics. Like the classical Pitman-Noether efficiency, $\mathrm{ARE}_{\mathrm{mx}}$ allows an obvious interpretation in terms of sample sizes and minimum powers. For its evaluation it suffices to compute $\mathrm{ARE}_{\mathrm{mx}}$ $((\psi_n):(\psi_n^*))$ for a sequence (ψ_n) of one-sample rank test for H_0 at as. level α, of the form employed in Corollary 5.2, based on some scores generating function a, and for the as. maximin test (ψ_n^*) of Corollary 5.3. We have to assume that (ψ_n) is as. unbiased, i.e.

$$2\eta a(1) \leqq F_0(IC\cdot\Lambda) \tag{5.14}$$

for IC given by (4.20). Since, by virtue of (5.9)–(5.11),

$$F_0(IC_*\cdot\Lambda) - 2\eta a_*(1) = \lambda(a_*^2), \tag{5.15}$$

$\mathrm{ARE}_{\mathrm{mx}}((\psi_n):(\psi_n^*))$ has the following form.

COROLLARY 5.4. *Under the assumptions of Corollary* 5.3 *and* (4.10)–(4.12), (5.14), *the* as. *minimax relative efficiency of (ψ_n) with respect to (ψ_n^*) is given by*

$$\mathrm{ARE}_{\mathrm{mx}}((\psi_n):(\psi_n^*)) = \frac{(F_0(IC\cdot\Lambda) - 2\eta a(1))^2}{\lambda(a^2)\lambda(a_*^2)}$$

where IC is defined by (4.20).

The results concerning the as. maximum size of rank tests have been independent of the particular F_0, due to distribution freeness (Theorem 3.3). Similarly, uniform power results follow if nonparametric alternatives are employed. Then the robustness results about rank tests take on an intrinsically nonparametric form.

Let $\{\Pi_\theta \mid \theta \in \Theta\}$ be a parametric family of pm's on (0, 1) whose parameter space is a subset of $[-\infty, +\infty]$ and contains 0 in its interior, and which has the following properties (5.16)–(5.18), (5.21).

$$\Pi_0 = \lambda \tag{5.16}$$

$$\Pi_\theta \ll \lambda, \qquad \theta \in \Theta. \tag{5.17}$$

If π_θ denotes the λ-density of Π_θ, then there is a nondegenerate function $\Xi \in L_2(d\lambda)$ such that

$$\frac{\pi_\theta^{1/2} - 1}{\theta} \to \frac{1}{2} \Xi \text{ in } L_2(d\lambda) \text{ as } \theta \to 0. \tag{5.18}$$

Then, for any $F_0 \in \mathscr{M}_{cs}$, define the family $\{F_\theta \mid \theta \in \Theta\}$ by

$$dF_\theta = \pi_\theta \circ F_0 \, dF_0, \qquad \theta \in \Theta. \tag{5.19}$$

The nonparametric form that the preceding corollaries now take is based on the fact that

$$F_0(IC \cdot \Lambda) = \int a(s) \, \Xi \, (½ + ½ \, s) \lambda(ds), \tag{5.20}$$

independently of $F_0 \in \mathscr{M}_{cs}$, provided that we assume

$$\Xi(1 - s) = -\Xi(s), \qquad s \in (0, 1), \tag{5.21}$$

which is also in accordance with (5.7). Moreover, the defining equations for the minimax solution simplify in this case to

$$\lambda((\Xi - d_1)^+) = \eta, \tag{5.22}$$

$$a_*(s) = \Xi(½ + ½s) \wedge d_1, \qquad s \in [0, 1). \tag{5.23}$$

To guarantee (4.10)–(4.12) for this a^* it suffices to assume that Ξ has a continuous, nonnegative derivative on $[0, 1)$.

REMARK. While the case of Kolmogorov nbd's is automatically covered by the case of total variation nbd's, the asymptotics of this section also apply to Lévy and Prokhorov nbd's. Given $\{F_\theta^0\}$, one has to pass to the shifted df's $F_{0n}(t) = F_0^0(t - \delta_n)$, $t \in [-\infty, +\infty]$, under the null hypothesis, and to $F_{\tau_n}(t) = F_{\tau_n}^0(t + \delta_n)$, $t \in [-\infty, +\infty]$, under the alternative, and consider total variation nbd's of radius δ_n centered at these shifted df's; also the differentiability assumptions refer to these shifted df's. In the case of a location parameter family $\{F_\theta^0\}$, the whole modification amounts to decreasing τ by 2δ, when compared with the total variation case. The argument for the finite sample case can be found in Huber (1968); for the asymptotic case, it has been spelled out explicitly in [21] and Quang (1976).

EXAMPLES 1. *The normal location alternatives*, $F_\theta(t) = \Phi(t - \theta)$; $t, \theta \in [-\infty, +\infty]$. For (F_0^n) vs. $(F_{\tau_n}^n)$ at as. level α, the as. normal scores rank test $(\hat{\varphi}_n)$ which is based on the scores generating function $\hat{a}(s) = \Phi^{-1}(½ + ½ \, s)$, $s \in [0, 1)$, is optimal; we have that $\lim_n F_0^n(\hat{\varphi}_n) = \alpha$, $\lim_n F_{\tau_n}^n(\varphi_n) = 1 - \Phi(u_\alpha - \tau)$. Due to $\hat{a}(1) = +\infty$, $\lim_n \alpha_n(\hat{\varphi}_n) = 1$ and $\lim_n \beta_n(\hat{\varphi}_n) = 0$.

Assuming $\eta \in (0, 1/\sqrt{2\pi})$, a_* is given by $a_*(s) = \Phi^{-1}(½ + ½ \, s) \wedge d_1$, $s \in [0, 1)$, where d_1 is uniquely determined by the equation $\varphi(d_1) - d_1(1 - \Phi(d_1)) = \eta$; note that $\lambda(a_*^2) = 2(\Phi(d_1) - ½ - \eta \, d_1)$. For the as. maximin rank test (ψ_n^*) for H_0 vs. H_1 at as. level α (with as. critical value $u_\alpha(\lambda(a_*^2))^{1/2} + (\epsilon + 2\delta) \, d_1$) we have $\lim_n \alpha_n(\psi_n^*) = \alpha$ and $\lim_n \beta_n(\psi_n^*) = 1 - \Phi(u_\alpha - \tau(\lambda(a_*^2))^{1/2})$.

A limiting form of (ψ_n^*) as $\eta \to 1/\sqrt{2\pi}$, or equivalently $d_1 \to 0$, is the as. sign-test (ψ_n^s), with as. critical value $u_\alpha + (\epsilon + 2\delta)$ (consider $d_1^{-1} a_*$). It satisfies $\lim_n \alpha_n(\psi_n^s) = \alpha$ and $\lim_n \beta_n(\psi_n^s) = 1 - \Phi(u_\alpha - \tau(\sqrt{(2/\pi)} - 2\eta)) \cdot (\psi_n^s)$ is never biased. Moreover,

$$\text{ARE}_{\text{mx}}((\psi_n^s):(\psi_n^*)) = \frac{(\sqrt{(2/\pi)} - 2\eta)^2}{\lambda(a_*^2)};$$

this is an increasing function of η which, departing from its value $2/\pi$ in the uncontaminated model, tends to 1 as $\eta \to 1/\sqrt{2\pi}$. For the unmodified as. sing-test (φ_n^s), with critical value u_α, we have $\lim_n \alpha_n(\varphi_n^s) = 1 - \Phi(u_\alpha - (\epsilon + 2\delta))$, $\lim_n \beta_n(\varphi_n^s) = 1 - \Phi(u_\alpha - \tau(\sqrt{(2/\pi)} - \eta))$.

If (φ_n^w) denotes the classical as. Wilcoxon test (scores generating function $a(s) = s$, $s \in [0, 1)$, as. critical value $1/\sqrt{3} \, u_\alpha$), then $\lim_n \alpha_n(\varphi_n^w) = 1 - \Phi(u_\alpha - \sqrt{3} \, (\epsilon + 2\delta))$, $\lim_n \beta_n(\varphi_n^w) = 1 - \Phi(u_\alpha - \tau(\sqrt{(3/\pi)} - \sqrt{3}\eta))$. For the modified Wilcoxon test (ψ_n^w) with as. critical value $1/\sqrt{3} \, u_\alpha + (\epsilon + 2\delta)$ we have $\lim_n \alpha_n(\psi_n^w) = \alpha$ and $\lim_n \beta_n(\psi_n^w) = 1 - \Phi(u_\alpha - \tau(\sqrt{(3/\pi)} - 2\sqrt{3} \, \eta))$. When compared with (ψ_n^s), (ψ_n^w) loses its superiority, which in terms of asymptotic relative efficiency amounts to 3/2 in the uncontaminated model, as soon as η exceeds a certain value; to be precise, $\mathrm{ARE}_{\mathrm{mx}}((\psi_n^s):(\psi_n^w)) > 1$ iff $\eta > (1/\sqrt{2\pi})\cdot((\sqrt{3/2} - 1)/(\sqrt{3} - 1))$. Furthermore,

$$\mathrm{ARE}_{\mathrm{mx}}((\psi_n^w):(\psi_n^*)) = \frac{(\sqrt{(3/\pi)} - 2\sqrt{3}\eta)^2}{\lambda(a_*^2)}, \qquad \eta \in \left[0, \frac{1}{2\sqrt{\pi}}\right].$$

As $\eta \to (1/2\sqrt{\pi})$, this function first increases, departing from its value $3/\pi$ in the uncontaminated model until $\eta = \tilde{\eta} = \varphi(\tilde{d}) - \tilde{d}(1 - \Phi(\tilde{d}))$, where $\tilde{d}$ is the unique solution of $d = \sqrt{\pi}(2\Phi(d) - 1) > 0$; at $\tilde{\eta}$ it attains a unique maximum of value $(3/\tilde{d})((1/\sqrt{\pi}) - 2\tilde{\eta})$; then it tends decreasingly to 0; beyond $\eta = 1/2\sqrt{\pi}$, (ψ_n^w) is biased.

The preceding results hold uniformly for the families $F_\theta^\sigma(t) = \Phi((t/\sigma) - \theta)$, $t, \theta \in [-\infty, +\infty]$, if the scale parameter $\sigma \in (0, +\infty)$ is unknown. For the mapping $G \to G^\sigma(G^\sigma(B) = G(\sigma^{-1}B), B \in \mathscr{B})$ sends the nbd $\mathscr{P}(F_\theta^1; \epsilon, \delta)$ onto $\mathscr{P}(F_\theta^\sigma; \epsilon, \delta)$ one-to-one, and rank tests are invariant under scale transforms. Thus the previous as. maximin rank test (ψ_n^*) is actually uniformly maximin and thus may be viewed as a robust version of the classical t-test. Unfortunately, its form depends on τ (given ϵ and δ). More generally, the preceding results hold uniformly for nonparametric alternatives with quadratic mean derivative $\frac{1}{2}\Xi$, $\Xi = \Phi^{-1}$.

2. *Nonparametric alternatives, induced by the family* $\pi_\theta(s) = 1 + \theta(s - \frac{1}{2})$, $s \in (0, 1)$, $|\theta| < 2$. *Fix* $F_0 \in \mathscr{M}_{cs}$. In the uncontaminated model, the as. Wilcoxon rank test $(\hat{\varphi}_n)(\hat{a}(s) = \frac{1}{2}s$, $s \in [0, 1)$; $k_\alpha(\hat{a}) = (1/\sqrt{12})u_\alpha)$ is optimal; $\lim_n F_0^n(\hat{\varphi}_n) = \alpha$, $\lim_n F_{\tau_n}^n(\hat{\varphi}_n) = 1 - \Phi(u_\alpha - (1/\sqrt{12})\tau)$. In the model $H_0 \cup H_1$, we have $\lim_n \alpha_n(\hat{\varphi}_n) = 1 - \Phi(u_\alpha - \sqrt{3}(\epsilon + 2\delta))$, $\lim_n \beta_n(\hat{\varphi}_n) = 1 - \Phi(u_\alpha - \tau((1/\sqrt{12}) - \sqrt{3}\eta))$. For the test $(\hat{\psi}_n)$ with adjusted critical values $(l_\alpha(\hat{a}) = (1/\sqrt{12})u_\alpha + \frac{1}{2}(\epsilon + 2\delta))$, we obtain that $\lim_n \alpha_n(\hat{\psi}_n) = \alpha$, $\lim_n \beta_n(\hat{\psi}_n) = 1 - \Phi(u_\alpha - \tau((1/\sqrt{12}) - \sqrt{12}\eta))$; this test turns biased for $\eta > 1/12$.

For $\eta < 1/8$ let $d_1 = \frac{1}{2} - \sqrt{2\eta}$; then $a_*(s) = (\frac{1}{2}s) \wedge d_1$, $s \in [0, 1)$; $\lambda(a_*^2) = \frac{1}{3}(1 + 4\sqrt{2\eta})d_1^2$. The as. rank test (ψ_n^*) based on a_*, with as. critical value $l_\alpha(a_*)$, is as. maximin, uniformly with respect to $F_0 \in \mathscr{M}_{cs}$; it has as. minimum power $1 - \Phi(u_\alpha - \tau(\lambda(a_*^2))^{1/2})$. In particular

$$\mathrm{ARE}_{\mathrm{mx}}((\hat{\psi}_n):(\psi_n^*)) = 1 - \eta^{3/2}\cdot\frac{32\sqrt{2} - 144\sqrt{\eta}}{1 - 24\eta + 32\eta\sqrt{2\eta}}, \qquad \eta \in \left[0, \frac{1}{12}\right].$$

REMARK. Behnen's (1972) general ARE-bounds are rendered invalid in our model, due to the possibility that for $G_n \in \mathscr{P}(F_{\tau_n}; \epsilon_n, \delta_n)$ the relation $G_n(-t) \leqq 1 - G_n(t)$, $t \in [-\infty, +\infty]$, may be violated for large values of t, although F_{τ_n} satisfies it by assumption (and hence G_n usually at least on some compact). In other words, the classical ARE-bounds depend too strongly on the tail area of df's, where in practice it does not matter and cannot be decided anyway, whether or how df's are ordered, in view of the small values of df's there, or values close to 1.

6. Further aspects.

6.1. *Asymptotic fine structure of shrinking* nbd's. As remarked at the end of Section 4, the reason why rank statistics were not yet covered by [23] is the occurrence of sequences that are noncontiguous to (F_0^n), which renders the approximation (4.18) insufficient. To bypass this difficulty one could formally make a restriction to the model $H_0'' \cup H_1''$ given by

$$H_j'' = \{(W_n) \in H_j' \mid (W_n) \ll (F_0^n)\}, \qquad j = 1, 2 \tag{6.1}$$

(H_j' defined by (4.13)). By virtue of Lemma 2.1 of [24], the extreme limit laws of the rank statistics would stay the same (if the difference between pointwise and F_0-essential extrema is neglected). Moreover, the sequences (Q_{0n}^n), (Q_{1n}^n) which are obtained from least favourable pairs (Q_{0n}, Q_{1n}) for $\mathscr{P}(F_0; \epsilon_n, \delta_n)$ vs. $\mathscr{P}(F_{\tau_n}; \epsilon_n, \delta_n)$ are contained in H_j'' by virtue of Lemma 3.4

of [27]. (The continuity of df's Q_{0n}, Q_{1n} is implied by the assumed continuity of the F_θ's and Theorem 5.2 of [22]. That also the df's Q_N and R_N, which are used in the proof of Lemma 2.1 of [24], are continuous, follows from their definition. Moreover, note that the proof of this lemma also applies to moving centers F_{τ_n}.) Thus the submodel $H_0'' \cup H_1''$ turns out to be representative for the full model $H_0 \cup H_1$ in every respect. It has the further technical advantage that condition (4.9) could be relaxed to

$$\lim_n \int (a_n([1 + ns]) - a(s))^2 \lambda(ds) = 0 \tag{6.2}$$

for some $a \in L_2(d\lambda)$, and that assumptions (4.10)–(4.12) and (5.12), (5.13) could be dispensed with completely.

The reason why the restriction to $H_0'' \cup H_1''$ appears to be artificial is that the contiguity assumption $(W_n) \ll (F_0^n)$ is purely asymptotic in nature; it has no interpretation for finite n (e.g., one could not tell which portion of $\mathscr{P}(F_0;\ \epsilon_n,\ \delta_n)$ is excluded). For this reason, approximations that cover the full nbd's are more honest. Certain undesirable aspects that go along with the nonuniformity of the contiguity of the sequences (W_n) in H_j'' have already been pointed out in the context of initial estimates at the end of [24]. For the present context, the following example may demonstrate the unreliability of approximations that are confined to contiguous sequences.

EXAMPLE. Let T_n be of form (4.28), with $IC \in L_2(dF_0)$, $F_0(IC^2) = 1$, and IC unbounded, $\sup IC = +\infty$ say. The sequence (T_n) shall be examined under nbd's $\mathscr{P}(F_0;\ \epsilon_n,\ 0)$ where $\epsilon_n = o(n^{-1/2})$.

Let $G_{0n} = \sup\{W_n \circ T_n^{-1} \mid W_n \in H_{0n}\}$ and $G_0 = \lim\sup\{(W_n \circ T_n^{-1}) | (W_n) \in H_0\} (H_{0n}, H_0$ given by (4.6), (4.7)). Then $G_{0n}(t) = (1 - \epsilon_n)^n F_0^n(T_n \leqq t)$, $t < +\infty$. Thus $G_0 = I_{+\infty}$ (one-point mass in $+\infty$) if $n\epsilon_n \to +\infty$. (If $\epsilon_n = o(n^{-1})$ then $G_0 = \Phi$; nbd's this size cannot have any influence asymptotically, as $\| W_n - F_0^n \| \leqq n\epsilon_n$ for $W_n \in H_{0n}$).

Now let $\tilde{T}_n$ be of form (4.28) based on a bounded IC_n such that the sup-norm satisfies $\| IC_n \| = o(n^{1/4})$ and moreover, $F_0((IC - IC_n)^2) \to 0$ as $n \to +\infty$. Then by Lindeberg-Feller theorem (the Lindeberg expression for $\tilde{T}_n$ vanishes eventually), $\mathscr{L}(\tilde{T}_n - n^{-1/2} \sum_{i=1}^n G_{ni}(IC_n)) \Rightarrow \mathscr{N}(0;\ 1)$ as $n \to +\infty$, under all sequences $(\otimes_{i=1}^n G_{ni}) \in H_0$. Therefore, the stochastic limit superior $\tilde{G}_0$ of $(\tilde{T}_n)$ under H_0 is given by $\tilde{G}_0(t) = \Phi(t - \lim\sup_n n^{1/2}\epsilon_n \sup IC_n)$, $t < +\infty$.

For any sequence $\epsilon_n = o(n^{-1/2})$, $n\epsilon_n \to +\infty$, an appropriate sequence (IC_n) can be chosen such that $\lim_n n^{1/2}\epsilon_n \sup IC_n = 0$. Then $\tilde{G}_0 = \Phi$ whereas $G_0 = I_{+\infty}$, even though $\lim_n F_0^n((T_n - \tilde{T}_n)^2) = 0$ and hence $T_n - \tilde{T}_n \to_{W_n} 0$ for all $(W_n) \ll (F_0^n)$.

REMARK. Nbd's that shrink faster than $n^{-1/2}$ can also be used to demonstrate the effect of the boundedness of the scores $a_n(i)$. By (4.21), $\Phi = \lim\sup\{(W_n \circ R_n^{-1}) | (W_n) \in H_0\}$ (if, without restriction, $\lambda(a^2) = 1$) as long as $\lim_n n^{1/2} \int_{(1-(\epsilon_n+2\delta_n))}^1 a\,d\lambda = 0$. The maximum of such shrinking rate, which cannot exceed $o(n^{-1/2})$, may actually be larger than $o(n^{-1})$, even if a is unbounded. For example, it equals $o((n \log n)^{-1/2})$ in the case of normal scores. (Incidentally, $\Phi^{-1}(\frac{1}{2} + \frac{1}{2}(n/(n+1))) = O((\log n)^{1/2})$; thus in view of the preceding example the normal scores rank statistic appears to be better approximated by $\tilde{T}_n$ for $IC_n(t) = \operatorname{sig}(t)\Phi^{-1}(F_0(|t|) \wedge (\frac{1}{2} + \frac{1}{2}(n/(n+1)))$, $t \in [-\infty, +\infty]$, which takes into account the boundedness of the scores at stage n.)

6.2. *The two-sample case.* The two-sample problem can be treated robustly in a similar fashion; so it shall only be briefly sketched here.

As for the definition of averaged and randomized scores rank statistics R_n in this case, the reader is referred to Hájek and Sidak (1967) and Behnen (1976). It can be shown as in Proposition 2.1 that, if the scores are increasing, then R_n is stochastically increasing. The bring-in effect of ranks can be stated as follows: let $x_1, \cdots, x_m, y_1, \cdots, y_n$ denote the two samples of observations, then the value of R_n remains unchanged if, e.g., $y_1, \cdots, y_n < k$ for some $k \in (-\infty, +\infty)$ and the x_i, $x_i > k$, are brought in up to k. (As in the one-sample problem, this property is a special case of the invariance of rank statistics under monotone (and, respectively, odd) transformations.) As in Proposition 2.2, a reduction to the continuous i.i.d. case can be achieved.

Based on a similar reasoning as in Section 3, two df's G_1 and G_2 are called approximately equal iff there is a $F \in \mathcal{M}_c$ such that (ϵ, δ given in advance) $G_1, G_2 \in \mathcal{P}(F; \epsilon, \delta)$. The nonparametric hypothesis of *approximate equality* is defined as $\mathcal{H}^{=}_{\epsilon,\delta} = \cup\{\mathcal{P}(F; \epsilon, \delta) \times \mathcal{P}(F; \epsilon, \delta) | F \in \mathcal{M}_c\}$. If G_1 denotes the common df of the x_i's and G_2 the common df of the y_i's, then $(G_1, G_2) \in \mathcal{H}^{=}_{\epsilon,0}$ means that the x_i's obey some df F with probability $1 - \epsilon$, and the y_i's independently obey the same F with (independent) probability $1 - \epsilon$; the observations being unspecified otherwise. In the case of total variation nbd's and if $G_1, G_2 \in \mathcal{M}_c$, then $(G_1, G_2) \in \mathcal{H}^{=}_{0,\delta}$ is equivalent to $\| G_1 - G_2 \| \leqq 2\delta$. As has been done for the classical null hypothesis of equality, we also enlarge the classical alternative of positive deviation of the first sample by passing from $(F_1, F_2) \in \mathcal{M}_c \times \mathcal{M}_c$ such that $F_1(t) \leqq F_2(t)$, $t \in [-\infty, +\infty]$, to $\mathcal{P}(F_1; \epsilon, \delta) \times \mathcal{P}(F_2; \epsilon, \delta)$. In the case of ϵ-contamination nbd's, the interpretation of $(G_1, G_2) \in \mathcal{P}(F_1; \epsilon, \delta) \times \mathcal{P}(F_2; \epsilon, \delta)$ is the usual one. As for total variation nbd's, assume that $F_1(t) \leqq F_2(t) - 2\delta$, $|t| \leqq k$, for some $k \in (0, +\infty)$ such that $F_1(-k) \leqq \delta$ and $F_2(k) \geqq 1 - \delta$. Then $(G_1, G_2) \in \mathcal{P}(F_1; 0, \delta) \times \mathcal{P}(F_2; 0, \delta)$ entails that $G_1(t) \leqq G_2(t)$, $|t| \leqq k$, and $G_1(-k) \leqq 2\delta$, $G_2(k) \geqq 1 - 2\delta$. Conversely, assume that two df's G_1, G_2 satisfy $G_1(t) \leqq G_2(t)$, $|t| \leqq k$, for some $k \in (0, +\infty)$ such that $G_1(-k) \leqq 2\delta$ and $1 - \delta \geq G_2(k) \geq 1 - 2\delta$. Define the df's F_1, F_2 by $F_1(t) = (G_1(t) - \delta)^+$ and $F_2(t) = (G_2(t) + \delta) \wedge 1$, $t \in (-\infty, +\infty)$. Then $(G_1, G_2) \in \mathcal{P}(F_1; 0, \delta) \times \mathcal{P}(F_2; 0, \delta)$ and $F_1(t) \leqq F_2(t)$, $t \in [-\infty, +\infty]$, $F_1(t) \leq (F_2(t) - 2\delta)^+$, $|t| \leqq k$, $F_1(-k) \leqq \delta$, $F_2(k) \geqq 1 - \delta$. Therefore, our alternatives represent, in a certain sense, positive stochastic deviation of the first sample df on some suitably large compact, which is determined by the lower and upper δ-quantiles; the tails of df's are left unspecified. Analogously to Theorem 3.1, two-sample rank statistics turn out to be distribution free for $\{(F, F) | F \in \mathcal{M}_c\}$, and more generally, for Behnen's (1972) nonparametric alternatives indexed by $F \in \mathcal{M}_c$, with respect to gross error and Kolmogorov nbd's; they are not distribution free with respect to Lévy and Prokhorov ndb's.

For the local asymptotic study, first the model of [23] has to be extended to the two-sample case, which is a routine affair. Two-sample alternatives are employed that are defined essentially as before (quadratic mean derivative ½ Λ) and are attached to some (F_0, F_0), $F_0 \in \mathcal{M}_c$. The parameters for the first sample are of the form $\xi_{1N} = (1/m)(mn/N)^{1/2}\xi$, and for the second sample $\xi_{2N} = (1/n)(mn/N)^{1/2}\xi$, where $\xi = \epsilon, \delta, \tau$ and $N = m + n$. Test statistics of the following kind are considered,

$$(6.3) \qquad T_N = \left(\frac{mn}{N}\right)^{1/2} \left(\frac{1}{m}\sum_{i=1}^{m} IC(x_i) - \frac{1}{n}\sum_{i=1}^{n} IC(y_i)\right), \qquad IC \in L_2(dF_0),\ F_0(IC) = 0.$$

We require that $m/N \to \rho$ as $N \to +\infty$, for some $\rho \in (0, 1)$. Then the extreme limit laws of the sequence (T_N) are $\mathcal{N}(s(IC); F_0(IC^2))$, respectively $\mathcal{N}(\tau F_0(IC \cdot \Lambda) - s(-IC); F_0(IC^2))$, where

$$(6.4) \qquad s(IC) = (1 - \rho)(-\delta \inf_{[F_0]} IC + (\epsilon + \delta)\sup IC) + \rho(-(\epsilon + \delta)\inf IC + \sup_{[F_0]} IC).$$

The optimal choice of IC is given by

$$(6.5) \qquad IC_* = d_0 \vee \Lambda \wedge d_1,$$

where

$$(6.6) \qquad F_0((d_0 - \Lambda)^+) = \eta = F_0((\Lambda - d_1)^+).$$

Using this IC_* we obtain a test which is as. maximin among all as. tests subject to some as. level, as is shown by an as. expansion of the log likelihood of least favorable pairs in the manner of Section 4 of [23].

To treat rank statistics in this model, which contains sequences as. orthogonal to (F_0^N) and, when restricted to contiguous sequences, allows for the same discrepancies as noted in Section

6.1, we appeal to Theorem 1, Theorem 2, and corollary of Govindarajulu, et al., (1966), or to Theorem 2.4 of Hajek (1968) and Theorem 1 of Hoeffding (1973). Then by means of calculations similar to those of Section 4 it is proved that rank statistics can indeed be subsumed formally under the foregoing results if we set

$$IC = a \circ F_0, \tag{6.7}$$

where a denotes the scores generating function. In particular, an as. maximin test is obtained from the function a_*,

$$a_* = d_0 \vee (\Lambda \circ F_0^{-1}) \wedge d_1. \tag{6.8}$$

The results concerning ARE_{mx} reproduce those previously derived, due to the relation

$$a_{\text{I}}(s) = a_{\text{II}}\,(\tfrac{1}{2} + \tfrac{1}{2}\,s), \qquad s \in [0, 1), \tag{6.9}$$

between the scores generating function in the one-sample case (a_{I}) and the two-sample case (a_{II}), which in view of the extreme limit laws determined by (6.4) extends to the present model (e.g., compare (5.11) and (6.8)).

6.3 *Other asymptotic approaches.* Although the infinitesimal nbd approach of this paper appears to be most canonical and should yield good approximation for (very) small nbd's, it has the possibly undesirable feature that, in the limit, there is zero contamination. A different approach, which avoids degeneration of nbd's has been sketched by Huber (1977), page 47: to each member of a fixed nbd, a classical shift alternative is attached. This method leads to essentially the same minimax solution. However, it may be argued on the grounds that, practically, all df's have to be assumed symmetric, and that the minimax solution is inadmissible; cf. [25]. It is obvious that Huber's approach is intimately related to Huber (1964), as is the infinitesimal nbd approach to Hampel's (1968) local robustness Lemma 5 (cf. also [24]). Still outstanding seems a counterpart to Hampel's (1968, 1971) theory of qualitative robustness. Such an approach, cf. [26], would employ a fixed null hypothesis and a fixed alternative, as well as constant, nondegenerating nbd's. In technical respects, consistency of tests and test statistics would be the main tools rather than local power or as. normality. One expects that, in such a set up, rank tests turn out to be qualitatively robust, generally.

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