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Γ-MINIMAX SELECTION PROCEDURES IN SIMULTANEOUS TESTING PROBLEMS[1]

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Suppose we have to decide on the basis of appropriately drawn samples which of k treatment populations are "good" compared to either given control values or to a control population from which an additional sample is available. The unknown parameters are assumed to vary randomly according to a prior distribution about which we only have the partial knowledge that it is contained in a given class Γ of priors. Though we derive in both cases (under the assumption of monotone likelihood ratios) Γ-minimax procedures which by definition attain minimal supremal risk over Γ, the emphases are different: while we try to demonstrate in the "known controls case" how well known results from the theory of testing hypotheses can be utilized to solve the problem, our main purpose in the "unknown control case" is to give a new proof for a theorem which was stated but only partially proved by Randles and Hollander.

1. Introduction. Suppose that for every $i \in \{0, 1, \cdots, k\}$ we are given a family $\{f_{i,\vartheta}\}_{\vartheta \in \Omega_i \subseteq \mathbb{R}}$ of densities with respect to either the Lebesgue measure ("continuous case") or any counting measure ("discrete case") on the real line $\mathbb{R}$, which have monotone nondecreasing likelihood ratios (M.L.R.) $f_{i,\vartheta'}(z)/f_{i,\vartheta}(z)$ in z for $\vartheta < \vartheta'$. Let $\mathbf{X}_i = (X_{i1}, \cdots, X_{in_i})$, $i = 0, 1, \cdots, k$, be independent samples from populations $\mathscr{P}_0, \mathscr{P}_1, \cdots, \mathscr{P}_k$ from which for every $i \in \{0, 1, \cdots, k\}$ we can deduce a sufficient statistic Z_i with density f_{i,ϑ_i}, where $\vartheta_i \in \Omega_i$ only is unknown.

Our goal is to select a subset of $\{\mathscr{P}_1, \cdots, \mathscr{P}_k\}$ containing "good" populations and excluding "bad" ones, to be defined more precisely in the sequel. Every selection procedure is viewed to be a probability measure (depending on the observations) over $S = \{s \mid s \subseteq \{1, \cdots, k\}\}$, where every $s \in S$ represents the indices of populations to be selected eventually.

In the "*known control case*" (Section 3) for $i = 1, \cdots, k$ we take values $\vartheta_{0i} \in \Omega_i$ and $\Delta_i > 0$ and call $\mathscr{P}_i$ "good" if $\vartheta_i \geq \vartheta_{0i} + \Delta_i$ and "bad" if $\vartheta_i \leq \vartheta_{0i}$. Assume that $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k)$, the parameter vector, a priori varies randomly over $\boldsymbol{\Omega} = \Omega_1 \times \cdots \times \Omega_k$ according to some prior distribution $\mathscr{T}$, about which we have only the partial knowledge that it is contained in Γ, the class of priors with property $\mathscr{T}\{\boldsymbol{\vartheta} \in \boldsymbol{\Omega} \mid \mid \vartheta_i \geq \vartheta_{0i} + \Delta_i\} = \pi_i$ and $\mathscr{T}\{\boldsymbol{\vartheta} \in \boldsymbol{\Omega} \mid \vartheta_i \leq \vartheta_{0i}\} = \pi_i'$, where $\pi_i, \pi_i' \geq 0$, $\pi_i + \pi_i' \leq 1$, $i = 1, \cdots, k$, are fixed.

Now for any specific loss function a Γ-minimax rule ψ^Γ is defined as having smallest supremal risk over Γ among all the competing procedures (cf. Blum and Rosenblatt (1967)). Let us adopt the following loss function:

$$L(\boldsymbol{\vartheta}, s) = \sum_{i \in s} L_{2i} I_{(-\infty, \vartheta_{0i}]}(\vartheta_i) + \sum_{i \notin s} L_{1i} I_{[\vartheta_{0i} + \Delta_i, \infty)}(\vartheta_i), \qquad s \in S, \quad \boldsymbol{\vartheta} \in \boldsymbol{\Omega},$$

where $L_{2i}, L_{1i} \geq 0$, $i = 1, \cdots, k$, represent the losses for including "bad" populations or excluding "good" ones, respectively.

For this loss function (and more generally for additive ones, cf. (1)) and any prior $\mathscr{T}$ the conditional risk—given $\mathbf{X}$—of a procedure depends only on its conditional probabilities of selecting $\mathscr{P}_i$, $i = 1, \cdots, k$. This was stated already by Lehmann (1957, 1961). An idea of the proof can be found in Nagel (1970). Thus we may restrict our considerations to the class $\mathscr{D}$ of

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procedures of the form $\psi(\mathbf{X}) = (\psi_1(\mathbf{X}), \cdots, \psi_k(\mathbf{X}))$, $\mathbf{X} = (\mathbf{X}_1, \cdots, \mathbf{X}_k)$, where $\psi_i(\mathbf{X})$ denotes the conditional probability—given $\mathbf{X}$—of selecting $\mathscr{P}_i$, $i = 1, \cdots, k$.

Though we generalize results of Randles and Hollander (1971) and Huang (1974) our main purpose is rather to demonstrate how one can utilize results from the (Neyman, Pearson) theory of testing hypotheses to find the Γ-minimax rules. Within the scope of selecting a best population this was shown already by Miescke (1979). Finally we point out that other interesting results concerning Γ-minimax rules can be found in Gupta and Huang (1975, 1977).

In the "*unknown control case*" (Section 4), instead of having the ϑ_{0i}'s as control values, $\mathscr{P}_0$ now acts as control population with which $\mathscr{P}_1, \cdots, \mathscr{P}_k$ are to be compared. Things which change with respect to the previous situation are obvious in nature: Now we have $\mathbf{X} = (\mathbf{X}_0, \mathbf{X}_1, \cdots, \mathbf{X}_k)$, $\boldsymbol{\theta} = (\theta_0, \theta_1, \cdots, \theta_k)$ and $\boldsymbol{\Omega} = \Omega_0 \times \Omega_1 \times \cdots \times \Omega_k$. Γ and L are given as before, but now $\vartheta_{01}, \cdots, \vartheta_{0k}$ coincide with ϑ_0, the realization of the random parameter θ_0 for $\mathbf{X}_0$.

A crucial assumption we have to add is that all populations are in the "continuous case" where $\vartheta_0, \vartheta_1, \cdots, \vartheta_k$ are location parameters. In this setup Randles and Hollander (1971) specified Γ-minimax rules within $\mathscr{D}^0$, the class of procedures of the form $\psi(\mathbf{X}) = (\psi_1(\mathbf{X}_0, \mathbf{X}_1), \cdots, \psi_k(\mathbf{X}_0, \mathbf{X}_k))$, $\mathbf{X} = (\mathbf{X}_0, \mathbf{X}_1, \cdots, \mathbf{X}_k)$. Here we give an alternative to the incomplete proof of their Theorem 4.1. Instead of using a Hunt-Stein argument our proof is based on a generalization of their Lemma 3.2 (Section 2) where an approach, due to Lehmann (1957), is applied which admits also the use of improper priors.

2. A lemma. Our considerations in this section are valid in both cases. In this sense $\mathbf{X}$ is understood either to be equal to $(\mathbf{X}_1, \cdots, \mathbf{X}_k)$ or $(\mathbf{X}_0, \mathbf{X}_1, \cdots, \mathbf{X}_k)$ throughout the following, and $\boldsymbol{\vartheta}$, $\boldsymbol{\Omega}$ and Γ are to be interpreted analogously.

Let us more generally admit any additive loss function:

$$L(\boldsymbol{\vartheta}, s) = \textstyle\sum_{i\in s} L_2^{(i)}(\boldsymbol{\vartheta}) + \sum_{i\notin s} L_1^{(i)}(\boldsymbol{\vartheta}), \qquad s \in S, \quad \boldsymbol{\vartheta} \in \boldsymbol{\Omega}, \tag{1}$$

where the $L_j^{(i)}$'s are nonnegative functions.

Then for $\psi \in \mathscr{D}$ the risk with respect to $\mathscr{T} \in \Gamma$ is given by $r(\mathscr{T}, \psi) = \sum_{i=1}^k r^{(i)}(\mathscr{T}, \psi_i)$, where for every $i \in \{1, \cdots, k\}$

$$r^{(i)}(\mathscr{T}, \psi_i) = \int_{\Omega} \{L_2^{(i)}(\boldsymbol{\vartheta}) E_{\boldsymbol{\vartheta}}\psi_i(\mathbf{X}) + L_1^{(i)}(\boldsymbol{\vartheta})[1 - E_{\boldsymbol{\vartheta}}\psi_i(\mathbf{X})]\}\, d\mathscr{T}(\boldsymbol{\vartheta})$$

is the risk of ψ in the corresponding ith component problem.

LEMMA. *A procedure* $\psi^\Gamma = (\psi_1^\Gamma, \cdots, \psi_k^\Gamma)$ *is Γ-minimax with respect to a class* $\mathscr{G} \subseteq \mathscr{D}$ *of rules if there exist priors* $\mathscr{T}_n \in \Gamma$, $n \in N = \{1, 2, \cdots\}$, *such that for every* $i \in \{1, \cdots, k\}$ *the following holds true: for the* ith *component problem there exist Bayes-rules* ψ_{in}^B *with respect to* $\mathscr{T}_n$, $n \in N$, *and* $\mathscr{G}$ *with*

$$\liminf_{n\to\infty} r^{(i)}(\mathscr{T}_n, \psi_{in}^B) \geq \sup_{\mathscr{T}\in\Gamma} r^{(i)}(\mathscr{T}, \psi_i^\Gamma). \tag{2}$$

PROOF. Let $\psi = (\psi_1, \cdots, \psi_k) \in \mathscr{G}$ be a selection procedure. Then

$$\begin{aligned}\sup_{\mathscr{T}\in\Gamma} r(\mathscr{T}, \psi) &= \sup_{\mathscr{T}\in\Gamma} \textstyle\sum_{i=1}^k r^{(i)}(\mathscr{T}, \psi_i) \geq \sup_n \sum_{i=1}^k r^{(i)}(\mathscr{T}_n, \psi_i)\\ &\geq \sup_n \textstyle\sum_{i=1}^k r^{(i)}(\mathscr{T}_n, \psi_{in}^B) \geq \liminf_{n\to\infty} \sum_{i=1}^k r^{(i)}(\mathscr{T}_n, \psi_{in}^B)\\ &\geq \textstyle\sum_{i=1}^k \liminf_{n\to\infty} r^{(i)}(\mathscr{T}_n, \psi_{in}^B)\\ &\geq \textstyle\sum_{i=1}^k \sup_{\mathscr{T}\in\Gamma} r^{(i)}(\mathscr{T}, \psi_i^\Gamma) \geq \sup_{\mathscr{T}\in\Gamma} \sum_{i=1}^k r^{(i)}(\mathscr{T}, \psi_i^\Gamma)\\ &= \sup_{\mathscr{T}\in\Gamma} r(\mathscr{T}, \psi^\Gamma).\end{aligned}$$

3. Known controls. Within the framework given in Section 1 we derive now Γ-minimax procedures with respect to $\mathscr{D}$. For every $i \in \{1, \cdots, k\}$ let $\varphi_{i,\alpha}^*$, $\alpha \in [0, 1]$, denote the best

(U.M.P.) level α test—based on $\mathbf{X}_i$—for the testing problem

$$H_i: \vartheta_i \leq \vartheta_{oi} \quad \text{versus} \quad K_i: \vartheta_i > \vartheta_{oi}, \tag{3}$$

and let A_i denote the set of values $\beta \in [0, 1]$ minimizing the term $L_{2i}\pi_i'\beta - L_{1i}\pi_i E_{\vartheta_{oi}+\Delta_i} \varphi_{i,\beta}^*(\mathbf{X}_i)$.

THEOREM 1. *Every $\psi^\Gamma \in \mathscr{D}$ of the following type is Γ-minimax with respect to $\mathscr{D}$: $\psi_i^\Gamma(\mathbf{X}) = 1(0)$ as $Z_i \geq (<) c_i$, where c_i satisfies*

$$[L_{2i}\pi_i' f_{i,\vartheta_{oi}}(c) - L_{1i}\pi_i f_{i,\vartheta_{oi}+\Delta_i}(c)](c_i - c) \geq 0, \qquad c \neq c_i, i = 1, \cdots, k.$$

PROOF. We apply our lemma in the simple version of Randles and Hollander (1971) where the sequences of priors and Bayes-rules reduce to a single prior $\mathscr{T}_0$ and to single Bayes-rules $\psi_1^B, \cdots, \psi_k^B$. Under $\mathscr{T}_0$ let $\theta_1, \cdots, \theta_k$ be independent where θ_i equals $\vartheta_{oi} + \Delta_i(\vartheta_{oi}, \vartheta_{oi} + \Delta_i/2)$ with probability $\pi_i(\pi_i', 1 - \pi_i - \pi_i')$, $i = 1, \cdots, k$. Now we fix $i \in \{1, \cdots, k\}$.

For every $\psi \in \mathscr{D}$ the Bayes-risk for the ith component problem is given by

$$r^{(i)}(\mathscr{T}_0, \psi_i) = L_{2i}\, \pi_i' E_{\vartheta_{oi}} \tilde{\psi}_i(\mathbf{X}_i) + L_{1i}\pi_i[1 - E_{\vartheta_{oi}+\Delta_i}\tilde{\psi}_i(\mathbf{X}_i)],$$

where $\tilde{\psi}_i(\mathbf{X}_i) = E_{\mathscr{T}_0} E_{(\theta_1, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_k)}\, \psi_i(\mathbf{X})$ can be viewed as being a test for (3). Obviously, the Bayes-rules for the ith component problem are just those φ_{i,α_i}^* with $\alpha_i \in A_i$. And since $E_{\vartheta_i}\varphi_{i,\alpha_i}^*$ is nondecreasing in ϑ_i, $\mathscr{T}_0$ is least favorable to them. Thus every procedure $(\varphi_{1,\alpha_1}^*, \cdots, \varphi_{k,\alpha_k}^*)$ with $\alpha_i \in A_i$, $i = 1, \cdots, k$, is Γ-minimax with respect to $\mathscr{D}$. Finally by choosing the $\alpha_i \in A_i$ appropriately to get nonrandomized tests, the proof is completed.

Example 1. For $i = 1, \cdots, k$ let $\mathbf{X}_i$ (with sample mean $\bar{X}_i$) come from a normal distribution $N(\vartheta_i, \sigma_i^2)$ with known variance $\sigma_i^2 > 0$. Then by Theorem 1

$$\psi_i^\Gamma(\mathbf{X}) = 1 \text{ iff } \bar{X}_i \geq \vartheta_{oi} + \Delta_i/2 + (n_i\Delta_i)^{-1}\sigma_i^2 \ln(L_{2i}\pi_i'/L_{1i}\pi_i), \qquad i = 1, \cdots, k.$$

Now suppose that for certain $\mathscr{P}_i$ with $i \in I$, say, instead of $\mathbf{X}_i$ only incomplete information is available: $Y_i = \#\{X_{ij} \mid X_{ij} \geq b_i, j = 1, \cdots, n_i\}$, where b_i is fixed. Thus instead of $\mathbf{X}$ we have at hand only $\hat{\mathbf{X}}$ which can be obtained from $\mathbf{X}$ if for every $i \in I$ $\mathbf{X}_i$ is replaced by Y_i. Since the M.L.R. property in a natural way carries over from the normal to the corresponding binomial distributions, Theorem 1 still is applicable, this time leading to $\hat{\psi}^\Gamma$, say, where for $i \notin I$ $\hat{\psi}_i^\Gamma(\hat{\mathbf{X}}) = \psi_i^\Gamma(\mathbf{X})$ but where for $i \in I$

$$\hat{\psi}_i^\Gamma(\mathbf{X}) = 1 \text{ iff } Y_i \geq [\ln((1 - q)q'/q(1 - q'))]^{-1}[n_i \ln(q'/q) + \ln(L_{2i}\pi_i'/L_{1i}\pi_i)],$$

where

$$q = \Phi(n_i^{1/2}(b_i - \vartheta_{oi} - \Delta_i)/\sigma_i), \qquad q' = \Phi(n_i^{1/2}(b_i - \vartheta_{oi})/\sigma_i),$$

and Φ denotes the cdf of $N(0, 1)$.

4. Unknown control. Since we are dealing now with location parameter M.L.R.-families we can utilize the fact (cf. Randles and Hollander (1971)) that for every $i \in \{1, \cdots, k\}$ $W_i = Z_i - Z_0$ has the density $g_i(w - (\vartheta_i - \vartheta_0)) = \int_{\mathbb{R}} f_i(w + u - \vartheta_i) f_0(u - \vartheta_0)\, du$, $w \in \mathbb{R}$, which likewise has the M.L.R. property if $\delta_i = \vartheta_i - \vartheta_0$ is considered as location parameter for W_i.

For every $i \in \{1, \cdots, k\}$ let $\bar{\varphi}_{i,\alpha}^*$, $\alpha \in [0, 1]$ denote the U.M.P. level α test – based on W_i – for the testing problem

$$\bar{H}_i: \delta_i \leq 0 \quad \text{versus} \quad \bar{K}_i: \delta_i > 0, \tag{4}$$

and let $\bar{A}_i$ denote the set of values $\beta \in [0, 1]$ minimizing the term $L_{2i}\pi_i'\beta - L_{1i}\pi_i E_{\Delta_i} \bar{\varphi}_{i,\beta}^*(W_i)$.

THEOREM 2. *Every $\psi^\Gamma \in \mathscr{D}^\circ$ of the following type is Γ-minimax with respect to $\mathscr{D}^\circ$: $\psi_i^\Gamma(\mathbf{X})$*

$= 1(0)$ *as* $W_\iota \geq (<) c_\iota$, *where* c_ι *satisfies*

$$[L_{2\iota}\pi_i' g_i(c) - L_{1\iota}\pi_i g_\iota(c - \Delta_\iota)](c_\iota - c) \geq 0, \qquad c \neq c_\iota, i = 1, \cdots, k.$$

PROOF. Again, it suffices to prove that every $(\bar{\varphi}^*_{1,\alpha_1}, \cdots, \bar{\varphi}^*_{k,\alpha_k})$ with $\alpha_i \in \bar{A}_\iota$, $i = 1, \cdots, k$, is Γ-minimax.

Let us represent every $\mathscr{T} \in \Gamma$ by $\mathscr{T} = (T, t)$, where t denotes the conditional distribution of $(\theta_1, \cdots, \theta_k)$—given θ_0—, and T the marginal distribution of θ_0. For $n \in \mathbb{N}$ let $\mathscr{T}_n = (T_n, t_0)$, where T_n is the uniform distribution over $[-n, n]$ and where under t_0—given $\theta_0 = \vartheta_0 - \theta_1$, $\cdots, \theta_k$ are independent, each θ_ι assuming the value $\vartheta_0 + \Delta_i(\vartheta_0, \vartheta_0 + \Delta_\iota/2)$ with probability $\pi_i(\pi_\iota', 1 - \pi_\iota - \pi_\iota')$, $i = 1, \cdots, k$. Now we fix $i \in \{1, \cdots, k\}$.

For $\psi \in \mathscr{D}^\circ$ the Bayes-risk with respect to $\mathscr{T}_n$, $n \in \mathbb{N}$, for the ith component problem is given by $r^{(\iota)}(\mathscr{T}_n, \psi_\iota) =$

$$\int_{\mathbb{R}} \{L_{2i}\pi_\iota' E_{(\vartheta_0,\vartheta_0)}\psi_\iota(\mathbf{X}_0, \mathbf{X}_i) + L_{1i}\pi_\iota[1 - E_{(\vartheta_0,\vartheta_0+\Delta_\iota)}\psi_i(\mathbf{X}_0, \mathbf{X}_i)]\}\, dT_n(\vartheta_0).$$

Thus the Bayes-rules with respect to $\mathscr{T}_n$, $n \in \mathbb{N}$, and $\mathscr{D}^\circ$ turn out to be

$$\psi^B_{in}(\mathbf{X}_0, \mathbf{X}_\iota) = 1(h, 0) \quad \text{if} \quad L_{2i}\pi_i' \int_{\mathbb{R}} f_0(Z_0 - \vartheta_0)f_\iota(Z_i - \vartheta_0)\, dT_n(\vartheta_0)$$

$$< (=, >) L_{1\iota}\pi_\iota \int_{\mathbb{R}} f_0(Z_0 - \vartheta_0)f_i(Z_\iota - \vartheta_0 - \Delta_i)\, dT_n(\vartheta_0),$$

where $h = h(Z_0, Z_\iota) \in [0, 1]$ may be chosen arbitrarily since it has no influence upon the risk. Standard analysis leads us to

$$r^{(\iota)}(\mathscr{T}_n, \psi^B_{in}) = \int_{\mathbb{R}}\int_{\mathbb{R}} \min\{L_{2\iota}\pi_i'(2n)^{-1} \int_{-n}^{n} f_0(z_0 - \eta)f_i(z_\iota - \eta)\, d\eta,$$

$$L_{1\iota}\pi_i(2n)^{-1} \int_{-n}^{n} f_0(z_0 - \eta)f_i(z_i - \Delta_\iota - \eta)\, d\eta\}\, dz_0\, dz_i.$$

Substituting first $z_0 = nv + u$ and $z_i = nv - u$ in the outer integrals and then $\eta = -\xi + nv$ in the interior ones, we arrive at

$$r^{(\iota)}(\mathscr{T}_n, \psi^B_{in}) = \int_{\mathbb{R}}\int_{\mathbb{R}} \min\{L_{2i}\pi_\iota' \int_{n(v-1)}^{n(v+1)} f_0(\xi + u)f_\iota(\xi - u)\, d\xi,$$

$$L_{1\iota}\pi_i \int_{n(v-1)}^{n(v+1)} f_0(\xi + u)f_i(\xi - \Delta_i - u)\, d\xi\}\, dv\, du$$

$$\geq \int_{\mathbb{R}}\int_{-1}^{1} \min\{L_{2\iota}\pi_i' \int_{n(v-1)}^{n(v+1)} f_0(\xi + u)f_i(\xi - u)\, d\xi,$$

$$L_{1i}\pi_\iota \int_{n(v-1)}^{n(v+1)} f_0(\xi + u)f_i(\xi - \Delta_\iota - u)\, d\xi\}\, dv\, du.$$

Now we can apply Lebesgue's dominated convergence theorem to the last double integral, since the "min"-terms converge pointwise in $(u, v) \in \mathbb{R} \times (-1, 1)$ to $\min\{L_{2i}\pi_i' g_\iota(2u), L_{1\iota}\pi_\iota g_i(2u - \Delta_\iota)\}$ which at the same time is an integrable upper bound for them. Thus we have

$$\text{(5)} \qquad \liminf_{n\to\infty} r^{(i)}(\mathscr{T}_n, \psi^B_{in}) \geq \int_{\mathbb{R}} \min\{L_{2\iota}\pi_i' g_i(u), L_{1\iota}\pi_\iota g_i(u - \Delta_\iota)\}\, du.$$

The right-hand side of (5), however, equals $L_{2i}\pi_i'\alpha_i + L_{1i}\pi_i[1 - E_{\Delta_\iota}\bar{\varphi}^*_{i,\alpha_\iota}(W_i)]$ for $\alpha_i \in \bar{A}_i$:

This follows from the fact that every $\bar{\varphi}^*_{i,\alpha_i}$ with $\alpha_i \in \bar{A}_i$ can be viewed as being a Bayes-test for an auxiliary problem, where W_i is known to have either the density $g_i(u)$ or $g_i(u - \Delta_i)$, the losses of errors of both kinds are unity and the prior probabilities are $L_{2i}\pi_i'/(L_{1i}\pi_i + L_{2i}\pi_i')$ and $L_{1i}\pi_i/(L_{1i}\pi_i + L_{2i}\pi_i')$. Thus (2) is verified and the proof is completed by noting that for $\alpha \in [0, 1]$

$$\begin{aligned}\sup_{\mathscr{T}\in\Gamma} r^{(i)}(\mathscr{T}, \bar{\varphi}^*_{i,\alpha}) &= \sup_T \sup_{t,(T,\,t)\in\Gamma} r^{(i)}((T, t), \bar{\varphi}^*_{i,\alpha}) \\ &= \sup_T \int_{\mathbb{R}} \{L_{2i}\pi_i' E_0 \bar{\varphi}^*_{i,\alpha}(W_i) \\ &\qquad + L_{1i}\pi_i[1 - E_{\Delta_i}\bar{\varphi}^*_{i,\alpha}(W_i)]\}\, dT(\vartheta_0) \\ &= L_{2i}\pi_i'\alpha + L_{1i}\pi_i[1 - E_{\Delta_i}\bar{\varphi}^*_{i,\alpha}(W_i)],\end{aligned}$$

which again follows from the monotonicity properties of U.M.P. tests in the testing parameters in case of monotone likelihood ratios.

REMARK. It is not difficult to see that in fact we have proved that

$$\lim_{n\to\infty} r^{(i)}(\mathscr{T}_n, \psi^B_{in}) = \sup_{\mathscr{T}\in\Gamma} r^{(i)}(\mathscr{T}, \psi_i^\Gamma), \qquad i = 1, \cdots, k.$$

Example 2. For $i = 1, \cdots, k$ let $\mathbf{X}_i$ be a sample from $N(\vartheta_i, \sigma^2)$ where the variance $\sigma^2 > 0$ is known. Then by Theorem 2 for $i = 1, \cdots, k$ we have

$$\bar{\psi}_i^\Gamma(\mathbf{X}) = 1 \quad \text{iff} \quad \bar{X}_i - \bar{X}_0 \geq \Delta_i/2 + \sigma^2\Delta_i^{-1}(n_i^{-1} + n_0^{-1}) \ln(L_{2i}\pi_i'/L_{1i}\pi_i).$$

Now the question arises whether in case of $n_0 \geq k$ it is better to split the sample $\mathbf{X}_0$ into k subsamples $\mathbf{X}_{0i}$ of sizes m_i, $m_1 + \cdots + m_k = n_0$, then to switch over to $\bar{X}_i - \bar{X}_{0i}$, $i = 1, \cdots, k$, and finally to take the rule ψ^Γ provided by Theorem 1, where ψ_i^Γ differs from $\bar{\psi}_i^\Gamma$ only in that $\bar{X}_0$ is replaced by $\bar{X}_{0i}$ (the sample mean of $\mathbf{X}_{0i}$) and n_0 by m_i, $i = 1, \cdots, k$.

If comparison is made in terms of supremal risks over the corresponding classes $\bar{\Gamma}$ and Γ, say, the answer turns out to be in favor of $\bar{\psi}^\Gamma$:

$$\sup_{\mathscr{T}\in\Gamma} r(\mathscr{T}, \psi^\Gamma) \geq \sup_{(T,t)\in\bar{\Gamma}} r((T, t), \bar{\psi}^\Gamma).$$

If, on the other hand, comparison is made in terms of risks, this time point-wise over pairs of "comparable priors", then it is not possible to give an unique answer in favor of one of the two competitors. But there is an exception:

Let $L_{1i}\pi_i = L_{2i}\pi_i'$, $i = 1, \cdots, k$, and let $\mathscr{T}$ be any prior of $(\theta_1 - \theta_0, \cdots, \theta_k - \theta_0)$ with respect to ψ^Γ. If (T, t) with respect to $\bar{\psi}^\Gamma$ is chosen in such a way that under (T, t) the conditional distribution of $\theta_i - \theta_0$—given $\theta_0 = \vartheta_0$— coincides with the marginal distribution of $\theta_i - \theta_0$ under $\mathscr{T}$, $i = 1, \cdots, k$, then

$$r^{(i)}((T, t), \bar{\psi}^\Gamma) \leq r^{(i)}(\mathscr{T}, \psi^\Gamma), \qquad i = 1, \cdots, k.$$

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