The Annals of Statistics
1981, Vol. 9, No. 1, 177–183

A CLASS OF NONLINEAR ADMISSIBLE ESTIMATORS IN THE ONE-PARAMETER EXPONENTIAL FAMILY

BY DAN RALESCU[1] AND STEFAN RALESCU

Indiana University

We are concerned with the admissibility of nonlinear estimators of the form $(aX + b)/(cX + d)$ in the one-parameter exponential family, in estimating $g(\theta)$ with quadratic loss. Our method will be reminiscent of that of Karlin who gave sufficient conditions for admissibility of linear estimators aX in estimating the mean in the one-parameter family. Our results generalize those of Ghosh and Meeden who studied admissibility of $aX + b$ for estimating an arbitrary function $g(\theta)$. Particular cases of estimators of the form, c/X are studied and several examples are given. We show that $(n - 2)/(X + a)$, $a \geq 0$ is admissible in estimating the parameter of an exponential density. We also discuss the case of truncated parameter space.

1. Introduction. Let us consider X a random variable, whose density with respect to some σ-finite measure μ belongs to the one-parameter exponential family: $f_\theta(x) = \beta(\theta)e^{\theta x}$. The natural parameter space $\Theta = \{\theta: \int e^{\theta x}\, d\mu(x) < \infty\}$ is known to be an interval in $\mathbb{R}$. The end points of this interval will be denoted by $\underline{\theta}$ and $\bar{\theta}$, respectively.

In Karlin (1958) sufficient conditions are given for the admissibility of linear estimators of the form aX, in estimating the mean $E_\theta X$. This result was generalized in several directions. More recently, Ghosh and Meeden (1977) gave sufficient conditions for the admissibility of estimators of the form $aX + b$, in estimating an arbitrary piecewise continuous, locally integrable function $g(\theta)$.

These sufficient conditions mentioned above, are expressed in terms of divergence of some improper integrals.

There are, however, important examples when admissible estimators are naturally of the form c/X, rather than aX or $aX + b$. The presence of such nonlinear admissible estimators arises especially in estimating a function of a scale parameter, e.g., in a gamma density (see Ghosh and Singh (1970)). We shall see in this paper that Karlin's method can be developed to give sufficient conditions for the admissibility of nonlinear estimators of the form $(aX + b)/(cX + d)$, in estimating an arbitrary function $g(\theta)$, where $X \sim f_\theta$.

The main theorem which is presented in Section 2 includes both the results of Karlin (1958) and Ghosh and Meeden (1977) as particular cases. As a corollary, we give sufficient conditions for the admissibility of estimators of the form c/X.

In Section 3 we present several examples of admissible estimators of the form $(aX + b)/(cX + d)$ and c/X. These examples come especially from estimating a function $g(\lambda)$ in an exponential density $\lambda e^{-\lambda x} I_{(0, \infty)}(x)$ and $1/\sigma^2$ in a normal density $N(0, \sigma^2)$.

In Section 4 we derive admissible estimators of the form $(aX + b)/(cX + d) + \phi(X)$ (where $\phi(X)$ is a "correction") in the case when the parameter space is truncated.

2. Admissibility of $(aX + b)/(cX + d)$. Let us consider a function $g(\theta)$ which is piecewise continuous; further restrictions will be imposed later on g.

As in Ghosh and Meeden (1977) we first write $(aX + b)/(cX + d)$ as a formal Bayes estimator, with respect to some (generally improper) prior π. For more details of this kind of approach, see Zidek (1970). If $\pi(\theta)$ is the Radon-Nikodym derivative of the prior distribution

---

Received June 1979; revised October 1979.

[1] Work supported by the Air Force Office of Scientific Research, AFSC USAF, Grant No. AFOSR 76-2927.

AMS 1970 *subject classifications.* 62C15, 62F10.

Key words and phrases. Nonlinear admissible estimators, quadratic loss, formal Bayes estimators, scale parameter, truncated parameter space.

with respect to the Lebesgue measure, we can write

$$(2.1) \qquad (ax+b)/(cx+d) = \left(\int g(\theta)\beta(\theta)e^{\theta x}\pi(\theta)\,d\theta\right)\Big/\left(\int e^{\theta x}\beta(\theta)\pi(\theta)\,d\theta\right).$$

Integrating by parts, we get:

$$(2.2) \qquad -a\int e^{\theta x}(\beta\pi)'\,d\theta + b\int e^{\theta x}\beta\pi\,d\theta = -c\int (g\beta\pi)'\,e^{\theta x}\,d\theta + d\int g\beta\pi e^{\theta x}\,d\theta$$

and, by the unicity of the Laplace transform,

$$(2.3) \qquad -a(\beta\pi)' + b(\beta\pi) = -c(g\beta\pi)' + d(g\beta\pi).$$

The above differential equation has the solution:

$$(2.4) \qquad \pi(\theta) = \frac{1}{\beta(\theta)\,|\,cg(\theta)-a\,|}\exp\left(\int_{\alpha}^{\theta}\frac{dg(t)-b}{cg(t)-a}\,dt\right)$$

where α is an interior point of Θ.

Throughout the remainder of this paper, we shall make the following assumptions:

(A1) $cg(\theta) - a > 0$, for any $\theta \in \Theta$.

(A2) $\displaystyle\int_u^v \frac{dg(t)-b}{cg(t)-a}\,dt$ exists, for any $[u, v] \subset \Theta$.

(A3) $\displaystyle\int_{-\infty}^{\infty}\frac{e^{\theta x}}{(cx+d)^2}\,d\mu(x) < \infty.$

The main result is contained in the following

THEOREM. *Let $X \sim f_\theta(x) = \beta(\theta)e^{\theta x}$, and $\underline{\theta}$, $\bar{\theta}$ be the endpoints of Θ. Suppose that conditions* (A1)–(A3) *are satisfied. Denote by*

$$(2.5) \qquad \sigma(\theta) = \pi(\theta)\beta(\theta)(cg(\theta)-a)^2\int_{-\infty}^{\infty} e^{\theta x}(cx+d)^{-2}\,d\mu(x),$$

where $\pi(\theta)$ is given by (2.4). *If*

$$(2.6) \qquad \lim_{v\to\bar{\theta}}\int_u^v \sigma^{-1}(\theta)\,d\theta = \infty, \qquad \lim_{u\to\underline{\theta}}\int_u^v \sigma^{-1}(\theta)\,d\theta = \infty$$

then $(aX+b)/(cX+d)$ is admissible for estimating $g(\theta)$, with quadratic loss.

PROOF. Suppose that $(aX+b)/(cX+d)$ is not admissible; then there exists an estimator δ such that

$$(2.7) \qquad \int_{-\infty}^{\infty}(\delta(x)-g(\theta))^2 f_\theta(x)\,d\mu(x) \le \int_{-\infty}^{\infty}[(ax+b)/(cx+d)-g(\theta)]^2 f_\theta(x)\,d\mu(x).$$

We will show that $\delta(x) = (ax+b)/(cx+d)$ a.e. First, (2.7) is equivalent to:

$$(2.8) \qquad \int [\delta(x)-(ax+b)/(cx+d)]^2 f_\theta(x)\,d\mu(x)$$
$$\le 2\int_{-\infty}^{\infty}[(ax+b)/(cx+d)-\delta(x)][(ax+b)/(cx+d)-g(\theta)]f_\theta(x)\,d\mu(x).$$

Multiplying both sides by π, integrating over $[u, v] \subset \Theta$, and using Fubini's theorem, we get:

$$\int_u^v \left(\int_{-\infty}^{\infty} [\delta(x) - (ax + b)/(cx + d)]^2 \beta(\theta) e^{\theta x} \, d\mu(x) \right) \pi(\theta) \, d\theta$$

$$\leq 2 \int_{-\infty}^{\infty} [(ax + b)/(cx + d) - \delta(x)] \tag{2.9}$$

$$\times \left\{ \int_u^v [(ax + b)/(cx + d) - g(\theta)] \beta(\theta) e^{\theta x} \pi(\theta) \, d\theta \right\} d\mu(x).$$

By using (2.4) and assumption (A1), the inner integral in the right-hand side of (2.9) simplifies, after some calculations, to:

$$\int_u^v [(ax + b)/(cx + d) - g(\theta)] \beta(\theta) e^{\theta x} \pi(\theta) \, d\theta \tag{2.10}$$

$$= (cx + d)^{-1} \left[\exp\left(ux + \int_\alpha^u \frac{dg(t) - b}{cg(t) - a} \, dt \right) - \exp\left(vx + \int_\alpha^v \frac{dg(t) - b}{cg(t) - a} \, dt \right) \right].$$

Denote by $T(\theta) = \int_{-\infty}^{\infty} [\delta(x) - (ax + b)/(cx + d)]^2 \beta(\theta) e^{\theta x} \, d\mu(x)$; it is enough to show that $T(\theta_0) = 0$, for some θ_0.

By using (2.9), (2.10) and the Schwarz inequality, we get:

$$\int_u^v T(\theta) \pi(\theta) \, d\theta$$

$$\leq 2 \int_{-\infty}^{\infty} \left(\frac{ax + b}{cx + d} - \delta(x) \right) \frac{1}{cx + d}$$

$$\cdot \left\{ \exp\left(ux + \int_\alpha^u \frac{dg(t) - b}{cg(t) - a} \, dt \right) - \exp\left(vx + \int_\alpha^v \frac{dg(t) - b}{cg(t) - a} \, dt \right) \right\} d\mu(x)$$

$$\leq 2T^{1/2}(u) \beta^{-1/2}(u) \left(\int_{-\infty}^{\infty} \frac{e^{ux}}{(cx + d)^2} \, d\mu(x) \right)^{1/2} \exp\left(\int_\alpha^u \frac{dg(t) - b}{cg(t) - a} \, dt \right)$$

$$+ 2T^{1/2}(v) \beta^{-1/2}(v) \left(\int_{-\infty}^{\infty} \frac{e^{vx}}{(cx + d)^2} \, d\mu(x) \right)^{1/2} \exp\left(\int_\alpha^v \frac{dg(t) - b}{cg(t) - a} \, dt \right)$$

$$= 2T^{1/2}(u) \beta^{1/2}(u) \pi(u) (cg(u) - a) \left(\int \frac{e^{ux}}{(cx + d)^2} \, d\mu(x) \right)^{1/2}$$

$$+ 2T^{1/2}(v) \beta^{1/2}(v) \pi(v) (cg(v) - a) \left(\int \frac{e^{vx}}{(cx + d)^2} \, d\mu(x) \right)^{1/2} \tag{2.11}$$

Let us consider the following cases:

CASE 1. $\liminf_{v \to \bar{\theta}} \pi(v) T^{1/2}(v) \beta^{1/2}(v) (cg(v) - a) (\int (e^{vx}/(cx + d)^2) \, d\mu(x))^{1/2} > 0$.

By using this and (2.11), we get:

$$M(v) = \int_u^v T(\theta) \pi(\theta) \, d\theta \leq K \pi(v) T^{1/2}(v) \beta^{1/2}(v) (cg(v) - a) \left(\int \frac{e^{vx}}{(cx + d)^2} \, d\mu(x) \right)^{1/2} \tag{2.12}$$

for v in some neighborhood V of $\bar{\theta}$, and K is a generic constant, possibly depending on u, whose exact value plays no role in the subsequent analysis. Then

$$M(v) \leq K[M'(v)\pi(v)]^{1/2}\beta^{1/2}(v)(cg(v) - a)\left(\int \frac{e^{vx}}{(cx+d)^2}\, d\mu(x)\right)^{1/2} \tag{2.13}$$

or

$$\frac{M'(v)}{M^2(v)} \geq \frac{1}{K\pi(v)\beta(v)(cg(v) - a)^2 \displaystyle\int \frac{e^{vx}}{(cx+d)^2}\, d\mu(x)}. \tag{2.14}$$

Choose $v_1, v_2 \in V$, $v_1 < v_2$, and assume $M(v_1) > 0$. Then

$$\begin{aligned} \frac{1}{M(v_1)} - \frac{1}{M(v_2)} &= \int_{v_1}^{v_2} \frac{M'(v)}{M^2(v)}\, dv \\ &\geq \int_{v_1}^{v_2} \frac{dv}{K\pi(v)\beta(v)(cg(v) - a)^2\left(\displaystyle\int \frac{e^{vx}}{(cx+d)^2}\, d\mu(x)\right)}. \end{aligned} \tag{2.15}$$

Since the left-hand side is bounded by $[M(v_1)]^{-1}$ and the right-hand side equals $K^{-1} \int_{v_1}^{v_2} \sigma^{-1}(v)\, dv$ we get a contradiction, by letting $v_2 \to \bar{\theta}$ and using the hypothesis (2.6).

CASE 2. $\liminf_{v\to\bar{\theta}} \pi(v)T^{1/2}(v)\beta^{1/2}(v)(cg(v) - a)(\int (e^{vx}/(cx+d)^2)\, d\mu(x))^{1/2} = 0.$

Then, by using Fatou's lemma, we get

$$\int_u^{\bar{\theta}} T(\theta)\pi(\theta)\, d\theta \leq 2\pi(u)T^{1/2}(u)\beta^{1/2}(u)(cg(u) - a)\left(\int \frac{e^{ux}}{(cx+d)^2}\, d\mu(x)\right)^{1/2} \tag{2.16}$$

If we denote by $N(u) = \int_u^{\bar{\theta}} T(\theta)\pi(\theta)\, d\theta$, we can write:

$$N^2(u) \leq 2(-N'(u))\pi(u)\beta(u)(cg(u) - a)^2\left(\int \frac{e^{ux}}{(cx+d)^2}\, d\mu(x)\right) \tag{2.17}$$

Thus:

$$\frac{-N'(u)}{N^2(u)} \geq \frac{1}{2\pi(u)\beta(u)(cg(u) - a)^2\left(\displaystyle\int \frac{e^{ux}}{(cx+d)^2}\, d\mu(x)\right)}. \tag{2.18}$$

If $N(u_0) = 0$ for some u_0, then $T(\theta)\pi(\theta) = 0$ a.e. on $[u_0, \bar{\theta}]$, therefore $T(\theta_0) = 0$ for some θ_0, and we are done.

If we assume $N(u) \neq 0$ for any u, then, by using the same argument as in Case 1, and the second half of the hypothesis (2.6), we are led to a contradiction.

REMARK. The assumption (A1) can be replaced throughout, by $cg(\theta) - a < 0$ for any $\theta \in \Theta$.

Observe that the above theorem includes the result of Ghosh and Meeden (1977), if we take $c = 0$, $d = 1$.

As a particular case of our theorem, we have the following

COROLLARY. *Suppose that $g(\theta) > 0$ for any $\theta \in \Theta$, $\int_u^v (dt/g(t))$ exists for any $[u, v] \subset \Theta$, and $\int_{-\infty}^{\infty} (e^{\theta x}/x^2)\, d\mu(x) < \infty$. If:*

$$\lim_{v\to\bar{\theta}} \int_u^v \left[g(\theta)\int_{-\infty}^{\infty} e^{\theta x}/x^2\, d\mu(x)\right]^{-1} \cdot \exp\left(c\int_\alpha^\theta \frac{1}{g(t)}\, dt\right) d\theta = \infty \tag{2.19}$$

(2.20) $$\lim_{u\to\underline{\theta}} \int_u^v \left[g(\theta) \int_{-\infty}^{\infty} e^{\theta x}/x^2 \, d\mu(x) \right]^{-1} \cdot \exp\left(c \int_\alpha^\theta \frac{1}{g(t)} dt \right) d\theta = \infty,$$

then c/X is admissible for estimating $g(\theta)$ with quadratic loss.

3. Examples. The examples to be presented below are related to the estimation of a function of the scale parameter in a gamma density.

EXAMPLE 1. Suppose that $X_1, X_2, \cdots, X_n$ are i.i.d with exponential density $\lambda e^{-\lambda x} I_{(0,\infty)}(x)$, where $\lambda > 0$. We want to estimate $g(\lambda) = \lambda$.

Since $X = \sum_{i=1}^n X_i$ is a sufficient statistic for λ, we can consider estimators based on X. The density of X is gamma, of the form

(3.1) $$f_\lambda(x) = \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} I_{(0,\infty)}(x).$$

By changing the parameter into $\theta = -\lambda$, we get:

(3.2) $$f_\theta(x) = \frac{(-\theta)^n}{\Gamma(n)} x^{n-1} e^{\theta x} I_{(0,\infty)}(x), \qquad \theta < 0$$

and we estimate $g(\theta) = -\theta$.

It is easy to see that conditions (2.19), (2.20) in the corollary above are satisfied for $c = n - 2$. Thus, if $n \geq 3$, the estimator $(n-2)/X$ is admissible in estimating λ. This is a well-known result (see Ghosh and Singh (1970)).

EXAMPLE 2. Consider again $X_1, \cdots, X_n \sim \lambda e^{-\lambda x} I_{(0,\infty)}(x)$, $\lambda > 0$. We want to estimate $g(\lambda) = \lambda$.

It is easy to see that if $X = \sum_{i=1}^n X_i$, the estimator $(n-2)/(X+k)$ is admissible in estimating λ, for any $k \geq 0$.

This result does not seem to be known. Of course, Example 1 is a particular case, for $k = 0$.

Also note that the estimators $(n-2)/(X+k)$, $k > 0$, and $(n-2)/X$ are not equivalent (i.e., the risk of $(n-2)/(X+k)$ depends on k), and, therefore, at some points $\lambda > 0$, it is possible to improve upon the risk of $(n-2)/X$.

EXAMPLE 3. In this example we consider $X_1, X_2, \cdots, X_n$ normally distributed with mean 0 and variance $\sigma^2 > 0$. The function to be estimated is $1/\sigma^2$.

Since $X = \sum_{i=1}^n X_i^2$ is sufficient for σ^2, our admissible estimator will be a function of X.

It is well known that $(\sum_{i=1}^n X_i^2)/\sigma^2$ is χ_n^2. If we denote by $\theta = -(2\sigma^2)^{-1}$, then $\theta < 0$ and the density of X is

(3.3) $$f_\theta(x) = \frac{(-\theta)^{n/2}}{\Gamma(n/2)} e^{\theta x} x^{(n/2)-1} I_{(0,\infty)}(x).$$

Also $g(\theta) = -2\theta > 0$. In looking for an admissible estimator of the form c/X, it is easily seen that conditions (2.19), (2.20) are satisfied for $c = n - 4$.

Thus, if $n \geq 5$, the estimator $(n-4)/(\sum_{i=1}^n X_i^2)$ is admissible in estimating $1/\sigma^2$ in an $N(0, \sigma^2)$ population.

EXAMPLE 4. Let us consider again $X_1, X_2, \cdots, X_n \sim N(0, \sigma^2)$ and we want to estimate $g(\sigma^2) = \sigma^2$. If $X = \sum_{i=1}^n X_i^2$, it is well known that $X/(n+2)$ is admissible in estimating σ^2.

By applying the theorem above, we see that $(X+k)/(n+2)$ is admissible in estimating σ^2, for any $k \geq 0$. We have here a surprising property, showing that even if $X/(n+2)$ is admissible, we can strictly improve upon its risk, on "almost the whole parameter space."

To make this more precise, let us denote by $Y_k = (X+k)/(n+2)$, $k \geq 0$. The risk of Y_k

(with quadratic loss) is

$$R(Y_k, \sigma^2) = \frac{k^2 - 4\sigma^2 k + 2(n+2)\sigma^4}{(n+2)^2} \tag{3.4}$$

and the risk of the classical estimator $Y_0 = X/(n+2)$ is $R(Y_0, \sigma^2) = 2\sigma^4/(n+2)$. Therefore we see that $R(Y_k, \sigma^2) < R(Y_0, \sigma^2)$, if $\sigma^2 > k/4$.

Informally speaking, if k goes to 0, then the set on which $R(Y_k, \sigma^2) < R(Y_0, \sigma^2)$ will "approach" the whole parameter space $(0, \infty)$. Thus $X/(n+2)$ is "almost inadmissible."

EXAMPLE 5. Suppose that $X_1, \cdots, X_n$ is a sample from the gamma density: $\frac{\beta^\infty}{\Gamma(\alpha)} x^{\alpha-1}e^{-\beta x}I_{(0,\infty)}(x)$, where $\alpha > 0$ is known, and $\beta > 0$ is unknown. We want to estimate $g(\beta) = \beta$. Since $X = \sum_{i=1}^n X_i$ is sufficient for β and X is also gamma with parameters $n\alpha$ and β, by using the same technique as in Example 1, we find that $(n\alpha - 2)/X$ is admissible for estimating β. If $\alpha = m$ (integer) and $n = 1$, we get the estimator $(m-2)/X$ obtained by Ghosh and Singh (1970).

EXAMPLE 6. In this example we consider again $X_1, \cdots, X_n \sim \lambda e^{-\lambda x}I_{(0,\infty)}(x)$, $\lambda > 0$, and we want to estimate $g(\lambda) = (\lambda - 1)/(\lambda + 1)$.

We shall find here two admissible estimators which have the most general form $(aX+b)/(cX+d)$. We denote again by $X = \sum_{i=1}^n X_i$, $\theta = -\lambda$. The density of X is given by (3.2), and $g(\theta) = (\theta+1)/(\theta-1)$. We claim that, if $n \geq 3$, the estimators

$$[1 - X/(n-1)][1 + X/(n-1)]^{-1} \tag{3.5}$$

$$[1 - X/(n-2)][1 + X/(n-2)]^{-1} \tag{3.6}$$

are both admissible in estimating $g(\theta)$ with quadratic loss.

It is easy to see that assumptions (A1)–(A3) are satisfied. Consider the estimator (3.5), and the integral

$$\int_{-\infty}^{\infty} \frac{e^{\theta x}}{(cx+d)^2}\, d\mu(x) = \int_0^{\infty} \frac{e^{\theta x}}{(x+n-1)^2} x^{n-1}\, dx. \tag{3.7}$$

Clearly

$$\int_0^{\infty} \frac{e^{\theta x}}{(x+n-1)^2} x^{n-1}\, dx \leq \int_0^{\infty} x^{n-3}e^{\theta x}\, dx = \frac{\Gamma(n-2)}{(-\theta)^{n-2}}. \tag{3.8}$$

By using this inequality, it is easy to show that the hypotheses (2.6) of the theorem are satisfied.

The estimator (3.6) is handled in a similar way.

4. Truncated parameter space. By using the same prior (2.4) we can give an explicit formula for the admissible estimator, in the case of truncated parameter space.

Let us suppose that $\theta \in \Theta_0 = \{\theta \leq \theta_0\} \subset \Theta$. Then the generalized Bayes estimator with prior $\pi(\theta)$ is

$$\begin{aligned}\delta(X) = {} & (aX+b)/(cX+d) + (cX+d)^{-1}\exp\left(\theta_0 + \int_\alpha^{\theta_0} \frac{dg(t)-b}{cg(t)-a}\,dt\right) \\ & \cdot\left[\int_{-\infty}^{\theta_0} \frac{1}{cg(\theta)-a}\exp\left(\theta X + \int_\alpha^{\theta} \frac{dg(t)-b}{cg(t)-a}\,dt\right)d\theta\right]^{-1}.\end{aligned} \tag{4.1}$$

In obtaining formula (4.1) we need the following fact: if $f \in L^1(\mathbb{R})$, f is absolutely continuous on any interval of $\mathbb{R}$, and $f' \in L^1(\mathbb{R})$, then $\lim_{x\to\pm\infty} f(x) = 0$ (see Benedetto (1976)).

In our case $f(\theta) = \exp\left(\theta x + \int_\alpha^\theta \frac{dg(t) - b}{cg(t) - a}\,dt\right)$ satisfies these assumptions, since we supposed that the quotient in the right-hand side of (2.1) exists.

It can be shown, as in the proof of the previous theorem, that (4.1) is admissible in estimating $g(\theta)$, $\theta \le \theta_0$.

This result can be seen as a generalization of a corresponding result of Ghosh and Meeden (1977) who found admissible estimators of the form $aX + b + \phi(X)$ (where $\phi(X)$ is the "correction" due to the truncation). Also (4.1) generalizes a theorem of Katz (1961).

Note that in proving the admissibility of (4.1), only the second condition in (2.6) is needed, due to the truncation of the parameter space.

EXAMPLE. Consider $X_1, \cdots, X_n$ i.i.d with density $\lambda e^{-\lambda x} I_{(0,\infty)}(x)$. For the natural parameter space $\Theta = (0, \infty)$, the estimator $(n - 2)/X$ is admissible in estimating $g(\lambda) = \lambda$, as in Example 1 above.

If we know that $\lambda \ge 1$ and want to estimate the same function $g(\lambda) = \lambda$, then the admissible estimator is:

$$\delta(X) = \frac{n-2}{X} + \left(Xe^X \int_1^\infty t^{n-3} e^{-tX}\,dt\right)^{-1}.$$

REFERENCES

[1] BENEDETTO, J. J. (1976). *Real Variable and Integration.* Teubner, Stuttgart.

[2] GHOSH, J. K. and SINGH, R. (1970). Estimation of the reciprocal of scale parameter of a gamma density. *Ann. Inst. Statist. Math* **22** 51–55.

[3] GHOSH, M. and MEEDEN, G. (1977). Admissibility of linear estimators in the one parameter exponential family. *Ann. Statist.* **5** 772–778.

[4] KARLIN, S. (1958). Admissibility for estimation with quadratic loss. *Ann. Math. Statist.* **29** 406–436.

[5] KATZ, M. W. (1961). Admissible and minimax estimates of parameters in truncated spaces. *Ann. Math. Statist.* **32** 136–142.

[6] ZIDEK, J. V. (1970). Sufficient conditions for the admissibility under squared error loss of formal Bayes estimators. *Ann. Math. Statist.* **41** 446–456.

DEPARTMENT OF MATHEMATICS
SWAIN HALL—EAST
INDIANA UNIVERSITY
BLOOMINGTON, INDIANA 47405