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GENERALIZED ASSOCIATION, WITH APPLICATIONS IN MULTIVARIATE STATISTICS

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Association of random variables [*Ann. Math. Statist.* **38** 1466–1474] is generalized by considering increasing functions on certain partially ordered spaces. Generalized association enjoys many of the basic properties possessed by association. An essential connection links the two notions. Generalized association is inherited under certain mixing processes. Applications of this result follow for absolute value ordering, unimodal ordering, and linear ordering of the real numbers.

1. Introduction and Summary. Esary, Proschan, and Walkup (1967) introduce the notion of association, a type of positive dependence, among random variables, develop its properties, and demonstrate its usefulness in establishing bounds and inequalities in reliability, hypothesis testing, and applied probability. (See Section 2 for the exact definition of terms used in this section and throughout the paper.) A key property of association that makes it valuable in a variety of applications is the following: If $X_1, \cdots, X_n$ are associated, then

$$P[\cap_{i=1}^{n} \{X_i \leq (>)\, x_i\}] \geq \prod_{j=1}^{k} P[\cap_{i\in C_j}\{X_i \leq (>)\, x_i\}] \tag{1.1}$$

for all reals $x_1, x_2, \cdots, x_n$ and all partitions $C_1, C_2, \cdots, C_k$ of $\{1, 2, \cdots, n\}$. In fact, (1.1) holds also when X_i is replaced by $f_i(X_1, \cdots, X_n)$ where f_i is any nondecreasing function. Thus, implicit in a conclusion that a set of random variables is associated is a wealth of inequalities, often of direct use in various statistical problems.

Further properties and applications of association are presented in Esary and Proschan (1968), Šidák (1973), Barlow and Proschan (1975), Shaked (1977), Jogdeo (1977, 1978), Hameed and Sampson (1978), among others.

In the present paper, we exploit a generalized notion of association which considers increasing functions on certain partially ordered spaces. The resulting generalized association enjoys the same basic properties ($P_1 - P_4$ of Section 2) possessed by the original notion of association. An essential connection between the original notion of association and the generalized version is pointed out at the end of Section 2.

In Section 3 we establish a basic theorem showing how generalized association is "inherited" under certain commonly encountered mixing processes. Further theoretical consequences of the basic theorem are derived in Section 3.

In Section 4, we present applications in probability and statistical theory by considering particular types of ordering. Thus absolute value ordering yields the useful main result of Jogdeo (1977), (Theorem 4.1.1 below). We also show that absolute value association is preserved under certain integral transformations, thus extending its potential for application; an immediate application shows that i.i.d. normal random variables after mixture on random

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variance and/or random means enjoy absolute value association; see Examples 4.1.3 and 4.1.9 for precise statements.

In Section 4.2 we present applications of generalized association based on unimodal ordering, motivated by the key moving set inequality of Anderson (1955). In particular, an inequality is obtained for the distribution of aiming errors.

In Section 4.3 we show that a number of well known multivariate distributions govern random variables associated in the standard sense; the proofs make use of the theory developed in Section 3. These remarks are then applied to yield generalizations of results of Šidák (1973) and Hameed and Sampson (1978).

In Section 5, we present applications of generalized association to several basic statistical models, such as (i) simultaneous comparisons of treatments with a control, (ii) multiple decision problems, and (iii) analysis of variance problems.

2. Preliminaries. In this section we present definitions, notation, and basic facts used throughout the paper.

We use "increasing" in place of "nondecreasing" and "decreasing" in place of "nonincreasing" throughout.

2.1. Definition. A set of n random variables $Z_1, \cdots, Z_n$ (or equivalently, an n-variate random vector $\mathbf{Z}$) is said to be associated in the sense of Esary-Proschan-Walkup (EPW) (1967) if

$$\mathrm{Cov}[f(\mathbf{Z}), g(\mathbf{Z})] \geq 0$$

for every pair of increasing functions f and g for which the covariance exists; we write $\mathbf{Z}$ is associated (EPW).

(A function f defined on a subset of $R^k \to R$ is said to be *increasing* if it is increasing in each of its arguments.)

Note that association (EPW) coincides with the standard concept of association as used in the literature.

To generalize the EPW notion of association in such a way that its fundamental properties hold (see (P_1)–(P_4) below and the statement concerning independence that follows), we consider *linearly ordered* spaces $(\mathscr{X}_i, \leq^i)$, $i = 1, \cdots, n$. For $\mathbf{a}, \mathbf{b} \in X_{i=1}^n \mathscr{X}_i$, $\mathbf{a} <^* \mathbf{b}$ if $a_i <^i b_i$ for $i = 1, \cdots, n$.

2.2. Definition. A real valued function f defined on $X_{i=1}^n \mathscr{X}_i$ is said to be *increasing with respect to* $<^*$ if for $\mathbf{a}, \mathbf{b} \in X_{i=1}^n \mathscr{X}_i$, $\mathbf{a} <^* \mathbf{b} \Rightarrow f(a) \leq f(b)$; we write f is increasing $(<^*)$.

2.3. Definition. An n-variate random variable Z is said to *associated with respect to* $<^*$ if

$$\mathrm{Cov}[f(Z), \quad g(Z)] \geq 0$$

for every pair f, g of increasing $(<^*)$ functions for which the covariance exists; we write Z is *associated* $(<^*)$.

The assumption of *linearly ordered* spaces $\mathscr{X}_i$, $i = 1, \cdots, n$, is crucial. As a consequence, the fundamental properties of EPW association (see EPW, 1967, or Barlow and Proschan, 1975, page 30) carry over to association $(<^*)$:

(P_1) Any subset of associated $(<^*)$ random variables is associated $(<^*)$.

(P_2) The set consisting of a single random variable is associated $(<^*)$.

(P_3) Increasing functions of associated $(<^*)$ random variables are associated $(<^*)$.

(P_4) If two sets of associated $(<^*)$ random variables are independent of one another, then their union is a set of associated $(<^*)$ random variables.

Note also that (P_2) and (P_4) together imply that independent random variables are associated $(<^*)$.

The proofs of properties (P_1)–(P_4) just above parallel those in the case of EPW association and thus are omitted. (See Barlow and Proschan, 1975, page 30.)

It is easy to display examples in which the omission of the linear ordering requirement on the $\mathscr{X}_i$ no longer yields the basic properties (P_1)–(P_4), on which the theory of association rests. In fact, even the very elementary property (P_2) can fail to hold.

An interesting connection exists between association (EPW) and association ($<^*$):

2.4. REMARK. (a) If $(Z_1, \cdots, Z_n)$ is associated ($<^*$), then $(g_1(Z_1), \cdots, g_n(Z_n))$ is associated (EPW) for all g_i increasing ($\leq^i$), $i = 1, \cdots, n$. (b) Conversely, $(Z_1, \cdots, Z_n)$ is associated ($<^*$) if there exists a set of functions $\{h_1, \cdots, h_n\}$, h_i is strictly increasing ($\leq^i$), $i = 1, \cdots, n$, for which $(h_1(Z_1), \cdots, h_n(Z_n))$ is associated (EPW).

3. Association with respect to a partial ordering. In this section we present theoretical results governing association with respect to product partial orderings; the linearly ordered space in each case is the reals or R^n, with an appropriately chosen linear ordering.

Let $\mathscr{X}$ be a fixed space. If $(\mathscr{X}_i, \leq^i) \equiv (\mathscr{X}, \leq')$ for $i = 1, \cdots, k$, we write $(\mathscr{X}^k, <')$ for $X_{i=1}^k \mathscr{X}_i, <^*)$. Similarly, if $(\mathscr{X}_i, \leq^i) \equiv (\mathscr{X}, \leq')$ for $i = 1, \cdots, k$ and $(\mathscr{X}_i, \leq^i) \equiv (\mathscr{Y}, \leq)$ for $i = k + 1, \cdots, k + l \equiv n$, we write $(\mathscr{X}^k \times \mathscr{Y}^l, <' \times <)$ for $(X_{i=1}^n \mathscr{X}_i, <^*)$. It is worth mentioning that, in spite of the fact that all the forthcoming theorems are true even if every component of the random vector $\mathbf{Z}$ belongs to a different linearly ordered space, we confine ourselves (in the applications of this paper) to the case where $\mathbf{X} = (X_1, \cdots, X_k)$ and $\mathbf{Y} = (Y_1, \cdots, Y_l)$ are random vectors belonging to $(\mathscr{X}^k, <')$ and $(\mathscr{Y}^l, <)$, respectively.

3.1. DEFINITION. A random vector $\mathbf{Y}$ is said to be *stochastically increasing in the random vector* $\mathbf{X}$ *with respect to* $<'$ if $E[f(\mathbf{Y}) \mid \mathbf{X} = \mathbf{x}]$ is increasing in $\mathbf{x}$ for every f increasing ($<$) for which the expectation exists; we write $\mathbf{Y} \uparrow (<')$ st. in $\mathbf{X}$.

Next we present a more general version of theorems of Esary and Proschan (1968), Khatri (1974), and Jogdeo (1978); the theorem yields many applications. We omit the proof since it parallels that of Jogdeo (1978).

3.2. THEOREM. *Let* (a) $\mathbf{X}$ *be associated* ($<'$), (b) $\mathbf{Y}$, *given* $\mathbf{X}$, *be conditionally associated* ($<$), *and* (c) $\mathbf{Y} \uparrow (<')$ st. *in* $\mathbf{X}$. *Then* (1) $(\mathbf{X}, \mathbf{Y})$ *is associated* ($<^*$), (2) *in particular*, $\mathbf{Y}$ *is associated* ($<$).

Note that when $\mathbf{X}$ is a scalar random variable, the bound given by Khatri (1974) follows.

Theorem 3.2 is of special interest in the particular case in which $Y_1, \cdots, Y_n$ are conditionally independent, given $\mathbf{X}$. See Subsections (4.1.1–4.1.4) and 4.3 (ii) for interesting applications. To obtain the desired result, we need the following lemma.

3.3. LEMMA. *Let* $Y_1, \cdots, Y_n$ *be conditionally independent, given* $\mathbf{X}$. *For* $i = 1, \cdots, n$, *let* $Y_i \uparrow (<')$ st. *in* $\mathbf{X}$. *Then* $\mathbf{Y} \uparrow (<')$ st. *in* $\mathbf{X}$.

PROOF. We prove the lemma for $n = 2$. The proof for $n > 2$ is similar and is omitted.

We need to show that $E[f(Y_1, Y_2) \mid \mathbf{X} = \mathbf{x}]$ is increasing ($<'$) in $\mathbf{x}$ for f increasing ($<$). This follows since by the conditional independence of Y_1, Y_2, given $\mathbf{X}$, we have

$$E[f(Y_1, Y_2) \mid \mathbf{X} = x] = E[g(Y_2, \mathbf{x}) \mid \mathbf{X} = \mathbf{x}],$$

where $g(y_2, \mathbf{x}) = E[f(Y_1, y_2) \mid \mathbf{X} = \mathbf{x}]$. □

Using Lemma 3.3.2 we obtain the following corollary to Theorem 3.2.

3.4. COROLLARY. *Let* (a) $\mathbf{X}$ *be associated* ($<'$), (b) $Y_1, \cdots, Y_n$ *be conditionally independent given* $\mathbf{X}$, *and* (c) *for* $i = 1, \cdots, n$, $Y_i \uparrow (<')$ st. *in* $\mathbf{X}$. *Then* (1) $(\mathbf{X}, \mathbf{Y})$ *is associated* ($<^*$)*m and* (2) *in particular*, $\mathbf{Y}$ *is associated* ($<$).

We shall find the following special case of Theorem 3.4 of particular use in applications. See Subsection 4.3 [(i) and (ii)].

3.5. REMARK. Let (a) X be a scalar random variable, (b) $Y_1, \cdots, Y_n$, given X, be conditionally independent, and (c) $Y_i \uparrow (<')$ st. in X for $i = 1, \cdots, n$. Then Y is associated ($<$).

PROOF. The result is an immediate consequence of Theorem 3.4 and property (P_2). □

3.6. REMARK. The conclusion of Corollary 3.5 may *not* hold if the assumption $Y_i \uparrow (<')$ st. in X for $i = 1, \cdots, n$ is dropped. To see this, let Z_1, Z_2, X be mutually independent. Then $Y_1 = Z_1 + X$ and $Y_2 = Z_2 + X$ are associated (EPW) while $Y_1 = Z_1 + X$ and $Y_2 = Z_2 - X$ are not.

4. Applications to statistical theory. In this section we present applications of value in probability and statistical theory. In the next section we present applications of value in statistical methodology.

SUBSECTION 4.1. *Absolute value ordering.* Throughout this subsection, let $\mathscr{X} = R$ and $<^a$ denote *absolute value ordering*: For $\mathbf{a}, \mathbf{b}, \varepsilon R^k$, $\mathbf{a} <^a \mathbf{b}$ if $|a_i| \le |b_i|$ for $i = 1, \cdots, k$.

By Remark 2.4, $\mathbf{X}$ is associated ($<^a$) if $(|X_1|, \cdots, |X_k|)$ is associated (EPW).

Next we state an interesting result of Jogdeo (1977) from which he obtains a number of useful inequalities for the multivariate normal distribution, the symmetric multivariate t distribution, and the Wishart matrix distribution.

4.1.1. THEOREM (Jogdeo). *Let $\mathbf{Z} = (Z_1, \cdots, Z_n)$ be a vector with mutually independent components, each having a symmetric unimodal density. Let $\mathbf{Y} = \mathbf{Z} + \mathbf{X}$, where $\mathbf{X}$ is independent of $\mathbf{Z}$ and $|\mathbf{X}| = (|X_1|, \cdots, |X_n|)$ is associated. Then $|\mathbf{Y}|$ is associated* (EPW).

We obtain Jogdeo's result as an immediate consequence of Lemma 4.1.2 below and Theorem 3.4. The lemma is an immediate consequence of Wintner's (1938) result that the convolution of two symmetric and unimodal densities is symmetric and unimodal.

4.1.2. LEMMA. *Let $\mathbf{X}$ and $\mathbf{Y}$ be as specified in Theorem 4.1.1. Then for $i = 1, \cdots, n$,*

$$Y_i \downarrow (<^a) \quad \text{st. } in \quad \mathbf{X}.$$

4.1.3. EXAMPLE. Let $X_1, \cdots, X_n$, given σ, a random variable, be conditionally independent. Let $X_i \mid \sigma$ be distributed according to ($\sim$) $N(0, \sigma^2)$, $i = 1, \cdots, n$. Then the unconditional vector $(|X_1|, \cdots, |X_n|)$ is associated (EPW).

PROOF. Clearly $X_i \uparrow (<^a)$ st. in σ. Appealing to Corollary 3.5, the result follows. □

Since a function g is increasing ($\le^a$) if and only if $g(x) = g(-x)$ and g is increasing for $x > 0$, a useful consequence of Jogdeo's theorem is presented in:

4.1.4. COROLLARY. *Let $X_1, \cdots, X_n$, given θ, a random variable, be conditionally independent random variables, where X_i has symmetric unimodal density $f_i(x - \theta)$, $i = 1, \cdots, n$. Let g: $R \to R$ be such that $g(x) = g(-x)$ and g is increasing in x for $x > 0$. Then the unconditional random variables $g(X_1), \cdots, g(X_n)$ are associated* (EPW).

Simple applications of Corollary 4.1.4 are obtained when (i) $X_i \mid \theta$ is $N(\theta, \sigma_i^2)$, $i = 1, \cdots, N$, or (ii) $f_i(x \mid \theta) = \frac{1}{2}\lambda_i \exp\{-\lambda_i(x - \theta)\}$, $i = 1, \cdots, n$.

Next we show that absolute value association is preserved under certain integral transformations.

Let A and B be subsets of R such that $(A, +)$ and $(B, +)$ are semigroups. Let μ be a σ-finite measure, invariant under translation and sign changes, and let ν be a sigma-finite measure.

4.1.5. DEFINITION. A function $K: A \times B \to R$ is said to be *decomposable* (μ, ν) if for $\lambda_1, \lambda_2 \in A$, $x \in B$, $K(\lambda_1 + \lambda_2, x)$ may be expressed as follows:

$$K(\lambda_1 + \lambda_2, x) = \int \int K_1^{(\omega)}(\lambda_1, x - y) K_2^{(\omega)}(\lambda_2, y)\, d\mu(y)\, d\nu(\omega),$$

where $K_i^{(\omega)}$ obeys (a) $K_i^{(\omega)}(\lambda, -x) = K_i^{(\omega)}(-\lambda, -x)$ for $i = 1, 2$ and each ω, and (b) $K_i^{(\omega)}(\lambda, x) \geq K_i^{(\omega)}(\lambda, -x)$ when λ and x have the same sign for $i = 1, 2$, and for each ω.

A special case of a theorem of Conlon, León, Proschan, and Sethuraman (1977) is the following:

4.1.6. THEOREM. *Let* (i) $K: A \times B \to R$ *be decomposable* (μ, ν), *and* (ii) $f: B \to R$ *be increasing* $(<^a)$. *Then the transformation*

$$h(\lambda) = \int_B K(\lambda, x) f(x)\, d\mu(x)$$

is $\uparrow (<^a)$.

A useful application of Theorem 4.1.6 is contained in:

4.1.7. COROLLARY. *For each* $i = 1, 2, \cdots, n$, *let* (a) $K_i(\cdot, \cdot)$ *satisfy the hypotheses of Theorem* 4.1.6 *and* (b) $\mathbf{Y}$ *have a density given by*

$$f_{\mathbf{Y}}(\mathbf{y}) = \int \cdots \int \prod_{i=1}^n K_i(x_i, y_i)\, dG(\mathbf{x}),$$

where G *is the distribution of* $\mathbf{X}$, *associated* $(<^a)$. *Then* $\mathbf{Y}$ *is associated* $(<^a)$; i.e., $(|Y_1|, \cdots, |Y_n|)$ *is associated* (EPW).

PROOF. The result follows immediately from Theorem 4.1.6. and Theorem 3.4. □

4.1.8. EXAMPLE. Given $(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$, let $Y_1, \cdots, Y_n$ be conditionally independent, with $Y_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, \cdots, n$. Let $\boldsymbol{\mu} \sim H$ be associated $(<^a)$ and $\boldsymbol{\sigma}^2 \sim G$. Then the (unconditional) joint density function of $Y_1, \cdots, Y_n$ is given by

$$f_{Y_1, \cdots, Y_n}(y_1, \cdots, y_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdot \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^n \phi_i(y_i; \mu_i, \sigma_i^2)\, dG(\boldsymbol{\sigma}^2)\, dH(\boldsymbol{\mu}),$$

where ϕ_i denotes the density of $N(\mu_i, \sigma_i^2)$.

Clearly ϕ is decomposable by a convolution and mixture. From Corollary 4.1.7, it follows that $\mathbf{Y}$ is associated $(<^a)$, i.e., $(|Y_1|, \cdots, |Y_n|)$ is associated (EPW).

SUBSECTION 4.2. *Unimodal ordering.* In this subsection we study unimodal ordering motivated by the following definition of Anderson (1955).

4.2.1. DEFINITION. Let $h: R^n \to R$ be a nonnegative function such that (a) $h(\mathbf{x}) = h(-\mathbf{x})$ for all $\mathbf{x}$ (h is symmetric), and (b) $\{\mathbf{x}: h(\mathbf{x}) \geq u\}$ is convex for each u, $0 < u < \infty$ (h is unimodal). We say h is *symmetric unimodal.*

A symmetric unimodal function h induces a corresponding linear ordering $<^h$ of R^n as follows:

4.2.2. DEFINITION. For all $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in R^n$ we say $\mathbf{x}^{(1)} <^h \mathbf{x}^{(2)}$ if $h(\mathbf{x}^{(1)}) \leq h(\mathbf{x}^{(2)})$ for the symmetric unimodal function h.

The following theorem has an interesting application to problems involving aiming errors; see Application 4.2.10.

4.2.3. THEOREM. *Let* (a) *h be symmetric unimodal,* (b) *$[\mathbf{Z}] = (\mathbf{Z}_1, \cdots, \mathbf{Z}_k)$ be a k-tuple of mutually independent n-dimensional random vectors, each having a symmetric unimodal density* (c) *$[\mathbf{Y}] = [\mathbf{Z}] + (X_1\mathbf{u}, \cdots, X_k\mathbf{u})$, where $\mathbf{u}$ is a fixed n-dimensional vector,* (d) *$[\mathbf{Z}]$ is independent of $\mathbf{X}$, and* (e) *$\mathbf{X}$ is associated ($<^a$). Then $[\mathbf{Y}]$ is associated ($<^h$)*; i.e., *$(h(\mathbf{Y}_1), \cdots, h(\mathbf{Y}_k))$ is associated* (EPW).

We obtain Theorem 4.2.3 as an immediate consequence of Lemma 4.2.4 below and Theorem 3.4.

4.2.4. LEMMA. *Let $[\mathbf{Y}]$ satisfy the hypotheses of Theorem* 4.2.3. *Then $\mathbf{Y}_i \uparrow (<^a)$* st. *in $\mathbf{X}$, $i = 1, \cdots, k$.*

The above lemma is an immediate consequence of the following theorem of Anderson.

4.2.5. THEOREM (Anderson). *Let E be a convex set in R^n such that $\mathbf{x} \in E \Rightarrow -\mathbf{X} \in E$. Let f be a nonnegative symmetric unimodal Lebesgue integrable function. Then for $0 \leq |k| \leq 1$,*

$$\int_E f(\mathbf{x} + k\mathbf{y})\, d\mathbf{x} \geq \int_E f(\mathbf{x} + \mathbf{y})\, d\mathbf{x}.$$

4.2.6. DEFINITION. Let h': $R^n \to R$ be a nonnegative function such that (a) $h'(\mathbf{x}) = h'(-\mathbf{x})$ for all $\mathbf{x}$ (h' is symmetric), and (b) $\{\mathbf{x}: h'(\mathbf{x}) \leq u\}$ is convex for all $0 < u < \infty$. We say that h' is *symmetric reverse unimodal.*

It is easy to verify the following corollary.

4.2.7. COROLLARY. *Under the hypotheses of Theorem* 4.2.3., *$(h'(\mathbf{Y}_1), \cdots, h'(\mathbf{Y}_k))$ is associated* (EPW) *for all h' symmetric reverse unimodal.*

Let $\| \mathbf{x} \| = (\sum_{i=1}^n x_i^2)^{1/2}$.

4.2.8. DEFINITION. Let h: $R^n \to R$ be a nonnegative function such that $\| \mathbf{x} \| \leq \| \mathbf{y} \| \Rightarrow$ $\mathrm{h}(\mathbf{x}) \geq \mathrm{h}(\mathbf{y})$. We say h is a *radially symmetric unimodal* function.

Since the convolution of two radially symmetric unimodal functions is a radially symmetric unimodal function, we can use the ideas of the proof of Theorem 4.1.1 to obtain:

4.2.9. THEOREM. *Let* (a) *$[\mathbf{Z}] = (\mathbf{Z}_1, \cdots, \mathbf{Z}_k)$ be a k-tuple of mutually independent n-dimensional random vectors, each having a radially symmetric unimodal density,* (b) $[\mathbf{Y}] = [\mathbf{Z}] + [\mathbf{X}]$, (c) *$[\mathbf{Z}]$ is independent of $[\mathbf{X}]$, and* (d) *$(\| \mathbf{X}_1 \|, \cdots, \| \mathbf{X}_k \|)$ is associated* (EPW). *Then $(\| \mathbf{Y}_1 \|, \cdots, \| Y_k \|)$ is associated* (EPW).

4.2.10. APPLICATION TO AIMING ERRORS. A useful application of Theorem 4.2.3 may be made in studying the effect of a random environment (say the wind) on the errors in aiming at a target. For simplicity, let $n = 2$, so that the target is fixed at the origin in 2-dimensional space. Let Z_{ij} be the error assuming no wind along axis j ($j = 1, 2$) of the ith projectile aimed at the target, $i = 1, \cdots, k$. For each $i = 1, \cdots, k$ we assume (Z_{i1}, Z_{i2}) is symmetric about the origin. Since small errors are more likely to occur than large errors, it is reasonable to assume that the joint density h_i of (Z_{i1}, Z_{i2}) is symmetric and unimodal for $i = 1, \cdots, k$.

Let $\mathbf{u}$ be the vector indicating the direction of the wind, which is blowing in a fixed direction. Then for the ith projectile aimed at the target ($i = 1, \cdots, k$) the actual error Y_{ij} along axis j ($j = 1, 2$) is given by $Y_{ij} = Z_{ij} + X_i U_i$ where X_i is a function of the wind force. Under the assumptions of Theorem 4.2.3 and Corollary 4.2.7, the random variables $h(\mathrm{Y}_1), \cdots, h(\mathrm{Y}_k)$ are associated (EPW) for each h, symmetric and unimodal (or reverse symmetric unimodal). In particular (a) the distances from the target of the k-projectiles are associated (EPW), (b) the jth coordinates vector $(Y_{ij}, \cdots, Y_{kj})$ is associated ($<^a$); i.e., $(|Y_{1j}|, \cdots, |Y_{kj}|)$ is associated (EPW) for each j ($j = 1, 2$), and (c) the decreasing "scores" designating the successively larger rings in a typical bullseye target are associated (EPW).

Clearly, the result may be extended to Euclidean space of any number of dimensions.

A similar application to aiming errors can be obtained from Theorem 4.2.9.

SUBSECTION 4.3. *Multivariate distributions of associated* (EPW) *random variables.* A number of inequalities have been derived recently for various multivariate distributions, and then used to obtain conservative simultaneous confidence or prediction regions. See, for example, Khatri (1974), Shaked (1975), and Šidák (1973). A simple unified method of obtaining such inequalities is by showing that the random variables, treated are associated (EPW).

In this subsection, we show that association (EPW) exists among the random variables and among certain functions of them, governed by some additional multivariate distributions. We rely on the theory developed in Section 3 to demonstrate such association.

4.3.1. RESULT. Let $\mathbf{Z} = (Z_1, \cdots, Z_k) \sim N(\mathbf{0}_{k\times 1}, \sum_{k\times k})$, where $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \rho\sigma_i\sigma_j$, $0 < \rho < 1$, for $i \neq j$, $i, j = 1, \cdots, k$. Then (a) $\mathbf{Z}$ is associated (EPW), (b) $g(Z_1), \cdots, g(Z_k)$ is associated (EPW) for every function g, satisfying the conditions of Corollary 4.1.4 and (c) in particular, for $r \geq 1$, $(|Z_1|^r, \cdots, |Z_k|^r)$ is associated (EPW).

PROOF. (a) For $i = 1, \cdots, k$, let $Z_i = \sigma_i[(1-\rho)^{1/2} Y_i - \rho^{1/2} Y]$, where $Y, Y_1, \cdots, Y_k$ are i.i.d. $\sim N(0, 1)$. Then $\mathbf{Z} \sim N(\mathbf{0}_{k\times 1}, \sum_{k\times k}$. By Corollary 3.5, $\mathbf{Z}$ is associated (EPW). (b) and (c) follow from Corollary 4.1.4. □

The normal distribution specified in Result 4.3.1 is encountered in many statistical applications of interest, such as in statistical design models for equicorrelated random variables and in problems of ranking and selection. See, for example, Rinott and Santner (1977) and Tong (1977).

Result 4.3.1 may be used to generalize a result of Šidák (1973) in that the restrictions above on the covariance matrix Σ are quite mild and practical.

Hameed and Sampson (1978) obtain total positivity properties for the absolute bivariate and trivariate normal. These contain Result 4.3.1 above for $k = 2, 3$. However for $k \geq 4$ our result is not contained in their result.

4.3.2. RESULT. Corollary 3.5 can also be used to show that $\mathbf{X}$ is associated (EPW) whenever $\mathbf{X}$ is distributed according to one of the following multivariate distributions arising in statistics:

(a) A particular multivariate F distribution considered by Marshall and Olkin (1974),

(b) The multivariate logistic distribution developed by Malik and Abraham (1973),

(c) The multivariate Burr distribution arising in the theory of reliability and life testing (see, for example, Takahasi, 1965),

(d) A multivariate exchangeable FGM distribution (see Shaked, 1975, and Johnson and Kotz, 1975).

5. Applications to statistical practice.

(i) *Simultaneous comparisons of treatments with a control.* Suppose that treatment population i has distribution $F_i(x, \theta_i)$, $i = 1, \cdots, k$, and the control population has distribution $F_0(x, \theta_0)$. Having observed the outcome $t = (t_0, t_1, \cdots, t_k)$ of the statistic $T = (T_0, T_1, \cdots, T_k)$, we wish to construct a joint one-sided confidence region (see Tong, 1977, for a discussion of a special case, namely, the i.i.d. case) for the parameters $\theta_i - \theta_0$, i = 1, $\cdots$, k, based on the random variables X_i with outcome $x_i = t_i - t_0$, $i = 1, \cdots, k$, or alternatively, for the parameters θ_i/θ_0, i = 1, $\cdots$, k. based on the random variable X_i' with outcome $x_i' = t_i/t_0$, $i = 1, \cdots, k$. (Assume, for convenience, that $\theta_0 > 0$.) If T is location (scale) invariant, then $\mathbf{X}(\mathbf{X}')$ satisfies the conditions of Corollary 3.5, and a conservative bound for the joint probability of coverage of the parameters of interest is readily obtained by using the fact that associated (EPW) random variables are positively orthant dependent (see Theorem 3.2 of Barlow and Proschan, 1975, page 33), and consequently for every $\mathbf{a}_{1\times k} = [c, \cdots, c]$

$$P[\mathbf{X} \leq \mathbf{a}] \geq \textstyle\prod_{i=1}^{k} P[X_i \leq c], \qquad \text{and} \qquad P[\mathbf{X}' \leq \mathbf{a}] \geq \prod_{i=1}^{k} P[X_i' \leq c].$$

(ii) *Multiple decision problems.* Let $\pi_1, \cdots, \pi_k$ be k normal populations, where π_i is governed by distribution $N(\theta_i, \sigma^2)$; assume σ^2 is known exactly, while θ_i is known only to belong to an open interval Ω which may be infinite, $i = 1, \cdots, k$. Let $\theta_{[1]} \leq \cdots \leq \theta_{[k]}$ be the ordered (unknown) values of the k θ's, larger values of θ being regarded as "better."

Our aim is to select a subset of the populations which includes the populations having the largest θ value, $\theta_{[k]}$. To this end, we take a random sample $X_{ij} (j = 1, \cdots, n)$ from π_i $(i = 1, \cdots, k)$. Based on the observations, our goal is to select a subset of the k-populations with a probability that the subset contains $\pi_{[k]}$ is at least a specified probability p^*, $(1/k < p^* < 1)$. The probability of correct selection will depend on the spacing $\theta_{[k]} - \theta_{[k-1]}$ between the best and second best populations. Gupta and Panchapakesan (1972) specify a decision rule based on the sample $\{X_{ij}\}$; Tong (1977) establishes that the infinum of the probability of correct selection can be written as

$$\inf_{\theta\epsilon\Omega} P[CS \mid R] = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \Phi\left(z + \frac{(n\delta^*)^{1/2}}{\sigma}\right) d\Phi(z),$$

where Φ is the cumulative standard normal distribution function and δ^* is the minimum value of $\theta_{[k]} - \theta_{[k-1]}$ which must be detected. An application of Corollary 3.5 yields the lower bound:

$$\inf_{\theta\epsilon\Omega} P[CS \mid R] = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \Phi\left(z + \frac{(n^{1/2}\delta^*)}{\sigma}\right) d\Phi(z) > \prod_{i=1}^{k-1} \left[\int_{-\infty}^{\infty} \Phi\left(z + \frac{(n^{1/2}\delta^*)}{\sigma}\right) d\Phi(z)\right].$$

(iii) *Applications to analysis of variance problems such as in Esary and Proschan (1972) can be generalized.*

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