The Annals of Statistics
1980, Vol. 8, No. 6, 1391-1394

SIMULTANEOUS CONFIDENCE BOUNDS FOR THE TAIL OF AN INVERSE DISTRIBUTION FUNCTION[1]

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Confidence bounds are derived for the upper $100p$ percentile of an inverse distribution function. These bounds are considerably less conservative than similar bounds which apply to the entire function.

1. Introduction. If F is a cumulative distribution function (cdf), then define F^{-1} by $F^{-1}(y) = \inf\{x : F(x) \geq y\}$, for $0 \leq y \leq 1$. If F_n is the empirical distribution function of a random sample of size n with common continuous cdf F, then F_n^{-1} may be used to estimate F^{-1}. The random function $G_n(x) = F_n(x) - n^{-1/2}c_1(\alpha)$ will be called a $1 - \alpha$ lower confidence bound for F if

$$\Pr\{G_n(x) \leq F(x); -\infty < x < \infty\} = 1 - \alpha. \tag{1}$$

If G_n is a $1 - \alpha$ lower confidence bound for F, then the random function G_n^{-1} defined by

$$\begin{aligned} G_n^{-1}(y) &= F_n^{-1}(y - n^{-1/2}c_1(\alpha)); y \geq n^{-1/2}c_1(\alpha) \\ &= \quad 0 \quad ; y < n^{-1/2}c_1(\alpha) \end{aligned}$$

gives a $1 - \alpha$ upper confidence bound for F^{-1} in the sense that

$$\Pr\{G_n^{-1}(y) \geq F^{-1}(y); 0 \leq y \leq 1\} = 1 - \alpha. \tag{2}$$

The tables in Smirnov (1948) give constants $c_1(\alpha)$ such that (1), and consequently (2), hold approximately for large n. The constant $c_1(\alpha)$ is the $1 - \alpha$ quantile of the asymptotic distribution of $\sup_{-\infty<x<\infty} n^{1/2}(F_n(x) - F(x))$.

Upper confidence bounds for F^{-1} are needed when F is the null distribution of a test statistic and the critical points for statistical tests at many significance levels are to be estimated by Monte Carlo methods. If one wishes to guarantee with high confidence that all the critical points so determined are conservative, the critical points should be those of a simultaneous upper confidence bound for F^{-1}. However, often only the upper tail of F^{-1} is of interest and it is only necessary to bound $F^{-1}(y)$ for $1 - \gamma \leq y \leq 1$, say for $\gamma = 0.1$ or $\gamma = 0.2$. What is then required is a constant $c_\gamma(\alpha)$ such that

$$\Pr\{F_n(x) - n^{-1/2}c_\gamma(\alpha) \leq F(x); F^{-1}(1 - \gamma) \leq x < \infty\} = 1 - \alpha \tag{3}$$

or in other words such that

$$\Pr\{F_n^{-1}(y - n^{-1/2}c_\gamma(\alpha)) \geq F^{-1}(y), 1 - \gamma \leq y \leq 1\} = 1 - \alpha. \tag{4}$$

The constant $c_\gamma(\alpha)$ is the $1 - \alpha$ quantile of the asymptotic distribution of

$$\sup_{F^{-1}(1-\gamma)\leq x<\infty} n^{1/2}(F_n(x) - F(x)). \tag{5}$$

In this paper the asymptotic distribution of (5) is derived by modifying the method of Doob (1949) for finding $c_1(\alpha)$. Tables of $c_\gamma(\alpha)$ for selected values of γ and α are given below. For γ

Received March 1979; revised August 1979.

[1] Portions of the work were supported by Air Force Grant #AFOSR 72-2386. The author is currently at the U.S. Bureau of the Census.

AMS 1970 *subject classification.* 62E25, 62F25.

Key words and phrases. Empirical distribution function, confidence bounds.

= 0.1, $c_\gamma(\alpha)$ is about one-third as large as $c_1(\alpha)$. Using $c_{0.1}(\alpha)$ in place of $c_1(\alpha)$ therefore narrows the confidence bound on the tail of F^{-1} about as much as a ninefold increase in the sample size.

2. Derivation of confidence bounds. Let γ be fixed. The distribution of $\sup_{F^{-1}(1-\gamma)\le x<\infty} n^{1/2}(F_n(x) - F(x))$ is the same for all continuous F. Therefore assume that F is the uniform distribution function, i.e.,

$$F(x) = 0;\ x \le 0;$$
$$= x;\ 0 < x < 1;$$
$$= 1;\ x \ge 1.$$

Let Z_n be the stochastic process defined by $Z_n(x) = n^{1/2}(F_n(x) - x)$, $0 \le x \le 1$. Let W^0 be the zero-mean Gaussian process with covariance function $E(W^0(s)W^0(t)) = s(1-t)$, $0 \le s \le t \le 1$. Z_n converges weakly to W^0 on the metric space $D[0, 1]$ of all functions on $[0, 1]$ which are right-continuous and have left-hand limits, under the Skorohod topology. It follows that $\sup_{F^{-1}(1-\gamma)\le x<\infty} Z_n(x)$ converges in distribution to $\sup_{F^{-1}(1-\gamma)\le x<\infty} W^0(x)$. (In general, $f(Z_n)$ converges in distribution to $f(W^0)$, provided that for any sequence of functions g_n in $D[0, 1]$ which converge uniformly to a continuous function g, $f(g_n)$ converges to $f(g)$—see Billingsley (1968), page 34 for details.)

The following finds the distribution of $\sup_{F^{-1}(1-\gamma)\le x<\infty} W^0(x)$. The final answer is equation (7).

Proposition 1. *Let Y be a 0-mean Gaussian process with covariance function $\sigma^2 s(1-t)$, $0 \le s \le t \le 1$. Then*

(a) *for $A \ge 0$ and $B \ge -A$,*

$$\Pr\{Y(s) \ge A + Bs, \textit{for some } s \in [0, 1]\} = \exp\{-2A(A+B)/\sigma^2\}$$

(b) $\Pr\{Y(s) \ge A + Bs, \textit{for some } s \in [0, 1]\} = 1$, *for* $-A > B$.

PROOF.

(a) $\Pr\{Y(s) \ge A + Bs, \text{ for some } s \in [0, 1]\} = \Pr\left\{Y\left(\frac{t}{t+1}\right) \ge A + B\left(\frac{t}{t+1}\right);\ \text{for some } t \in [0, \infty)\right\} = \Pr\left\{(t+1)Y\left(\frac{t}{t+1}\right) \ge A + (A+B)t;\ \text{for some } t \in [0, \infty)\right\}$. In Doob (1949) it is shown that this probability is $\exp\{-2A(A+B)/\sigma^2\}$.

(b) If $B < -A$, then a.s. $Y(1) = 0 > A + B\cdot 1$.

Proposition 2. *The process $Y(s) = W^0(\gamma s) - sW^0(\gamma)$, $0 \le s \le 1$ is a zero-mean Gaussian process with covariance function $\gamma s(1-t)$, $0 \le s \le t \le 1$ and is independent of $W^0(r)$, $\gamma \le r \le 1$.*

PROOF. (a) $Y(s)$ is normal and $EY(s) = EW^0(\gamma s) - sEW^0(\gamma) = 0$.

(b) $E(Y(s)Y(t)) = E(W^0(\gamma s) - sW^0(\gamma))(W^0(\gamma t) - tW^0(\gamma)) = \gamma s(1-\gamma t) - t\gamma s(1-\gamma) - s\gamma t(1-\gamma) + st\gamma(1-\gamma) = \gamma s(1-t)$.

(c) (Independence.) For $0 \le s \le 1$ and $\gamma \le r \le 1$, $E(W^0(\gamma s) - sW^0(\gamma))W^0(r) = \gamma s(1-r) - s\gamma(1-r) = 0$. Since $Y(s)$ and $W^0(r)$ are uncorrelated and normally distributed, they are independent. Since this is true for each s, the Gaussian process Y is independent of $W^0(r)$.

Proposition 3. *Let $l(x) = a + b(1-x)$, for $1 - \gamma \le x \le 1$, where $a > 0$ and $b > 0$. Then* $\Pr\{W^0(x) \le l(x);\ 1 - \gamma \le x \le 1\} =$

$$(6)\quad \Phi\left(\frac{b + a/\gamma}{(\gamma(1-\gamma))^{1/2}}\right) - \Phi\left(\frac{b + a/\gamma - 2a\gamma(1-\gamma)}{(\gamma(1-\gamma))^{1/2}}\right)\exp(2a^2\gamma(1-\gamma) - 2ab - 2a^2/\gamma).$$

PROOF.

$$
\begin{aligned}
\Pr\{W^0(x) \le l(x);\ 1-\gamma \le x \le 1\}
&= \Pr\{W^0(x) \le a + bx;\ 0 \le x \le \gamma\}\\
&= 1 - \Pr\{W^0(x) \ge a + bx;\ \text{for some } x \in [0,\gamma]\}\\
&= 1 - \Pr\{W^0(x) - xW^0(\gamma) \ge a + (b - W^0(\gamma))x;\ \text{for some } x \in [0,\gamma]\}\\
&= 1 - \Pr\{W^0(\gamma s) - \gamma s W^0(\gamma) \ge a + (b - W^0(\gamma))\gamma s;\ \text{for some } s \in [0,1]\}\\
&= 1 - \Pr\{Y(s) \ge a + (b - W^0(\gamma))\gamma s;\ \text{for some } s \in [0,1]\}\\
&= 1 - \int_{-\infty}^{\infty} \Pr\{Y(s) \ge a + (b-t)\gamma s;\ \text{for some } s \in [0,1]/W^0(\gamma) = t\}\\
&\qquad \cdot \frac{1}{(2\pi\gamma(1-\gamma))^{1/2}} \exp\frac{-t^2}{2\gamma(1-\gamma)}\, dt.
\end{aligned}
$$

By Proposition 1(b) and 2(c) this is equal to

$$
\Phi\left(\frac{b + a/\gamma}{(\gamma(1-\gamma))^{1/2}}\right) - \int_{-\infty}^{b+a/\gamma} \Pr\{Y(s) \ge a + (b-t)\gamma s;\ \text{for some } s \in [0,1]\} \frac{1}{(2\pi\gamma(1-\gamma))^{1/2}} \exp\frac{-t^2}{2\gamma(1-\gamma)}\, dt.
$$

By Proposition 1(a) and Proposition 2, the integral is equal to

$$
\begin{aligned}
&\int_{-\infty}^{b+a/\gamma} \frac{1}{(2\pi\gamma(1-\gamma))^{1/2}} \exp\left(\frac{-2a(a + b\gamma - t\gamma)}{\gamma} - \frac{t^2}{2\gamma(1-\gamma)}\right) dt\\
&= \int_{-\infty}^{b+a/\gamma} \frac{1}{(2\pi\gamma(1-\gamma))^{1/2}}\\
&\qquad \cdot \exp\left(-\frac{t^2}{2\gamma(1-\gamma)} + 2at - 2a^2\gamma(1-\gamma) + 2a^2\gamma(1-\gamma) - 2ab - 2a^2/\gamma\right) dt\\
&= \int_{-\infty}^{b+a/\gamma} \frac{1}{(2\pi\gamma(1-\gamma))^{1/2}}\\
&\qquad \cdot \exp\left(-\left(\frac{t}{(2\gamma(1-\gamma))^{1/2}} - a(2\gamma(1-\gamma))^{1/2}\right)\right)\\
&\qquad \cdot \exp(2a^2\gamma(1-\gamma) - 2ab - 2a^2/\gamma)\, dt\\
&= \exp(2a^2\gamma(1-\gamma) - 2ab - 2a^2/\gamma) \int_{-\infty}^{b+a/\gamma} \frac{1}{(2\pi\gamma(1-\gamma))^{1/2}}\\
&\qquad \cdot \exp\left\{\frac{(t - 2a\gamma(1-\gamma))^2}{2\gamma(1-\gamma)}\right\} dt\\
&= \exp(2a^2\gamma(1-\gamma) - 2ab - 2a^2/\gamma)\Phi\left(\frac{b + a/\gamma - 2a\gamma(1-\gamma)}{(\gamma(1-\gamma))^{1/2}}\right). \qquad \square
\end{aligned}
$$

In (6), set $b = 0$; then

$\Pr\{W^0(x) \le a;\ 1-\gamma \le x \le 1\} =$

$$
\Phi\left(\frac{a/\gamma}{(\gamma(1-\gamma))^{1/2}}\right) - \Phi\left(\frac{a/\gamma - 2a\gamma(1-\gamma)}{(\gamma(1-\gamma))^{1/2}}\right)\exp(2a^2\gamma(1-\gamma) - 2a^2/\gamma). \tag{7}
$$

Let $c_\gamma(\alpha)$ be the value of a for which this probability is $1 - \alpha$. Estimate the qth quantile of F by the $q + c_\gamma(\alpha)n^{-1/2}$ quantile of F_n. For large n, the probability is approximately $1 - \alpha$ that no true quantile (for quantiles above $1 - \gamma$) exceeds the estimated quantile. For $\gamma = 1$, $1 - \exp(-2a^2)$ is used in place of (7). Since the length of the confidence band decreases as $n^{-1/2}$, considerable savings result. For example, for each α, $c_{.1}(\alpha)$ is less than one-third as large as $c_1(\alpha)$.

Values of $c_\gamma(\alpha)$

γ \ α	.9	.95	.99	.995	.999
.01	.10730	.12239	.15175	.16277	.18586
.05	.24022	.27399	.33972	.36435	.41606
.1	.34084	.38878	.48203	.51703	.59037
.2	.48771	.55631	.68974	.73983	.84476
.3	.60712	.69251	.85861	.92097	1.05179
.4	.71370	.81410	1.00938	1.08268	1.23623
.5	.80182	.92507	1.14706	1.23036	1.40486
1	1.07298	1.22387	1.51743	1.62762	1.85845

Acknowledgments. The author wishes to thank Gordon Simons for his encouragement and Van L. Parsons for several useful suggestions.

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