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EXACT NONCENTRAL DISTRIBUTIONS OF WILKS' Λ AND WILKS-LAWLEY *U* CRITERIA AS MIXTURES OF INCOMPLETE BETA FUNCTIONS: FOR THREE TESTS

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In this paper it is shown that in the general case the exact noncentral distributions of Wilks' Λ and Wilks-Lawley *U* can be obtained in a very straightforward manner. This completely eliminates the need for the more complicated inverse Mellon transform. It is first shown that any random variable whose moments satisfy Wilks' Type B integral equation (Type B random variables) has a distribution that can be represented as a mixture of incomplete beta functions. Then it is shown that the moments of Wilks' Λ and Wilks-Lawley *U* criteria can be written as mixtures of the moments of Type B random variables. Combining these results yields the noncentral distribution of Wilks' Λ and Wilks-Lawley *U* criteria as mixtures of incomplete beta functions for the following tests: equality of two dispersion matrices; MANOVA; and canonical correlation.

1. Introduction. A popular trend in the derivation of the exact distributions of multivariate test criteria has been the use of inverse Mellin transforms. Mathai (1973a) has pointed out some of the shortcomings of this approach. Mathai (1973b) has also presented 'computable' general series expansions for the exact distribution of various multivariate tests that do not rely on inverse Mellin transforms. The mixture representations presented here, are not only simple to derive but possess the fewest computational difficulties of any representation that has surfaced to date. The most important advantage of the mixture representation over Mathai's series is the ease with which absolute error bounds can be derived. The only problem with the mixture representation is the computation of zonal polynomials. This problem is not unique to the mixture representation. If the recent work of Gupta and Richards (1979) can be extended, zonal polynomials will yield to tractable recursion relations. Although the methods of this paper are quite straightforward, extension of the mixture representation in the noncentral linear case (Tretter and Walster, 1975) to the general case presented here, is not obvious.

2. Type B random variables. Let us define the random variable, W, as a Type B random variable if its moments satisfy Wilks' Type B integral equation, Wilks (1932):

$$E[W^h] = \int_0^1 w^h f(w)\,dw = \prod_{i=1}^{p} \frac{\Gamma[b(i)+h]\,\Gamma[c(i)]}{\Gamma[b(i)]\,\Gamma[c(i)+h]}; \tag{2.1}$$

where $b(i)$ and $c(i)$ are real and positive such that $b(i) < c(i)$ for $i = 1, \cdots p$.

Tretter and Walster (1975) employed Wilks' solution to (2.1) for $f(w)$ to obtain the central distribution of Wilks' Λ criterion for MANOVA as a mixture of incomplete beta functions. Following the same derivation employed in that case, it is possible to find a general representation for the cdf of a Type B random variable:

$$F(w) = \sum_{j=0}^{\infty} \sigma_j B[b(p), \nu(p) - \beta(p) + j] I[b(p), \nu(p) - \beta(p) + j \colon w]; \tag{2.2}$$

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where

$$K = \prod_{i=1}^{p} B[b(i), c(i) - b(i)]^{-1},$$

$$\nu(i) = \sum_{j=0}^{i-1} c(p - j), \quad \text{and} \quad \beta(i) = \sum_{j=0}^{i-1} b(p - j) \quad \text{for} \quad i = 1, \cdots p;$$

$$\sigma_j = \sum_J \left(\prod_{i=1}^{p-1} B_{j(i)}\right) \sum_{r(p-1)=0}^{j(p-1)} C[p - 1, j(p - 1), r(p - 1)]$$

$$\times \sum_{r(p-2)=0}^{j(p-2)+r(p-1)} C[p - 2, j(p - 2) + r(p - 1), r(p - 2)]$$

$$\vdots$$

$$\times \sum_{r(2)=0}^{j(2)-r(3)} C[2, j(2) + r(3), r(2)]C[1, j(1) + r(2), 0];$$

$\sum_J$ denotes the sum over all $j(i)$ such that $\sum_{i=1}^{p-i} j(i) = j$;

$$B_{j(i)} = \frac{\Gamma[c(i + 1) - b(i) + j(i)]}{\Gamma[c(i + 1) - b(i)]\,\Gamma[j(i) + 1]};$$

$$C[i, j(i), r(i)] = \binom{j(i)}{r(i)} B[c(i) - b(i) + j(i) - r(i), \nu(p - i) - \beta(p - i) + r(i)];$$

$$B(a, b) = \frac{\Gamma[a]\,\Gamma[b]}{\Gamma[a + b]}, \text{ the beta function; and}$$

$$I[a, b : w] = B[a, b]^{-1} \int_0^w u^{a-1}(1 - u)^{b-1}\, du, \text{ the incomplete beta function.}$$

3. Noncentral distribution of Wilks' Λ and Wilks-Lawley U criteria. Following the notation of Pillai and Nagarsenker (1972), the moments of Wilks' Λ and Wilks-Lawley U criteria for testing the hypothesis $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$ can be written:

$$E[Y^h] = |\mathbf{\Lambda}|^{-n_1/2} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_\kappa C_\kappa(\mathbf{M})(n_1/2)_\kappa E[W^h], \tag{3.1}$$

where the parameters $c(i)$ of the Type B random variable, W, are: $c(i) = (n - i + 1)/2 + k(i)$. For definitions of the symbols $\sum_\kappa$, $C_\kappa(\mathbf{M})$, and $(n)_\kappa$ in equation (3.1), see James (1964). If y is Wilks' Λ criterion, $b(i) = (n_2 - i + 1)/2$. If, however, Y is Wilks-Lawley U criterion, then $b(i) = (n_1 - i + 1)/2 + k(i)$.

For the test of equality of mean vectors in MANOVA, we have

$$E[Y^h] = \text{etr}\{\mathbf{\Omega}\} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_\kappa C_\kappa(\mathbf{\Omega}) E[W^h], \tag{3.2}$$

where the parameters $c(i)$ of the Type B random variable, W, are: $c(i) = (n - i + 1)/2 + k(i)$. If Y is Wilks' Λ, $b(i) = (n_2 - i + 1)/2$. If Y is Wilks-Lawley U, $b(i) = (n_1 - i + 1)/2 + k(i)$.

For the test of independence (canonical correlation), we have:

$$E[Y^h] = |\mathbf{I}_p - \mathbf{P}^2|^{n/2} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_\kappa C_\kappa(\mathbf{P}^2)(n/2)_\kappa E[W^h], \tag{3.3}$$

where the parameters $c(i)$ of the Type B random variable, W, are: $c(i) = (n - i + 1)/2 + k(i)$. If Y is Wilks' Λ, $b(i) = (n_2 - i + 1)/2$. If Y is Wilks-Lawley U, $b(i) = (q - i + 1)/2 + k(i)$. It should be noted in passing that the definition of $C_2(p, n, q, \mathbf{P}^2)$ given by Pillai and Nagarsenker (1972) following equation (3.12) is incorrect. It should read: $C_2(p, n, q, \mathbf{P}^2) = \Gamma_p(n/2) [\Gamma_p(q/2)\Gamma_p(n_2/2)]^{-1}$.

Now, James (1968) has shown that zonal polynomials of a positive definite matrix are positive. Thus equations (3.1), (3.2) and (3.3) are mixtures of moments of Type B random variables. It follows at once from the theory of mixtures, Robbins (1948) and Robbins and Pitman (1949) that the cdf of Wilks' Λ and Wilks-Lawley U criteria can be written as mixtures of incomplete beta functions by merely substituting equation (2.2) into (3.1), (3.2) and (3.3).

4. Conclusion. Two points are worth mentioning. First the mixture representations provide a straightforward and direct derivation of the distribution of the criteria without the unnecessary application of the inverse Mellin transform. Second, an absolute error bound on the remainder of the truncated series is an immediate consequence of the mixture representations.

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