The Annals of Statistics
1980, Vol. 8, No. 6, 1244–1251

THE EMPIRICAL DISTRIBUTION OF FOURIER COEFFICIENTS

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Suppose $X_1, X_2, \cdots$ are independent, identically distributed complex-valued L^2 random variables with $EX_1 = 0$ and $E(|X_1|^2) = 1$. Let Y_{nk} be the kth Fourier coefficient of $X_1, \cdots, X_n$:

$$Y_{nk} = \sum_{j=1}^{n} X_j \exp\left(\frac{2\pi(-1)^{1/2}kj}{n}\right).$$

Let μ_n be the empirical distribution of $\{n^{-1/2}Y_{nk}: k = 1, \cdots, n\}$. Then μ_n converges to the distribution of $U + iV$, where U and V are independent normal variables with mean 0 and variance ½. This theorem is derived from a similar result for the Fourier coefficients of random permutations of the coordinates of z^n, where z^n is a vector with n coordinates such that $\max_k |z_k^n| = o(n^{1/2})$, as $n \to \infty$.

I. Introduction. Suppose x is a vector in C^n, where C is the complex plane. That is, x has n coordinates, each a complex number. The *empirical distribution* of x is the probability measure on C which places mass n^{-1} on each coordinate of x; it will be denoted by μ_x. The *discrete Fourier transform* $\hat{x}$ is the vector in C^n whose coordinates are given by

$$\hat{x}_k = \sum_{j=1}^{n} x_j \exp\left(\frac{2\pi(-1)^{1/2}kj}{n}\right), \qquad 1 \le k \le n.$$

The coordinates of $\hat{x}$ are the *Fourier coefficients* of x.

Now suppose $X_1, \cdots, X_n$ are independent complex-valued random variables with a common L^2-distribution and suppose x is an observation on $(X_1, \cdots, X_n)$. It seems to be a well-known fact, at least in the case that the X_i's are real valued, that normal probability plots of the real and imaginary parts of the coordinates of $\hat{x}$ tend to be close to linear. This phenomenon is discussed, for example, in Brillinger (1975, pages 95–97) and Mallows (1969), and it is illustrated by some examples in Section 5.

If $X_1, \cdots, X_n$ have either real or complex normal distributions, there is a simple explanation for this phenomenon. (Recall that $Z = X + iY$ has a complex normal distribution if X and Y are independent real-valued normal variables with the same variance.) In the real case, the first $\left[\frac{n-1}{2}\right]$ Fourier coefficients of $(X_1, \cdots, X_n)$ are independent identically distributed complex normal variables, and $\hat{X}_i$ and $\hat{X}_{n-i}$ are conjugate for $1 \le i \le \left[\frac{n-1}{2}\right]$. In the complex case, the first $n - 1$ Fourier coefficients have independent, identical complex normal distributions. Thus in both cases, if x is an observation on $(X_1, \cdots, X_n)$ for n reasonably large, the empirical distribution of $\hat{x}$ should be close to complex normal. Consequently, normal probability plots of the coordinates of Re $\hat{x}$ and Im $\hat{x}$ should be close to linear.

If the distribution of the X_i's is not normal, the situation is not so simple. There are theorems which establish the asymptotic joint normality of a fixed finite number of the Fourier coefficients of $(X_1, \cdots, X_n)$; see, for example, Brillinger (1975, Theorem 4.4.1). However, these theorems do not by themselves prove that the empirical distribution of all the Fourier

Received January, 1979; revised May, 1980.
AMS 1970 *subject classification.* Primary 62E20; secondary 42A16.
Key words and phrases. Fourier coefficients, empirical distribution, discrete Fourier transform, random measures, complex normal distribution, permutation distribution, rankit plot.

coefficients will converge to a normal distribution. That seems to depend on the joint distribution of all n Fourier coefficients, which is hard to estimate. The purpose of this paper is to provide a rigorous mathematical proof for the asymptotic normality of the empirical distribution of the Fourier coefficients. This is the content of Theorem 2 below. We prove this theorem without characterizing the asymptotic joint distribution of all n Fourier coefficients.

Theorem 2 is a consequence of Theorem 1. Suppose z is a vector with complex coordinates, none of which has a particularly large modulus relative to the others. Consider all possible permutations of the coordinates of z. For each permutation, calculate the discrete Fourier transform. Theorem 1 shows that for most permutations, the empirical distribution of the Fourier coefficients will be close to complex normal. Theorem 1 applies in particular, of course, to the case in which z has only real coordinates.

A consequence of Theorems 1 and 2, pointed out in Theorem 3, is that the empirical distribution of the periodogram for data of the sort considered in the theorems should be close to exponential. This phenomenon has been observed empirically by Brillinger (1975, Figure 5.2.5., page 127).

We want to thank Christopher Bingham and David Brillinger for several helpful discussions. Bingham suggested the problem to us.

2. Preliminaries. Let C_o be the set of continuous real-valued functions on C with compact support. Give C_o the sup norm (denoted by $\| \|$), and let $f_1, f_2, \cdots$ be a dense countable subset of C_o. Let M denote the space of probability measures on C. Metrize M as follows: for μ, ν in M, let

$$d(\mu, \nu) = \sum_{i=1}^{n} \frac{\left| \int f_i \, d\mu - \int f_i \, d\nu \right|}{2^i \|f_i\|}.$$

d induces the weak topology on M, but M is not complete with respect to d. Let $\mathscr{M}$ denote the σ-field on M generated by the weak open sets. The space of probabilities on C^k may be given the weak topology in a similar way.

A *random measure* on C is a measurable map from some probability space into $(M, \mathscr{M})$. If μ is a random measure on $(\Omega, \mathscr{F}, P)$, and f a bounded Borel function on C, then $\int f \, d\mu$ is a random variable on $(\Omega, \mathscr{F}, P)$. The set function $E\mu$ is given by $E\mu(A) = \int \mu(A) \, dP$, for A a Borel subset of C. Thus, $E\mu$ is an element of M.

Suppose μ is a random measure satisfying $P(\mu = m) = 1$ for some m in M. Then μ is a *constant measure*, and the random measure μ will sometimes be identified with its value m. Lemma 1 provides a criterion for convergence in probability of a sequence of random measures to a constant measure; its easy proof is omitted.

LEMMA 1 *Suppose $\mu_1, \mu_2, \cdots$ are random measures on $(\Omega, \mathscr{F}, P)$ and m is an element of M. Suppose for each f in C_o:*

(i) $E \int f \, d\mu_n \to \int f \, dm$
(ii) $\mathrm{Var}(\int f \, d\mu_n) \to 0$.

Then μ_n converges in probability to m: that is, $P[d(\mu_n, m) > \epsilon] \to 0$ as $n \to \infty$ for each $\epsilon > 0$.

LEMMA 2 *Suppose m is an element of M. Let m^2 be the product of m with itself, a probability on C^2. Let $\{X_{nk}\}$ be an array of complex-valued random variables, with $n = 1, 2, \cdots$, and $k = 1, \cdots, k_n$. Let m_{nkl} be the joint distribution of X_{nk} and X_{nl}. Suppose the following conditions are satisfied:*

(i) $k_n \to \infty$ *as* $n \to \infty$;
(ii) $m_{nkl} \to m^2$ *as* $n \to \infty$, *uniformly in pairs* (k, l), *except for indices nkl in an exceptional set* E, *with* $\dfrac{\#\{(k, l): nkl \text{ in } E\}}{k_n^2} \to 0$ as $n \to \infty$.

For each n, let μ_n be the empirical distribution of $(X_{n1}, \cdots, X_{nk_n})$, so that μ_n is a random measure. Then μ_n converges in probability to m.

PROOF For f in C_o,

$$\int f\, d\mu_n = (k_n)^{-1} \textstyle\sum_{k=1}^{k_n} f(X_{nk}),$$

so

$$E\int f\, d\mu_n = (k_n)^{-1} \textstyle\sum_{k=1}^{k_n} \displaystyle\int f\, dm_{nk},$$

where m_{nk} is the distribution of X_{nk}. Now (ii) implies that $m_{nk} \to m$ as $n \to \infty$ uniformly in k, except for a set of k's with limiting density 0. Together with (i), this implies that

$$E\int f\, d\mu_n \to \int f\, dm.$$

Next,

$$E\left(\int f\, d\mu_n\right)^2 = (k_n)^{-2}\left[\textstyle\sum_{k\neq l} \displaystyle\int f(x)f(y)m_{nkl}\,(dx\; dy) + \textstyle\sum_l \displaystyle\int f^2\, dm_{nl}\right]$$

so

$$\mathrm{Var}\left(\int f\, d\mu_n\right) = T_1(n) + T_2(n),$$

where

$$T_1(n) = (k_n)^{-2} \textstyle\sum_{k\neq l} \left(\displaystyle\int f(x)f(y)m_{nkl}\,(dx\; dy) - \int f\, dm_{nk} \int f\, dm_{nl}\right)$$

and

$$T_2(n) = (k_n)^{-2} \textstyle\sum_{k=1}^{k_n} \left[\displaystyle\int f^2\, dm_{nk} - \left(\int f\, dm_{nk}\right)^2\right].$$

But $T_2(n)$ is bounded by $k_n^{-1} \| f \|^2$, which converges to 0 by (i). Furthermore, $T_1(n)$ converges to 0 by (ii). Thus, Lemma 1 implies that μ_n converges to m in probability. □

LEMMA 3 *Suppose X_{nk} and Y_{nk} are real-valued random variables, for $n = 1, 2, \cdots$, and $k = 1, \cdots, k_n$, where $k_n \to \infty$ as $n \to \infty$. Suppose that for each four-tuple of real numbers $\lambda_1, \cdots, \lambda_4$, the distribution of $\lambda_1 X_{nk} + \lambda_2 Y_{nk} + \lambda_3 X_{nl} + \lambda_4 Y_{nl}$ converges to $N(0, \frac{1}{2}(\lambda_1^2 + \cdots + \lambda_4^2))$ as $n \to \infty$, uniformly in pairs (k, l), except for indices nkl in an exceptional set E with $\dfrac{\#\{(k, l)\colon nkl \text{ in } E\}}{k_n^2} \to 0$ as $n \to \infty$. Set $Z_{nk} = X_{nk} + (-1)^{1/2} Y_{nk}$, and let μ_n be the empirical distribution of $(Z_{n1}, \cdots, Z_{nk_n})$. Then μ_n converges in probability to the standard complex normal distribution (that is, the distribution of $Z = U + iV$ where U and V are independent real-valued normal variables with mean 0 and variance ½).*

PROOF. Let γ be the distribution of four independent normal random variables, each with mean 0 and variance ½. Let γ_{nkl} denote the distribution of the random vector $(X_{nk}, Y_{nk}, X_{nl}, Y_{nl})$, for $n = 1, 2, \cdots$, and $k, l = 1, \cdots, k_n$. Now, for all indices nkl not in E, order the distributions γ_{nkl} to form a single sequence p_r, $r = 1, 2, \cdots$, in such a way that if γ_{nkl} corresponds to p_j and γ_{stv} corresponds to p_k, then $n > s$ implies $j > k$. Clearly, the characteristic function of p_r converges pointwise to the characteristic function of γ. Thus, if g is a metric on probabilities on R^4 inducing the topology of weak convergence, then $g(p_r, \gamma) \to 0$. But because of the ordering of the sequence $\{p_r\}$, this implies that $g(\gamma_{nkl}, \gamma) \to 0$ as $n \to \infty$ uniformly in pairs (k, l) with nkl not in E.

Now let m_{nkl} denote the distribution of (Z_{nk}, Z_{nl}) for $n = 1, 2, \cdots, k, l = 1, \cdots, k_n$. Let m denote the standard complex normal distribution. m_{nkl} is of course determined by γ_{nkl}, and so m_{nkl} converges to m^2 as $n \to \infty$ uniformly in pairs (k, l) with nkl not in E. By Lemma 2, then, μ_n converges in probability to m. □

Suppose x is a vector with n coordinates. Each permutation ψ on $\{1, \cdots, n\}$ yields a new vector x_ψ with coordinates $(x_\psi)_i = x_{\psi(i)}$. In the statement of the next lemma, Φ denotes the distribution function of a real-valued normal variable with mean 0 and variance 1.

LEMMA 4. *Suppose x, y, a, and b are vectors in R^n satisfying*:

(i) $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j = \sum_{j=1}^n a_j = \sum_{j=1}^n b_j = 0$

(ii) $\sum_{j=1}^n (x_j^2 + y_j^2) = n$,

$\sum_{j=1}^n a_j^2 = \sum_{j=1}^n b_j^2 = n,$ *and*

$\sum_{j=1}^n a_j b_j = 0.$

Set $V = \max\{|a_j|, |b_j|: 1 \le j \le n\}$ *and* $U = \max\{|x_j|, |y_j|: 1 \le j \le n\}$. *Let ρ be a random permutation on $\{1, \cdots, n\}$ taking on any particular permutation with probability $1/n!$. Set* $W = n^{-1/2}[\sum_{j=1}^n (x_{\rho(j)} a_j + y_{\rho(j)} b_j)]$. *If F is the distribution function of W then*

$$\sup_{t \in R} |F(t) - \Phi(t)| \le 48 n^{-1/2} V U + h(n),$$

where h does not depend on x, y, a, or b, and $h(n) \to 0$ as $n \to \infty$.

PROOF. Theorem 3.1 of Ho and Chen (1978) can be applied in a straightforward way to yield this lemma. Here is a sketch: using the notation of Ho and Chen (1978, pages 325–326), with

$$X_{ij} = \frac{a_i x_j + b_i y_j}{(n)^{1/2}} \quad \text{and} \quad \varepsilon = \frac{2UV}{(n)^{1/2}},$$

one has $L_n(\varepsilon) = 0$. Thus their result reduces to

$$\sup_{t in R} |\tilde{F}(t) - \phi(t)| \le 48 n^{-1/2} V U,$$

where $\tilde{F}$ is the distribution function of $(1 - 1/n)^{1/2} \times W$. To get the result, pick $h(n)$ to bound $\sup_{t \in R} |F(t) - \tilde{F}(t)|$; h can be taken independent of x, y, a, and b.

3. Theorems. The main result of this section is Theorem 1, which asserts that for vectors z in C^n, the empirical distribution of the Fourier coefficients of z_ψ is close to complex normal, for most permutations ψ. For each integer n, let $z^n = (z_1^n, \cdots, z_n^n)$ be a vector in C^n. Let ρ be a random permutation of $\{1, \cdots, n\}$, taking on any particular permutation with probability $1/n!$. Let $Z_k^n = n^{-1/2} \sum_{j=1}^n z_{\rho(j)}^n \exp\left(\frac{2\pi(-1)^{1/2} kj}{n}\right)$. Let μ_n be the empirical distribution of $(Z_1^n, \cdots, Z_n^n)$.

THEOREM 1. *Suppose the sequence $\{z^n\}$ satisfies*:

(i) $\sum_{k=1}^n z_k^n = 0$ *and* $\sum_{k=1}^n |z_k^n|^2 = n$

(ii) $\max_{1 \le k \le n} |z_k^n| = o(n^{1/2})$ *as* $n \to \infty$.

Then μ_n converges in probability to the standard complex normal distribution.

PROOF. The idea of the proof is to use Lemma 3 with $X_{nk} = \mathrm{Re}(Z_k^n)$, $Y_{nk} = \mathrm{Im}(Z_k^n)$ and $k_n = n$. The exceptional set E consists of all indices nkl such that either,

(a) $k = n$ or $l = n$

(b) $k = n - l$ or

(c) $k = l$.

Then

$$\frac{\#\{(k, l): nkl \text{ in } E\}}{n^2} \le \frac{4n}{n^2} \to 0 \qquad \text{as } n \to \infty.$$

Fix the real numbers $\lambda_1, \cdots, \lambda_4$, and set $\lambda^2 = \lambda_1^2 + \cdots + \lambda_4^2$. Fix k and l between 1 and n, with nkl not in E. For the rest of the proof, for notational convenience, drop the superscript n from z^n. Let $z = x + iy$. Then

$$\lambda_1 X_{nk} + \lambda_2 Y_{nk} + \lambda_3 X_{nl} + \lambda_4 Y_{nl}$$
$$= n^{-1/2} \sum_{j=1}^n \left(x_{\rho(j)} \left[\lambda_1 \cos\left(\frac{2\pi kj}{n}\right) + \lambda_2 \sin \frac{2\pi kj}{n} + \lambda_3 \cos \frac{2\pi lj}{n} + \lambda_4 \sin \frac{2\pi lj}{n} \right] \right)$$
$$+ n^{-1/2} \sum_{j=1}^n y_{\rho(j)} \left[-\lambda_1 \sin \frac{2\pi kj}{n} + \lambda_2 \cos \frac{2\pi kj}{n} - \lambda_3 \sin\left(\frac{2\pi lj}{n}\right) + \lambda_4 \cos\left(\frac{2\pi lj}{n}\right) \right].$$

So,

$$\lambda_1 X_{nk} + \lambda_2 Y_{nk} + \lambda_3 X_{nl} + \lambda_4 Y_{nl} = \left(\frac{\lambda^2}{2}\right)^{1/2} n^{-1/2} \sum_{j=1}^n (x_{\rho(j)} a_j + y_{\rho(j)} b_j),$$

where for $1 \le j \le n$,

$$a_j = \left(\frac{\lambda^2}{2}\right)^{-1/2} \left[\lambda_1 \cos\left(\frac{2\pi kj}{n}\right) + \lambda_2 \sin\left(\frac{2\pi kj}{n}\right) + \lambda_3 \cos\left(\frac{2\pi lj}{n}\right) + \lambda_4 \sin\left(\frac{2\pi lj}{n}\right) \right]$$

and $$b_j = \left(\frac{\lambda^2}{2}\right)^{-1/2} \left[-\lambda_1 \sin\left(\frac{2\pi kj}{n}\right) + \lambda_2 \cos\left(\frac{2\pi kj}{n}\right) - \lambda_3 \sin\left(\frac{2\pi lj}{n}\right) + \lambda_4 \cos\left(\frac{2\pi lj}{n}\right) \right].$$

Note that x, y, a, and b satisfy the conditions of Lemma 4, and

$$\sup_{1 \le j \le n} (|a_j|, |b_j|) \le \left(\frac{\lambda^2}{2}\right)^{-1/2} (|\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4|) \le 2(2)^{1/2}.$$

Thus, if F is the distribution of $\left(\frac{\lambda^2}{2}\right)^{-1/2} (\lambda_1 X_{nk} + \lambda_2 Y_{nk} + \lambda_3 X_{nl} + \lambda_4 Y_{nl})$, Lemma 4 allows us to conclude

$$\sup_{x \in R} |F(x) - \Phi(x)| \le 96(2)^{1/2}(n^{-1/2} \max_{1 \le k \le n} |z_k^n|) + h(n).$$

The right-hand side tends to 0 as $n \to \infty$ by condition (ii) and does not depend on k and l. Thus the distribution function of $\lambda_1 X_{nk} + \lambda_2 Y_{nk} + \lambda_3 X_{nl} + \lambda_4 Y_{nl}$ converges uniformly as $n \to \infty$ to the distribution function of a real-valued normal variable with mean 0 and variance $\frac{\lambda^2}{2}$, and this convergence is uniform for pairs (k, l) with nkl not in E. This implies that the distribution of $\lambda_1 X_{nk} + \lambda_2 Y_{nk} + \lambda_3 X_{nl} + \lambda_4 Y_{nl}$ converges to $N\left(0, \frac{\lambda^2}{2}\right)$ as $n \to \infty$, uniformly in pairs (k, l) with nkl not in E. Now apply Lemma 3 to conclude that μ_n converges in probability to the standard complex normal distribution. □

The following corollary will be used to prove Theorem 2.

COROLLARY 1. *Suppose the sequence* z^n *satisfies*:

(i) $\sum_{k=1}^n z_k^n = o(n)$ *and* $\sum_{k=1}^n |z_k^n|^2 = n + o(n)$ *as* $n \to \infty$

(ii) $\max_{1 \le k \le n} |z_k^n| = o(n^{1/2})$ *as* $n \to \infty$.

Let μ_n *correspond to* z^n *as in Theorem* 1. *Then* μ_n *converges in probability to the standard complex normal distribution.*

PROOF. Let $a_n = 1/n \sum_{k=1}^n z_k^n$ and $s_n^2 = 1/n \sum_{k=1}^n |z_k^n - a_n|^2$. Then $a_n \to 0$ and $s_n \to 1$ as $n \to \infty$. Apply Theorem 1 to $\frac{(z_i^n - a_n)}{s_n}$. □

THEOREM 2. *Suppose $X_1, X_2, \cdots$ are independent identically distributed complex-valued random variables with $E(X_1) = 0$ and $E|X_1|^2 = 1$. For each n, let $\hat{X}_n$ be the discrete Fourier transform of $(X_1, \cdots, X_n)$, and let μ_n be the empirical distribution of $n^{-1/2}\hat{X}_n$. Then μ_n converges in probability to the standard complex normal distribution.*

PROOF. Suppose $X_1, X_2, \cdots$ are defined on the probability triple $(\Omega, \mathscr{F}, P)$. Let the triple $(\Omega', \mathscr{F}', P')$ support a sequence $\rho_1, \rho_2, \cdots$ independent of $(X_1, X_2, \cdots)$, where ρ_n is a random permutation of $\{1, \cdots, n\}$ taking on each permutation with probability $1/n!$. A typical point of Ω' will be denoted by ω'.

Now consider $(\Omega \times \Omega', \mathscr{F} \times \mathscr{F}', P \times P')$. On this product space, let $Y_n(\omega, \omega')$ be the discrete Fourier transform of

$$X_{\rho_n(\omega', 1)}(\omega), \cdots, X_{\rho_n(\omega', n)}(\omega).$$

Let $\nu_n(\omega, \omega')$ be the empirical distribution of $n^{-1/2} Y_n(\omega, \omega')$. Fix $\epsilon > 0$. Let m be the standard complex normal distribution. Let

$$Z_n(\omega) = P'\{\omega': d(\nu_n(\omega, \omega'), m) > \epsilon\}.$$

Now, for almost all ω,

$$X_1(\omega) + \cdots + X_n(\omega) = o(n)$$

$$X_1^2(\omega) + \cdots + X_n^2(\omega) = n + o(n)$$

and

$$\max_{1 \le k \le n} |X_k(\omega)| = o(n^{1/2}).$$

The corollary to Theorem 1 implies that $Z_n(\omega) \to 0$ for almost all ω. By Fubini and dominated convergence,

$$(P \times P')\{d(\nu_n, m) > \epsilon\} = E(Z_n) \to 0.$$

Thus ν_n converges in probability to the standard complex normal distribution. Finally, note that the law of μ_n coincides with the law of ν_n. □

For a vector z in C^n, let $\hat{z}_k$ denote the kth Fourier coefficient. The kth *periodogram ordinate* is defined by

$$I(k) = (1/n)|\hat{z}_k|^2.$$

Under the conditions of Theorem 1 or Theorem 2, the empirical distribution of $n^{-1/2}\ \hat{z}$ is approximately standard complex normal: that is, the real and imaginary parts of the distribution are approximately independent $N(0, ½)$. Thus, if each mass point is squared and the moduli of the corresponding real and imaginary parts are summed, the resulting empirical distribution is approximately exponential with parameter 1. Formally:

THEOREM 3. *Under the conditions of Theorem 1 or Theorem 2, the empirical distribution of the periodogram ordinates converges in probability to an exponential distribution with parameter 1.*

This theorem provides an explanation for the linearity of the χ_2^2-probability plot of 500 periodogram ordinates noted by Brillinger (1975, page 127).

4. Notes and Questions. (1) Consider the $n \times n$ matrix F with entries $F_{jk} = n^{-1/2} \cdot \exp\left(\frac{2\pi(-1)^{1/2}kj}{n}\right)$. F is a unitary matrix, and for y in C^n, $n^{-1/2}\ \hat{y} = Fy$. The questions considered in this paper about the coordinates of Fy may be raised with arbitrary unitary matrices H, and in fact the theorems of Section 3 generalize immediately with the Fourier transformation replaced by arbitrary unitary transformation. That is, suppose for each n, H^n is an $n \times n$ unitary matrix, with $\max_{i,j} |H^n_{ij}| \le cn^{-1/2}$, where c is a constant which does not depend on n. For a triangular array X_{nk}, $n \ge 1$, $1 \le k \le n$, consider $y_n = (n)^{1/2} H^n x_n$. Then the theorems of Section 3 hold if $\hat{x}_n$ is replaced by y_n. Thus, broadly speaking, unitary transformations take arbitrary vectors into vectors with approximately normal empirical distribution. Some other aspects of this "normality-inducing" behavior of unitary transformations have been considered by Mallows (1969).

(2) In the setting of Theorem 3, do the empirical distributions μ_n converge almost surely? We have not been able to settle this question yet. Another related question of some statistical interest is to determine the distribution of the largest Fourier coefficient. Gersho, Gopinath and Odlyzko (1978), building on theoretical work of Halasz (1973), have shown that if $X_1, \cdots, X_n$ are independent with the same L^6-distribution and $\mathrm{Var}(X_i) = 1$, the maximum Fourier coefficient is with high probability close to $(n \log n)^{1/2}$. Can this result be extended to more general L^2-distributions?

(3) Suppose X_1, X_2, are independent complex-valued random variables with a common distribution, but $EX_1^2 = \infty$. In this case, is there any sequence c_n, such that if μ_n is the empirical distribution of $\{c_n Y_{nk}: h = 1, \cdots, n\}$, where $Y_{nk} = \sum_{j=1}^{n} X_j \exp\left(\frac{2\pi(-1)^{1/2}kj}{n}\right)$, then μ_n converges to some distribution?

If the X's have the distribution of a symmetric stable law of order p, $o < p < 2$, then such μ_n cannot converge to a constant measure. However, if $c_n = n^{-1/p}$, we can show that the corresponding μ_n converge in distribution but not in probability to a random measure, and plan to discuss this in a future paper.

5. Examples. Here are some normal probability plots to illustrate the results of Theorem 2. In each plot, the expected values of the order statistics from an appropriate-sized normal sample are plotted along the x-axis (these are denoted *rankits*), while various data or Fourier coefficients are plotted along the y-axis.

FIGURE 1—*Exponential data.* A pseudorandom sample of 1000 observations from an exponential distribution with parameter 1 was generated. Figure 1a is a normal probability plot of the original data, while Figure 1b is a normal probability plot of the real parts of the 2nd through 499th Fourier coefficients of the data.

FIGURE 2—*Uniform data.* The arrangement is the same as Figure 1, but the data is a pseudorandom sample of size 1000 from a uniform distribution on [0, 1].

FIGURE 3—*Cauchy data.* Here the data is a pseudorandom sample of size 1000 from a Cauchy distribution. The situation is quite different from the L^2 distributions illustrated in the

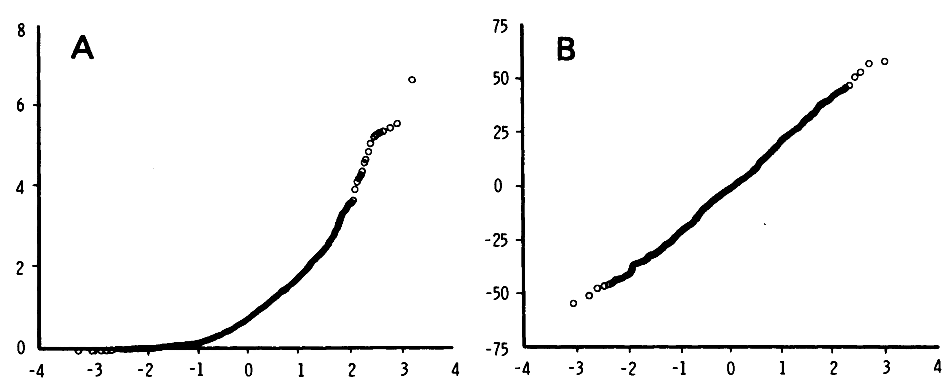

FIG. 1a. Y-axis: Pseudorandom sample of size 1000 from an exponential distribution with parameter 1. X-axis: Rankits.

FIG. 1b. Y-axis: Real parts of 2nd through 499th Fourier coefficients of data plotted in 1a. X-axis: Rankits.
NOTE. The normalization required by Theorem 2 for the coefficients would be to divide each of them by 23.6 to obtain an approximately standard normal empirical distribution.

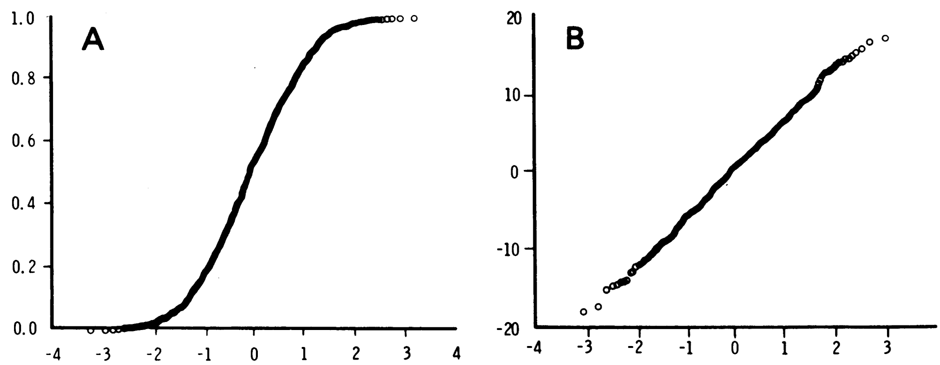

FIG. 2a. Y-axis: Pseudorandom sample of size 1000 from a uniform distribution on [0, 1]. X-axis: Rankits.
FIG. 2b. Y-axis: Real parts of 2nd through 499th Fourier coefficients of data plotted in 2a. X-axis: Rankits.
NOTE. The normalization required by Theorem 2 for the coefficients would be to divide each of them by 6.45.

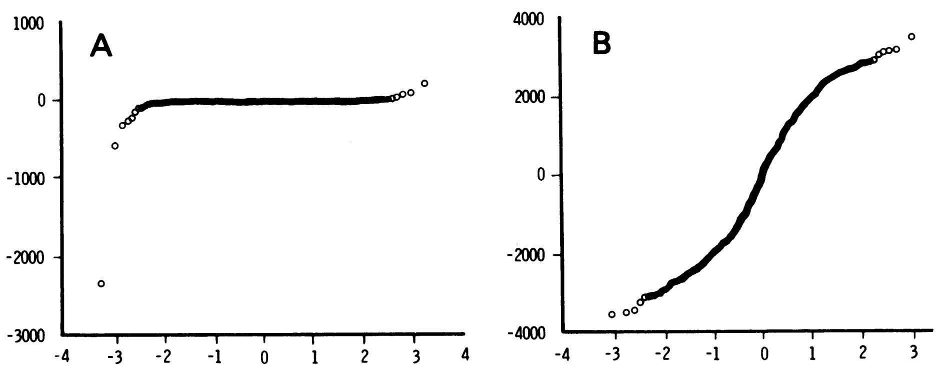

FIG. 3a. Y-axis: Pseudorandom sample of size 1000 from Cauchy data. X-axis: Rankits.
FIG. 3b. Y-axis: Real parts of 2nd through 499th Fourier coefficients of data plotted in 3a. X-axis: Rankits.

preceding three figures. Different Cauchy samples lead to differently shaped plots of the Fourier coefficients, but an S-shape, indicating a short-tailed empirical distribution, is common.

REFERENCES

BRILLINGER, DAVID R. (1975). *Time Series Data Analysis and Theory.* Holt, Rinehard, and Winston, New York.

GERSHO, A., GOPINATH, B. and ODLYZKO, A. M. (1978). Coefficient inaccuracy in transversal filtering. Bell Laboratories Tech. Memo.

HALASZ, G. (1973). On a result of Salem and Zygmund concerning random polynomials. *Studia Sci. Math. Hungar.* **8** 369–377.

HO, SOO-THONG and CHEN, LOUIS H. Y. (1978). An L_p bound for the remainder in a combinatorial central limit theorem," *Ann. Probability* **6** 231–249.

MALLOWS, COLIN L. (1969). Joint normality induced by orthogonal transformations. Bell Telephone Laboratories Tech. Memo.