

## A BAYESIAN APPROACH TO A PROBLEM IN SEQUENTIAL ESTIMATION<sup>1</sup>

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This paper considers the problem of sequentially estimating the mean of a normal distribution when the variance is unknown. A continuous time analogue of the discrete time problem is studied. For  $L$  in a class of loss functions, properties of the value function and optimal continuation region of  $L$  are presented. Asymptotic expansions are found for the value function and the optimal boundary function of the loss function  $L$ .

**1. Introduction.** A sequence of observations is made, generating a sequence  $X_1, X_2, \dots$  of independent normal  $N(\Delta, 1/\theta)$  random variables with mean  $\Delta$  and variance  $1/\theta$  both unknown. Let  $(\Delta, \theta)$  have the normal-gamma  $N - \Gamma(\alpha, \beta, r, m)$  conjugate prior, where  $\alpha, \beta, r > 0$  and  $-\infty < m < \infty$ . That is, the conditional prior distribution of  $\Delta$  given  $\theta$  is  $N(m, 1/(r\theta))$  and the marginal prior distribution of  $\theta$  is  $\Gamma(\alpha/2, \beta/2)$ , gamma with parameters  $\alpha/2$  and  $\beta/2$  and mean  $\alpha/\beta$ . The posterior distribution of  $(\Delta, \theta)$  given  $X_1, \dots, X_n$  is then  $N - \Gamma(\alpha_n, \beta_n, r_n, m_n)$  where  $\alpha_n = \alpha + n, r_n = r + n, m_n = (rm + n\bar{X}_n)/r_n, \bar{X}_n = (X_1 + \dots + X_n)/n$  and

$$\beta_n = \beta + \sum_{i=1}^n (X_i - \bar{X}_n)^2 + rn(\bar{X}_n - m)^2/(r + n).$$

See, for example, DeGroot, (1970), Chapter 9.

We seek a sequential procedure for estimating the mean  $\Delta$  of the  $X_i$ 's. The loss after taking  $n$  observations and estimating  $\Delta$  by a function  $\hat{\Delta}_n$  of the first  $n$  observations will be given by  $E[|\Delta - \hat{\Delta}_n|^k | X_1, \dots, X_n] + cn$ , which consists of a term for error of estimation and a term for cost of observation. For any number  $n$  of observations, the Bayes estimate  $m_n$  of  $\Delta$  minimizes the posterior expectation of  $|\Delta - \hat{\Delta}_n|^k$ , and so the loss will be given by:

$$\begin{aligned} L_n &= E[|\Delta - m_n|^k | X_1, \dots, X_n] + cn \\ &= \left(\frac{\beta_n}{r_n}\right)^{k/2} \frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{\alpha_n - k}{2}\right)}{\pi^{1/2} \Gamma\left(\frac{\alpha_n}{2}\right)} + cn. \end{aligned}$$

If  $r = 0$ , then we have an improper prior, indicating a lack of prior knowledge about  $\Delta$ . With  $r = 0, m_n$  and  $\beta_n$  simplify to  $m_n = \bar{X}_n$  and  $\beta_n = \beta + \sum_{i=1}^n (X_i - \bar{X}_n)^2$ , and  $L_n$  may be written as a function of  $\alpha_n$  and  $\beta_n$ . In the sequel, we will let  $r = 0$  and the analysis will be in terms of  $\alpha_n$  and  $\beta_n$ .

We seek an optimal stopping rule which will tell us when to stop taking observations, so as to minimize the expectation of  $L_n, E[L_n]$ . We would like to find a region  $\mathcal{C}$  in the  $(\alpha, \beta)$  plane such that the optimal stopping rule says to continue sampling as long as  $(\alpha_n, \beta_n) \in \mathcal{C}$  and stop as soon as  $(\alpha_n, \beta_n) \notin \mathcal{C}$ . If such a region exists, it is called an optimal continuation region.

Received June 1979; revised July 1979.

<sup>1</sup> This research was supported in part by a National Science Foundation Graduate Fellowship and was completed in partial fulfillment of the requirements for the Doctor of Philosophy degree in the University of Michigan Department of Statistics.

AMS 1970 subject classifications. 62L12, 62F10, 62L15, 60J25, 60J30, 62F15.

Key words and phrases. Bayesian sequential estimation, normal distribution, gamma distribution, normal-gamma prior distribution, loss function, value function, optimal continuation region, optimal stopping rule, asymptotic approximations.

For future use we note that  $\beta_n = \beta + S_{n-1}$  where  $S_{n-1} = Y_1 + \dots + Y_{n-1}$  is the sum of  $n - 1$  independent and identically distributed (i.i.d.) random variables  $Y_1, \dots, Y_{n-1}$ . Hence,  $Y_j = W_j^2, j \geq 1$ , where  $W_j = (X_1 + \dots + X_j - jX_{j+1})/(j(j + 1))^{1/2}$ .  $W_1, W_2, \dots$  are i.i.d.  $N(0, 1/\theta)$  and, therefore,  $Y_1, Y_2, \dots$  are i.i.d.  $\Gamma(1/2, \theta/2)$ .

Let  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$  denote the smallest  $\sigma$ -algebra with respect to which  $X_1, \dots, X_n$  are measurable. A *stopping rule* or *stopping time* is a random variable  $\tau$  taking values  $1, 2, \dots, +\infty$  such that  $\tau < \infty$  with probability one and  $\{\tau \leq n\} \in \mathcal{F}_n$  for  $n = 1, 2, \dots$ . For an *extended stopping time*  $\tau$  we drop the requirement that  $\tau$  be finite with probability one.

The *value function*  $V$  associated with the loss function  $L$  is defined as:

$$V(\alpha, \beta) = \inf_{\tau} E_{\alpha, \beta}[L(\alpha_{\tau}, \beta_{\tau})]$$

where the infimum is taken over all stopping times  $\tau$ .  $E_{\alpha, \beta}$  denotes expectation when the prior distribution on  $\theta$  is  $\Gamma(\alpha/2, \beta/2)$ . If there exists a stopping rule  $\sigma$  such that  $V(\alpha, \beta) = E_{\alpha, \beta}[L(\alpha_{\sigma}, \beta_{\sigma})]$ , then  $\sigma$  is called an *optimal stopping rule*.

Let  $\mathcal{C} = \{(\alpha, \beta): V(\alpha, \beta) < L(\alpha, \beta)\}$  and let  $\sigma = \inf\{n \geq 1: V(\alpha_n, \beta_n) = L(\alpha_n, \beta_n)\}$ . Chow, Robbins and Siegmund ((1971), page 70, Theorem 4.5) show that this  $\sigma$  is in fact an optimal stopping rule for the types of loss functions we are considering. Therefore,  $\mathcal{C}$  is an optimal continuation region and  $V(\alpha, \beta) = E_{\alpha, \beta}[L(\alpha_{\sigma}, \beta_{\sigma})]$ . Unfortunately, the definition of  $\sigma$  is too complicated to be of much practical use.

If  $V$  or  $\sigma$  cannot be determined exactly, we would like to approximate them closely. The approach will be to approximate  $V$  and  $\sigma$  for an analogous problem in continuous time. In future work, the author will relate these results back to the discrete time problem.

Alvo [1977] considers the discrete time version of the problem of Bayesian sequential estimation of the mean of a normal distribution when the mean and variance are both unknown. Error in estimation is measured by squared error loss and sampling cost is  $c$  units per observation. Alvo suggests a stopping rule for this problem and obtains upper and lower bounds on the risk associated with the stopping rule. He shows that the excess risk associated with his suggested stopping rule is bounded above by terms of order  $c$ . In the present paper, we consider the continuous time analogue of the discrete time problem. We approximate the optimal stopping rule by approximating the boundary of the optimal continuation region. We also approximate the risk or value function associated with the optimal stopping rule. In each case these approximations are obtained by means of asymptotic expansions to arbitrarily large powers of the sampling cost  $c$ .

We formulate an analogous problem in continuous time as follows. Let  $\theta$  be a random variable and  $\{Z_t, t \geq 0\}$  a stochastic process with right continuous sample paths. Suppose that  $\theta$  has a  $\Gamma(\alpha/2, \beta/2)$  distribution and that conditionally given  $\theta, \{Z_t, t \geq 0\}$  has stationary independent increments with  $Z_t$  distributed as a  $\Gamma(t/2, \theta/2)$  random variable. It may be verified that the conditional distribution of  $\theta$  given  $Z_s, 0 \leq s \leq t$ , is  $\Gamma(\alpha_t/2, \beta_t/2)$ , where the new parameters are  $\alpha_t = \alpha + t$  and  $\beta_t = \beta + Z_t$ . The random variables  $Z_t$  are the continuous time analogues of the sums  $S_{n-1}$  of independent gamma random variables in discrete time.

Let  $\mathcal{F}_t = \sigma\{Z_s, s \leq t\}$  represent the smallest  $\sigma$ -algebra with respect to which the random variables  $Z_s, 0 \leq s \leq t$ , are measurable. A *stopping time* or *stopping rule* is a nonnegative random variable  $\tau$  such that  $\tau < \infty$  with probability one and  $\{\tau \leq t\} \in \mathcal{F}_t$  for every  $t \geq 0$ . For an extended stopping time  $\tau$  we drop the requirement that  $\tau$  be finite with probability one.

As in the discrete case, the *value function*  $V$  associated with the loss function  $L$  is defined as  $V(\alpha, \beta) = \inf_{\tau} E_{\alpha, \beta}[L(\alpha_{\tau}, \beta_{\tau})]$ , where the infimum is taken over all stopping times  $\tau$ . Let  $\mathcal{C} = \{(\alpha, \beta): V(\alpha, \beta) < L(\alpha, \beta)\}$  and  $\sigma = \inf\{t \geq 0: V(\alpha_t, \beta_t) = L(\alpha_t, \beta_t)\}$ . Using Theorem 3, we show that  $\sigma$  is an optimal stopping rule for a class of loss functions in the continuous time case, so that  $V(\alpha, \beta) = E_{\alpha, \beta}[L(\alpha_{\sigma}, \beta_{\sigma})]$ .

In Section 2, we present some background material on optimal stopping in Markov processes. In Section 3, we show that  $V$  has certain smoothness properties in continuous time which help to characterize it. We show that  $V$  and  $\mathcal{C}$  have the properties

$$V < L \quad \text{and} \quad \mathcal{A}V = 0 \quad \text{in} \quad \mathcal{C},$$

$$V = L \quad \text{and} \quad V_{\alpha} = L_{\alpha} \quad \text{in} \quad \sim \mathcal{C},$$

where  $\mathcal{A}$  is a differential integral operator which is defined in Section 2 and  $\sim\mathcal{C}$  denotes the complement of the set  $\mathcal{C}$ . Moreover, we show that these conditions are sufficient for a function  $U$  and a region  $\mathcal{C}^*$  to be the value function  $V$  and the optimal continuation region  $\mathcal{C}$ .

In Section 4 we give asymptotic expansions for the value function  $V$  and the boundary of the optimal continuation region  $\mathcal{C}$  for large  $\alpha$  and for small  $c$ . We also look at some related problems to which the techniques developed may be applied.

**2. Preliminaries.**

2.1. *Results about Markov processes.* We use some of the theory of Markov processes. For notation and definitions, see Dynkin, (1965), Chapter 3.

Let  $X = (X_t, \mathcal{F}_t, P_x), t \geq 0$ , be a Markov process with sample space  $(\Omega, \mathcal{F})$  and state space  $(E, \mathcal{B})$ . A  $\mathcal{B}$ -measurable function  $f$  is in the domain of  $\mathcal{A}$ , denoted  $\mathcal{D}(\mathcal{A})$ , if the limit

$$\mathcal{A}f(x) = \lim_{t \downarrow 0} \frac{E_x f(X_t) - f(x)}{t}$$

exists for every  $x \in E$ . Call  $\mathcal{A}$  the *infinitesimal operator* of the Markov process  $X$ .

Theorem 1 gives what is known as *Dynkin's formula* (Dynkin (1965), page 133).

**THEOREM 1.** *Let  $\mathcal{A}$  be the infinitesimal operator of a strongly measurable strong Markov process  $X$ . Let  $f$  be a  $\mathcal{B}$ -measurable function such that  $f \in \mathcal{D}(\mathcal{A})$ . Let  $\tau$  be a stopping time such that  $E_x \tau < \infty$  and  $E_x f(X_\tau) < \infty$ . Let  $f_n, n \geq 1$ , be a sequence of bounded,  $\mathcal{B}$ -measurable functions such that  $f_n \in \mathcal{D}(\mathcal{A})$  for each  $n; f_n(x) \rightarrow f(x)$  and  $\mathcal{A}f_n(x) \rightarrow \mathcal{A}f(x)$  as  $n \rightarrow \infty$  for each  $x \in E$ ; there exists a function  $g_1$  such that  $|f_n(x)| \leq g_1(x)$  for each  $n \geq 1$  and  $x \in E$ , and  $E_x g_1(x_\tau) < \infty$ ; and there exists a function  $g_2$  such that  $|\mathcal{A}f_n(x)| \leq g_2(x)$  for each  $n \geq 1$  and  $x \in E$ , and  $E_x[\int_0^\tau g_2(x_t) dt] < \infty$ . Then  $E_x f(X_\tau) - f(x) = E_x[\int_0^\tau \mathcal{A}f(X_t) dt]$ .*

**PROOF.** The result for bounded  $f$  follows from Breiman, (1968), page 376. The result for general  $f$  is proved by a simple truncation argument.

Theorem 1 applies, for example, to the nonnegative loss functions described in 3.1 (Bartold (1976), pages 18, 19).

Let  $X = (X_t, \mathcal{F}_t, P_x)$  be a standard Markov process with Euclidean state space  $(E, \mathcal{B})$ . For any  $U \in \mathcal{B}$ , define a nonnegative random variable  $\tau(U) = \inf\{t \geq 0: X_t \notin U\}$ . Call  $\tau(U)$  the first exit time from the set  $U$ . Dynkin ((1965), pages 104–111) gives conditions under which  $\tau(U)$  is an extended stopping time. From these conditions we can deduce Theorem 2.

**THEOREM 2.** *Let  $X = (X_t, \mathcal{F}_t, P_x)$  be a standard Markov process with Euclidean state space  $(E, \mathcal{B})$ . If  $U \in \mathcal{B}$ , then  $\tau(U)$  is an extended stopping time.*

2.2 *An optimal stopping theorem.* Let  $X = (X_t, \mathcal{F}_t, P_x)$  be a standard Markov process on a sample space  $(\Omega, \mathcal{F})$ , with Euclidean state space  $(E, \mathcal{B})$ . Let  $\mathcal{T}$  denote the natural Euclidean topology on  $E$ . A  $\mathcal{B}$ -measurable function  $f = f(x)$  is  $\mathcal{T}$ -continuous if  $\lim_{t \downarrow 0} f(X_t) = f(x)$  a.e.  $P_x$  for every  $x \in E$ . Let  $\mathcal{L}$  denote the collection of all  $\mathcal{B}$ -measurable  $\mathcal{T}$ -continuous functions  $h = h(x)$  such that  $-\infty < h(x) \leq \infty$  and  $E_x h^-(X_t) < \infty$  for every  $t \geq 0, x \in E$ , where  $h^-$  denotes the absolute value of the negative part of  $h$ .

For any  $f \in \mathcal{L}$  and any (finite or extended) stopping time  $\tau$ , define the quantity  $f(X_\tau)$  by:

$$\begin{aligned} f(X_\tau) &= f(X_{\tau(\omega)}(\omega)) && \text{if } \omega \in \{\omega: \tau(\omega) < \infty\} \\ &= \limsup_{t \rightarrow \infty} f(X_t(\omega)) && \text{if } \omega \in \{\omega: \tau(\omega) = \infty\} \end{aligned}$$

A function  $f \in \mathcal{L}$  is *regular* if for any (finite or extended) stopping time  $\tau$  and  $x \in E, E_x f(X_\tau)$  is defined and for any two (finite or extended) stopping times  $\sigma$  and  $\tau$  such that  $P_x\{\sigma \leq \tau\} = 1, x \in E$ , the inequality  $E_x f(X_\tau) \leq E_x f(X_\sigma)$  is true for every  $x \in E$ .

A regular function  $f$  is a *regular majorant* of a function  $g$  if  $f(x) \geq g(x)$  for every  $x \in E$ . The function  $f$  is the *least regular majorant* of  $g$  if  $f$  is a regular majorant of  $g$  and  $f \leq h$  for any other regular majorant  $h$  of  $g$ .

For  $g \in \mathcal{L}$ , let  $s(x) = \sup E_x g(X_\tau)$  where the supremum is taken over all (finite) stopping times  $\tau$  such that  $E_x g^-(X_\tau) < \infty$ ,  $x \in E$ ; let  $\bar{s}(x) = \sup E_x g(X_\tau)$  where the supremum is taken over all finite and extended stopping times  $\tau$  such that  $E_x g^-(X_\tau) < \infty$ ,  $x \in E$ .

**THEOREM 3.** *Let  $g \in \mathcal{L}$  be an upper semicontinuous function satisfying condition A+:*

$$(A+) \quad E_x[\sup_{t \geq 0} g^+(X_t)] < \infty, \quad \forall x \in E$$

where  $g^+$  denotes the positive part of  $g$ . Then  $s$  is the least regular majorant of  $g$  and  $s(x) = \bar{s}(x)$  for every  $x \in E$ . If  $\sigma = \inf\{t \geq 0: s(X_t) = g(X_t)\}$ , then  $s(x) = E_x g(X_\sigma)$ . If, in addition,  $\lim_{t \rightarrow \infty} g(X_t) = -\infty$  a.e.  $P_x$ ,  $x \in E$ , then  $\sigma$  is finite a.e.  $P_x$ .

**PROOF.** See Sirjaev, (1976), page 176.

2.3. *The  $(\alpha_t, \beta_t)$  process.* It follows from elementary considerations that the process  $\{(\alpha_t, \beta_t), t \geq 0\}$  is a right continuous Markov process with Euclidean topological space  $(E, \mathcal{T}, \mathcal{B})$ , where  $E = R^+ \times R^+$ ,  $\mathcal{T}$  is the natural Euclidean topology on  $E$ , and  $\mathcal{B}$  is the  $\sigma$ -algebra of Borel subsets of  $E$ . The  $(\alpha_t, \beta_t)$  process is also a Feller process (Sirjaev (1973), page 18) since for every bounded,  $\mathcal{B}$ -measurable, continuous function  $f$  on  $(E, \mathcal{B})$  and for every  $t \geq 0$ , the function  $E_{\alpha, \beta}[f(\alpha_t, \beta_t)]$  is continuous in  $(\alpha, \beta)$ . Because every right continuous Feller process on a topological space  $(E, \mathcal{T}, \mathcal{B})$  is a strongly measurable strong Markov process (Dynkin (1965), pages 98, 99), the process  $\{(\alpha_t, \beta_t), t \geq 0\}$  is a strongly measurable strong Markov process.

We will determine the infinitesimal operator of the  $(\alpha_t, \beta_t)$  process. Let  $f = f(\alpha, \beta)$  be a measurable real valued function defined on  $E$  and continuous in  $\alpha$ . Let

$$(2.1) \quad \begin{aligned} Gf(\alpha, \beta) &= \lim_{t \downarrow 0} \frac{E_{\alpha, \beta}[f(\alpha + t, \beta + Z_t)] - f(\alpha + t, \beta)}{t} \\ &= \lim_{t \downarrow 0} \frac{1}{2} \int_0^\infty \frac{\Gamma\left(\frac{\alpha + t}{2}\right)}{\Gamma\left(\frac{t}{2} + 1\right)\Gamma\left(\frac{\alpha}{2}\right)} \frac{[f(\alpha + t, \beta + \beta y) - f(\alpha + t, \beta)]}{(1 + y)^{(\alpha+t)/2}} y^{(t/2)-1} dy \end{aligned}$$

provided this limit exists. If the integrand in this expression for  $Gf(\alpha, \beta)$  is bounded by an integrable function for  $0 \leq t \leq \delta$  for some  $\delta > 0$ , then by the dominated convergence theorem, the limit can be taken inside the integral, giving

$$Gf(\alpha, \beta) = \frac{1}{2} \int_0^\infty \frac{f(\alpha, \beta + \beta y) - f(\alpha, \beta)}{y(1 + y)^{\alpha/2}} dy.$$

Suppose that  $f$  is such that both  $Gf(\alpha, \beta)$  and  $f_\alpha^+(\alpha, \beta)$  exist and are finite, where  $f_\alpha^+$  denotes the positive part of the partial derivative of  $f$  with respect to  $\alpha$ . Then  $\mathcal{A}f(\alpha, \beta) = f_\alpha^+(\alpha, \beta) + Gf(\alpha, \beta)$ . If, in addition, the limit may be taken inside the integral in (2.1), then  $\mathcal{A}f(\alpha, \beta)$  may be written as

$$\mathcal{A}f(\alpha, \beta) = f_\alpha^+(\alpha, \beta) + \frac{1}{2} \int_0^\infty \frac{f(\alpha, \beta + \beta y) - f(\alpha, \beta)}{y(1 + y)^{\alpha/2}} dy.$$

As an example, suppose that  $L(\alpha, \beta) = \beta^\gamma h(\alpha) + \phi(\alpha)$  is a function defined on  $E$ , where  $\gamma > 0$ , and  $h$  and  $\phi$  are both nonnegative differentiable functions of  $\alpha$ . Then  $L_\alpha^+(\alpha, \beta) = L_\alpha(\alpha, \beta) = \beta^\gamma h'(\alpha) + \phi'(\alpha)$ . It can be shown that for  $\alpha > 2\gamma$ ,  $GL(\alpha, \beta) = \frac{1}{2} \beta^\gamma h(\alpha) (\Psi(\alpha | 2) - \Psi(\alpha | 2 - \gamma))$ , where  $\Psi$  is the digamma function defined by  $\Psi(x) = d \log \Gamma(x) | dx$  and  $\Gamma$  is the gamma function. Thus,  $\mathcal{A}L(\alpha, \beta)$  exists for every  $(\alpha, \beta) \in E$  such that  $\alpha > 2\gamma$ , and

$\mathcal{A}L(\alpha, \beta) = \beta^\gamma [h'(\alpha) + \frac{1}{2} h(\alpha) [\Psi(\alpha/2) - \Psi(\alpha/2 - \gamma)]] + \phi'(\alpha)$ . For future use, we note that  $\mathcal{A}L(\alpha, \beta)$  is decreasing in  $\beta$  for each fixed  $\alpha$  if  $h'(\alpha)/h(\alpha) < (\frac{1}{2})[\Psi(\alpha/2 - \gamma) - \Psi(\alpha/2)]$ .

2.4. *Three lemmas.* We use Lemma 1 in Sections 3 and 4.

LEMMA 1. *Suppose  $\mathcal{C}^*$  is an open set of the form  $\mathcal{C}^* = \{(\alpha, \beta): \beta > f(\alpha)\}$ , where  $f$  is a continuous nondecreasing function of  $\alpha$ . Let  $\tau = \tau_{\alpha, \beta}$  be defined by  $\tau = \inf\{t \geq 0: \beta_t \leq f(\alpha_t)\}$ , which we assume to be finite with probability one  $P_{\alpha, \beta}$ . Then  $\beta_\tau = f(\alpha_\tau)$  with probability one  $P_{\alpha, \beta}$  for every  $(\alpha, \beta) \in \mathcal{C}^*$ .*

PROOF. Since  $f$  is continuous and nondecreasing and  $\beta_t = \beta + Z_t$  is nondecreasing in  $t$  with probability one,  $\beta_\tau \leq f(\alpha_\tau) = f(\alpha_\tau) \leq \beta_\tau \leq \beta_\tau$ , which implies that  $\beta_\tau = f(\alpha_\tau)$ .

Note that this lemma implies that  $(\alpha_\tau, \beta_\tau) = (\alpha_\tau, f(\alpha_\tau))$  is on the boundary of  $\mathcal{C}^*$  with probability one  $P_{\alpha, \beta}$ . The stopping time  $\tau$  is the first exit time from the set  $\mathcal{C}^*$ .

Lemma 2 gives a set of sufficient conditions for a function  $U$  and a region  $\mathcal{C}^*$  to be the value function  $V$  and an optimal continuation region  $\mathcal{C}$  for a loss function  $L$ . The proof follows easily from Dynkin's formula. Essentially this result may be found in Sirjaev (1973), Chapter 3.

LEMMA 2. *Let  $L = L(\alpha, \beta)$  be a finite loss function defined on the state space  $E = R^+ \times R^+$ . Let  $V$  be the value function for  $L$  and let  $\mathcal{C} = \{(\alpha, \beta): V(\alpha, \beta) < L(\alpha, \beta)\}$ . Assume that  $\sigma$ , the optimal stopping rule for  $L$ , exists so that  $V(\alpha, \beta) = E_{\alpha, \beta}[L(\alpha_\sigma, \beta_\sigma)]$ . Suppose that  $\mathcal{C}^*$  is an open set of the form  $\mathcal{C}^* = \{(\alpha, \beta): \beta > f(\alpha)\}$ , where  $f$  is a continuous nondecreasing function of  $\alpha$ , and that for each  $(\alpha, \beta) \in E$  the first exit time  $\tau = \tau_{\alpha, \beta}$  from  $\mathcal{C}^*$  starting at  $(\alpha, \beta)$  is finite with probability one  $P_{\alpha, \beta}$ . Suppose that  $U$  is a real valued measurable function defined on  $E$  satisfying Dynkin's formula with both  $\tau$  and  $\sigma$ , for every  $(\alpha, \beta) \in E$ . Suppose also that:*

$$U(\alpha, \beta) < L(\alpha, \beta) \text{ and } \mathcal{A}U(\alpha, \beta) = 0 \text{ for } (\alpha, \beta) \in \mathcal{C}^*,$$

$$U(\alpha, \beta) = L(\alpha, \beta) \text{ and } \mathcal{A}U(\alpha, \beta) \geq 0 \text{ for } (\alpha, \beta) \notin \mathcal{C}^*.$$

Then  $U = V$  and  $\mathcal{C}^* = \mathcal{C}$  is an optimal continuation region.

PROOF. For  $(\alpha, \beta) \in \mathcal{C}^*$ , we see using Dynkin's formula that

$$E_{\alpha, \beta}[U(\alpha_\tau, \beta_\tau)] - U(\alpha, \beta) = E_{\alpha, \beta} \left[ \int_0^\tau \mathcal{A}U(\alpha_t, \beta_t) dt \right] = 0.$$

Therefore, for  $(\alpha, \beta) \in \mathcal{C}^*$ ,  $U(\alpha, \beta) = E_{\alpha, \beta}[U(\alpha_\tau, \beta_\tau)] = E_{\alpha, \beta}[L(\alpha_\tau, \beta_\tau)] \geq V(\alpha, \beta)$ . Also,  $U(\alpha, \beta) = L(\alpha, \beta) \geq V(\alpha, \beta)$  for  $(\alpha, \beta) \in \sim \mathcal{C}^*$ . Again using Dynkin's formula and the assumption that  $\mathcal{A}U(\alpha, \beta) \geq 0$  for every  $(\alpha, \beta)$ ,  $U(\alpha, \beta) \leq E_{\alpha, \beta}[U(\alpha_\sigma, \beta_\sigma)] \leq E_{\alpha, \beta}[L(\alpha_\sigma, \beta_\sigma)] = V(\alpha, \beta)$  so that  $U(\alpha, \beta) = V(\alpha, \beta)$  for every  $(\alpha, \beta) \in E$ . That  $\mathcal{C} = \mathcal{C}^*$  follows immediately. Therefore,  $\tau = \sigma$  is an optimal stopping rule and  $\mathcal{C}^* = \mathcal{C}$  is an optimal continuation region for  $L$ .

In Section 4 we use Lemma 3, which is a previously unpublished result of Michael Woodroffe.

LEMMA 3. *Let  $L_1$  and  $L_2$  be loss functions and let  $\mathcal{C}_i = \{(\alpha, \beta): V_i(\alpha, \beta) < L_i(\alpha, \beta)\}$  be an optimal continuation region for  $L_i, i = 1, 2$ . For  $(\alpha, \beta) \in \mathcal{C}_2$ , let  $\sigma_2$  be the first exit time from  $\mathcal{C}_2$  starting at the point  $(\alpha, \beta)$ . Assume that  $\sigma_2$  satisfies Dynkin's formula with both  $L_1$  and  $L_2$ , and that  $\sigma_2$  is an optimal stopping rule for  $L_2$ . If  $\mathcal{A}L_1(\alpha, \beta) \leq \mathcal{A}L_2(\alpha, \beta)$  for every  $(\alpha, \beta) \in \mathcal{C}_2$ , then  $\mathcal{C}_2 \subseteq \mathcal{C}_1$ .*

PROOF. Let  $V_i$  denote the value function associated with the loss function  $L_i, i = 1, 2$ . Suppose there is a point  $(\alpha, \beta) \in \mathcal{C}_2 \sim \mathcal{C}_1$ . Then  $0 = V_1(\alpha, \beta) - L_1(\alpha, \beta) \leq E_{\alpha, \beta}[L_1(\alpha_{\sigma_2}, \beta_{\sigma_2})] - L_1(\alpha, \beta) = E_{\alpha, \beta} \left[ \int_0^{\sigma_2} \mathcal{A}L_1(\alpha_t, \beta_t) dt \right] \leq E_{\alpha, \beta} \left[ \int_0^{\sigma_2} \mathcal{A}L_2(\alpha_t, \beta_t) dt \right] = E_{\alpha, \beta}[L_2(\alpha_{\sigma_2}, \beta_{\sigma_2})] - L_2(\alpha, \beta) = V_2(\alpha, \beta) - L_2(\alpha, \beta) < 0$ , which is a contradiction.

**3. Some properties of  $V$  and  $\mathcal{C}$ .**

3.1. *Introduction.* Let  $L$  be a loss function of the form  $L(\alpha, \beta) = \beta^\gamma h(\alpha) + \phi(\alpha)$  where  $\gamma > 0$ ;  $h$  is nonnegative, differentiable,  $\alpha^\gamma h(\alpha) \rightarrow 0$  as  $\alpha \rightarrow \infty$ ; and  $\phi$  is nonnegative, differentiable and increasing in  $\alpha$ ,  $\phi(\alpha) \rightarrow \infty$  as  $\alpha \rightarrow \infty$ . Suppose also that  $\mathcal{A}L(\alpha, \beta)$  is decreasing in  $\beta$  for each fixed  $\alpha$ . Such a loss function  $L$  will be said to satisfy Conditions A.

If a loss function  $L$  satisfies Conditions A, it can be shown that  $E_{\alpha,\beta}[L(\alpha_t, \beta_t)]$  is continuous in  $(\alpha, \beta)$  for  $t \geq 0$  and  $\alpha \geq \alpha_0$  where  $\alpha_0 > 2\gamma$ . We will consider only loss functions which satisfy Conditions A and consider only  $(\alpha, \beta) \in E$  such that  $\alpha > 2\gamma$ .

For a loss function  $L$  which satisfies Conditions A, let  $\mathcal{C} = \{(\alpha, \beta): V(\alpha, \beta) < L(\alpha, \beta)\}$  where  $V$  is the value function for  $L$ . For  $(\alpha, \beta) \in E$ , let  $\sigma = \sigma_{\alpha,\beta}$  be the first exit time from  $\mathcal{C}$ ,  $\sigma = \inf\{t \geq 0: V(\alpha_t, \beta_t) = L(\alpha_t, \beta_t)\}$ . Applying Theorem 3 to the function  $g(\alpha, \beta) = -L(\alpha, \beta)$ , we find that  $\sigma = \sigma_{\alpha,\beta}$  is finite a.e.  $P_{\alpha,\beta}$  and  $V(\alpha, \beta) = E_{\alpha,\beta}[L(\alpha_\sigma, \beta_\sigma)]$ . Therefore,  $\sigma$  is an optimal stopping rule and  $\mathcal{C}$  is an optimal continuation region for  $L$ .

In subsequent sections, we will prove the following properties of  $V$  and  $\mathcal{C}$ :

$V = V(\alpha, \beta)$  is an upper semicontinuous function of  $(\alpha, \beta)$ , a continuous nondecreasing function of  $\beta$  for fixed  $\alpha$ , and inside  $\mathcal{C}$  a continuous nonincreasing function of  $\alpha$  for fixed  $\beta$ .  $\mathcal{A}V(\alpha, \beta)$  exists for  $\alpha > 2\gamma$ ,  $\mathcal{A}V(\alpha, \beta) = 0$  for  $(\alpha, \beta)$  in the closure  $\bar{\mathcal{C}}$  of  $\mathcal{C}$ , and  $\mathcal{A}V(\alpha, \beta) > 0$  for  $(\alpha, \beta)$  in  $\sim \bar{\mathcal{C}}$ ,  $\alpha > 2\gamma$ . The set  $\mathcal{C}$  is an open set of the form

$$(3.1) \quad \mathcal{C} = \{(\alpha, \beta): \beta > g(\alpha)\}$$

where  $g$  is a continuous, nondecreasing function of  $\alpha$  defined by

$$(3.2) \quad g(\alpha) = \inf\{\beta: V(\alpha, \beta) < L(\alpha, \beta)\}.$$

$\mathcal{C}$  is unique in the sense that there is exactly one open continuation region  $\mathcal{C}$  satisfying:

$$(3.3) \quad (\alpha, \beta) \in \mathcal{C} \implies (\alpha, \beta') \in \mathcal{C} \text{ for } \beta' \geq \beta$$

such that if  $\sigma = \sigma_{\alpha,\beta}$  is the first exit time from  $\mathcal{C}$  starting at  $(\alpha, \beta)$ , then  $V(\alpha, \beta) = E_{\alpha,\beta}[L(\alpha_\sigma, \beta_\sigma)]$ .

The boundary condition  $V_\alpha = L_\alpha$  is true on the boundary  $\partial\mathcal{C}$  of  $\mathcal{C}$ .

For a measurable, real valued function  $U = U(\alpha, \beta)$  defined on  $E = R^+ \times R^+$  and  $\mathcal{C}^* \subseteq E$ , necessary and sufficient conditions are found such that  $U = V$  and  $\mathcal{C}^* = \mathcal{C}$ .

3.2. *Proofs of some properties of  $V$  and  $\mathcal{C}$ .*

LEMMA 4.

- (1)  $V = V(\alpha, \beta)$  is upper semicontinuous in  $(\alpha, \beta)$ .
- (2)  $\mathcal{C}$  is an open set.

PROOF. Let  $\mathcal{G}$  be the collection of all bounded stopping times taking values in the set of diadic rationals. Then

$$V(\alpha, \beta) = \inf_{\tau \in \mathcal{G}} E_{\alpha,\beta}[L(\alpha_\tau, \beta_\tau)].$$

Suppose  $\tau \in \mathcal{G}$  takes values  $k/2^n$  for  $k = 0, 1, \dots, m2^n$ . Then

$$E_{\alpha,\beta}[L(\alpha_\tau, \beta_\tau)] = \sum_{k=0}^{m2^n} E_{\alpha,\beta} \left[ L\left(\alpha_{\frac{k}{2^n}}, \beta_{\frac{k}{2^n}}\right) I\left\{\tau = \frac{k}{2^n}\right\} \right]$$

which is continuous in  $(\alpha, \beta)$ . Therefore,  $V$  is upper semicontinuous in  $(\alpha, \beta)$ . That  $\mathcal{C}$  is open follows from the upper semicontinuity of  $V$ . We will prove the following theorem.

THEOREM 4. *Let the loss function  $L = L(\alpha, \beta)$  satisfy Conditions A. Then:*

- (1) *there is an optimal continuation region  $\mathcal{C}$  of the form (3.1) with boundary function  $g = g(\alpha)$  defined by (3.2);*
- (2)  *$g$  is nondecreasing in  $\alpha$ ;*

- (3)  $g$  is a continuous function of  $\alpha$ ;
- (4) the value function  $V = V(\alpha, \beta)$  is, inside  $\mathcal{C}$ , a continuous nonincreasing function of  $\alpha$  for fixed  $\beta$ ;
- (5)  $V = V(\alpha, \beta)$  is a continuous nondecreasing function of  $\beta$  for fixed  $\alpha$ ;
- (6)  $\mathcal{A}V(\alpha, \beta)$  exists for every  $(\alpha, \beta)$ ;
- (7)  $\mathcal{A}V(\alpha, \beta) = 0$  for  $(\alpha, \beta)$  in the closure of  $\mathcal{C}$ .

The random variable  $Z_t$  has, conditionally given  $\theta$ , a  $\Gamma(t/2, \theta/2)$  distribution and  $\theta$  has a  $\Gamma(\alpha/2, \beta/2)$  distribution. For  $a > 0$ ,  $aZ_t$  has, conditionally given  $\theta$ , a  $\Gamma(t/2, \theta/(2a))$  distribution and  $\theta/a$  has a  $\Gamma(\alpha/2, a\beta/2)$  distribution. Therefore, for  $a \geq 1$ ,

$$\begin{aligned}
 V(\alpha, a\beta) &= \inf_{\tau} E_{\alpha, a\beta}[(a\beta + Z_{\tau})^{\gamma}h(\alpha_{\tau}) + \phi(\alpha_{\tau})] \\
 (3.4) \quad &= \inf_{\tau} E_{\alpha, \beta}[(a\beta + aZ_{\tau})^{\gamma}h(\alpha_{\tau}) + \phi(\alpha_{\tau})] \\
 &= a^{\gamma} \inf_{\tau} E_{\alpha, \beta}[L(\alpha_{\tau}, \beta_{\tau}) + (a^{-\gamma} - 1)\phi(\alpha_{\tau})] \\
 &\leq a^{\gamma}V(\alpha, \beta) + (1 - a^{\gamma})\phi(\alpha).
 \end{aligned}$$

If  $(\alpha, \beta) \in \mathcal{C}$  and  $a > 1$ , then  $V(\alpha, a\beta) \leq a^{\gamma}V(\alpha, \beta) + (1 - a^{\gamma})\phi(\alpha) < a^{\gamma}L(\alpha, \beta) + (1 - a^{\gamma})\phi(\alpha) = L(\alpha, a\beta)$ . Therefore,  $(\alpha, a\beta) \in \mathcal{C}$  and it follows that  $V(\alpha, \beta) < L(\alpha, \beta)$  for  $\beta > g(\alpha)$  and  $V(\alpha, \beta) = L(\alpha, \beta)$  for  $\beta < g(\alpha)$ . Using (3.4), for  $a \geq 1$ ,

$$\begin{aligned}
 (3.5) \quad V(\alpha, a\beta) &= \inf_{\tau} E_{\alpha, \beta}[a^{\gamma}\beta^{\gamma}h(\alpha_{\tau}) + \phi(\alpha_{\tau})] \\
 &\geq \inf_{\tau} E_{\alpha, \beta}[\beta^{\gamma}h(\alpha_{\tau}) + \phi(\alpha_{\tau})] = V(\alpha, \beta).
 \end{aligned}$$

Combining (3.4) and (3.5) shows that  $V(\alpha, \beta)$  is a continuous nondecreasing function of  $\beta$  for fixed  $\alpha$ , proving part (5) of Theorem 4. Combining these results with the fact that  $\mathcal{C}$  is open proves part (1) of Theorem 4.

Let  $(\alpha, \beta) \in \mathcal{C}$ . Because  $\mathcal{C}$  is open, there is an  $h_0 > 0$  such that  $(\alpha + h, \beta') \in \mathcal{C}$  and  $(\alpha - h, \beta') \in \mathcal{C}$  for  $\beta' \geq \beta$  and  $0 \leq h \leq h_0$ . For  $0 \leq h \leq h_0$ ,  $\sigma = \sigma_{\alpha, \beta} \geq h$  a.e.  $P_{\alpha, \beta}$ . Therefore,  $V(\alpha, \beta) = E_{\alpha, \beta}[L(\alpha_{\sigma}, \beta_{\sigma})] = E_{\alpha, \beta}[E_{\alpha, \beta, h}[L(\alpha_{\sigma}, \beta_{\sigma})]] = E_{\alpha, \beta}[V(\alpha + h, \beta + Z_h)]$ . Then  $V(\alpha + h, \beta) - V(\alpha, \beta) = -E_{\alpha, \beta}[V(\alpha + h, \beta + Z_h) - V(\alpha + h, \beta)] = -hG_h(\alpha, \beta)$  where

$$G_h(\alpha, \beta) = \int_0^{\infty} \frac{C_h[V(\alpha + h, \beta + \beta y) - V(\alpha + h, \beta)]y^{h/2}}{y(1+y)^{(\alpha+h)/2}} dy$$

and

$$C_h = \frac{1}{2} \Gamma\left(\frac{\alpha + h}{2}\right) / \left( \Gamma\left(\frac{h}{2} + 1\right) \Gamma\left(\frac{\alpha}{2}\right) \right).$$

By (3.4) and (3.5),  $0 \leq V(\alpha + h, \beta + \beta y) - V(\alpha + h, \beta) \leq ((1+y)^{\gamma} - 1)V(\alpha + h, \beta)$ . Therefore, the integrand in the expression for  $G_h(\alpha, \beta)$  is nonnegative and bounded by an integrable function for  $0 \leq h \leq 1$ . Therefore,  $V(\alpha + h, \beta) - V(\alpha, \beta) \leq 0$  and  $|V(\alpha + h, \beta) - V(\alpha, \beta)| \rightarrow 0$  as  $h \rightarrow 0$ . A similar argument shows that  $|V(\alpha, \beta) - V(\alpha - h, \beta)| \rightarrow 0$  as  $h \rightarrow 0$ . Therefore, inside  $\mathcal{C}$ ,  $V(\alpha, \beta)$  is a continuous nonincreasing function of  $\alpha$  for fixed  $\beta$ , proving part (4) of Theorem 4.

For  $(\alpha, \beta) \in \mathcal{C}$ ,  $GV(\alpha, \beta)$  may be written as

$$\begin{aligned}
 GV(\alpha, \beta) &= \lim_{h \downarrow 0} G_h(\alpha, \beta) \\
 &= \frac{1}{2} \int_0^{\infty} \frac{V(\alpha, \beta + \beta y) - V(\alpha, \beta)}{y(1+y)^{\alpha/2}} dy
 \end{aligned}$$

since by the dominated convergence theorem the limit can be taken inside the integral.

For  $(\alpha, \beta) \in \mathcal{C}$ ,  $V_{\alpha}^{+}(\alpha, \beta) = \lim_{h \downarrow 0} (V(\alpha + h, \beta) - V(\alpha, \beta))/h = \lim_{h \downarrow 0} -G_h(\alpha, \beta) = -GV(\alpha, \beta)$ , and by similar reasoning,  $V_{\alpha}^{-}(\alpha, \beta) = -GV(\alpha, \beta)$ . Therefore, for  $(\alpha, \beta) \in \mathcal{C}$ ,

$\mathcal{AV}(\alpha, \beta)$  exists and  $\mathcal{AV}(\alpha, \beta) = V_\alpha(\alpha, \beta) + GV(\alpha, \beta) = 0$ , proving part (7) of Theorem 4 for  $(\alpha, \beta) \in \mathcal{C}$ .

To show that the function  $g = g(\alpha)$  is a nondecreasing function of  $\alpha$ , fix  $\alpha^0$  and let  $\beta^0 = g(\alpha^0)$ . Define the region  $\mathcal{C}^*$  by  $\mathcal{C}^* = \mathcal{C} \cap \{(\alpha, \beta) : \alpha > \alpha^0, \beta > \beta^0\}$ . For  $(\alpha, \beta) \in E$ , let  $\tau = \tau_{\alpha, \beta} = \inf\{t \geq 0 : (\alpha_t, \beta_t) \notin \mathcal{C}^*\}$  and let  $U(\alpha, \beta) = E_{\alpha, \beta}[L(\alpha_\tau, \beta_\tau)]$  be the expected loss associated with  $\mathcal{C}^*$ . Let  $\sigma = \sigma_{\alpha, \beta}$  be the first exit time from  $\mathcal{C}$ , an optimal stopping rule, starting at  $(\alpha, \beta)$ .

For  $(\alpha, \beta) \in \mathcal{C}^*$ ,  $(\alpha_t, \beta_t)$  cannot be in  $\mathcal{C}$  without also being in  $\mathcal{C}^*$  since  $\alpha_t$  and  $\beta_t$  are both nondecreasing in  $t$ ,  $t \geq 0$ . Therefore,  $\tau = \sigma$  and  $U(\alpha, \beta) = V(\alpha, \beta)$  for  $(\alpha, \beta) \in \mathcal{C}^*$ .

Suppose there is an  $\alpha^* > \alpha^0$  such that  $g(\alpha^*) < \beta^0 = g(\alpha^0)$ . Then  $(\alpha^*, \beta^0)$  is on the boundary of  $\mathcal{C}^*$  and inside  $\mathcal{C}$ , so that  $U(\alpha^*, \beta^0) = L(\alpha^*, \beta^0) > V(\alpha^*, \beta^0)$ . Since  $\mathcal{C}^*$  is open, there is a  $t_0 > 0$  such that  $(\alpha^* + t, \beta) \in \mathcal{C}^*$  for  $0 < t \leq t_0$ ,  $\beta > \beta^0$ . For  $0 < t \leq t_0$ ,  $U(\alpha^* + t, \beta^0) = V(\alpha^* + t, \beta^0)$  a.e.  $P_{\alpha^* + t, \beta^0}$  and

$$E_{\alpha^* + t, \beta^0}[U(\alpha^* + t, \beta^0)] = E_{\alpha^* + t, \beta^0}[V(\alpha^* + t, \beta^0)] = E_{\alpha^* + t, \beta^0}[E_{\alpha^* + t, \beta^0}[L(\alpha^* + t, \beta^0)]] = E_{\alpha^* + t, \beta^0}[L(\alpha^* + t, \beta^0)].$$

That  $\lim_{t \downarrow 0} E_{\alpha^* + t, \beta^0}[U(\alpha^* + t, \beta^0)] = U(\alpha^*, \beta^0)$  follows from Corollary 2 of Theorem 4.9 and Theorem 4.10 of Dynkin ((1965), pages 123, 124) for loss functions  $L$  which are bounded above; the result follows for loss functions  $L$  satisfying conditions A by a truncation argument. This implies that  $U(\alpha^*, \beta^0) = E_{\alpha^* + t, \beta^0}[L(\alpha^* + t, \beta^0)] = V(\alpha^* + t, \beta^0)$ , which is a contradiction. Therefore,  $g$  is a nondecreasing function of  $\alpha$ , proving part (2) of Theorem 4.

We will now prove that  $\mathcal{AV}(\alpha, \beta) = 0$  for  $(\alpha, \beta) \in \partial\mathcal{C}$ . Let  $(\alpha, \beta) \in \partial\mathcal{C}$ . Because  $g$  is nondecreasing in  $\alpha$ ,  $V(\alpha', \beta) = L(\alpha', \beta)$  for  $\alpha' \geq \alpha$ , which implies that  $V_\alpha^+(\alpha, \beta) = L_\alpha(\alpha, \beta)$ . Therefore,  $\mathcal{AV}(\alpha, \beta)$  exists and  $\mathcal{AV}(\alpha, \beta) = V_\alpha^+(\alpha, \beta) + GV(\alpha, \beta)$ . Also, because  $g$  is nondecreasing in  $\alpha$ ,  $(\alpha - h, \beta) \in \bar{\mathcal{C}}$  for  $h > 0$ . Using Dynkin's formula of Theorem 1,

$$E_{\alpha - h, \beta}[V(\alpha, \beta + Z_h)] - V(\alpha - h, \beta) = E_{\alpha - h, \beta} \left[ \int_0^h \mathcal{AV}(\alpha - h + t, \beta + Z_t) dt \right] = 0$$

because  $\mathcal{AV} = 0$  inside  $\mathcal{C}$ . Therefore,

$$\begin{aligned} V_\alpha^-(\alpha, \beta) &= \lim_{h \downarrow 0} \frac{V(\alpha, \beta) - V(\alpha - h, \beta)}{h} \\ &= \lim_{h \downarrow 0} E_{\alpha - h, \beta} \frac{[V(\alpha, \beta) - V(\alpha, \beta + Z_h)]}{h} \\ &= -GV(\alpha, \beta). \end{aligned}$$

Since  $V(\alpha - h, \beta) \leq L(\alpha - h, \beta)$  and  $V(\alpha, \beta) = L(\alpha, \beta)$ , then  $V_\alpha^-(\alpha, \beta) \geq L_\alpha(\alpha, \beta) = V_\alpha^+(\alpha, \beta)$ . Therefore,  $\mathcal{AV}(\alpha, \beta) = V_\alpha^+(\alpha, \beta) + GV(\alpha, \beta) = V_\alpha^+(\alpha, \beta) - V_\alpha^-(\alpha, \beta) \leq 0$ . By Theorem 3,  $s(\alpha, \beta) = -V(\alpha, \beta)$  is a regular function, implying that  $V(\alpha, \beta) \leq E_{\alpha, \beta}[V(\alpha + h, \beta + Z_h)]$  for any  $(\alpha, \beta) \in E$  and  $h > 0$ . Therefore,  $\mathcal{AV}(\alpha, \beta) \geq 0$ , which implies that  $\mathcal{AV}(\alpha, \beta) = 0$  for  $(\alpha, \beta) \in \partial\mathcal{C}$ . This completes the proof of part (7) of Theorem 4.

Note that for  $(\alpha, \beta) \in \sim \mathcal{C}$ ,  $V_\alpha(\alpha, \beta) = L_\alpha(\alpha, \beta)$  exists. Therefore,  $\mathcal{AV}(\alpha, \beta) = V_\alpha(\alpha, \beta) + GV(\alpha, \beta)$  exists. This completes the proof of part (6) of Theorem 4.

It remains to be proved that  $g$  is continuous. Suppose  $\alpha_n \downarrow \alpha$  as  $n \rightarrow \infty$ . Since  $g$  is nondecreasing,  $g(\alpha) \leq \lim_{n \rightarrow \infty} g(\alpha_n) = K$ . Suppose  $K > g(\alpha)$  and let  $\beta$  be such that  $g(\alpha) < \beta < K$ . Then  $(\alpha, \beta) \in \mathcal{C}$  but  $(\alpha + h, \beta) \in \sim \mathcal{C}$  for  $h > 0$ . This contradicts the fact that  $\mathcal{C}$  is open. Therefore,  $g$  is right continuous.

We use Lemma 5 to prove that  $g$  is continuous.

LEMMA 5. Let  $\alpha$  be fixed,  $\alpha > 2\gamma$ . Then  $\mathcal{AV}(\alpha, \beta) - \mathcal{AL}(\alpha, \beta)$  is strictly decreasing in  $\beta$  for  $\beta \leq g(\alpha)$ .

PROOF. Suppose  $\beta \leq g(\alpha)$ . Then  $L(\alpha, \beta) = V(\alpha, \beta)$  and  $L_\alpha(\alpha, \beta) = V_\alpha(\alpha, \beta)$ , so



$$\begin{aligned} \mathcal{A}V(\alpha, \beta) - \mathcal{A}L(\alpha, \beta) &= \frac{1}{2} \int_0^\infty \frac{V(\alpha, \beta + \beta y) - L(\alpha, \beta + \beta y)}{y(1+y)^{\alpha/2}} dy \\ &= \frac{1}{2} \int_{\beta^{-1}g(\alpha)-1}^\infty \frac{V(\alpha, \beta + \beta y) - L(\alpha, \beta + \beta y)}{y(1+y)^{\alpha/2}} dy \end{aligned}$$

because  $V(\alpha, \beta + \beta y) = L(\alpha, \beta + \beta y)$  for  $\beta + \beta y \leq g(\alpha)$ , or  $y \leq \beta^{-1}g(\alpha) - 1$ . Therefore,

$$\mathcal{A}V(\alpha, \beta) - \mathcal{A}L(\alpha, \beta) = \frac{1}{2} \int_{g(\alpha)}^\infty \frac{V(\alpha, z) - L(\alpha, z)}{z - \beta} \frac{\beta^{\alpha/2}}{z^{\alpha/2}} dz$$

where  $z = \beta + \beta y$ . Since  $V(\alpha, z) - L(\alpha, z) < 0$  for  $z > g(\alpha)$ ,  $\mathcal{A}V(\alpha, \beta) - \mathcal{A}L(\alpha, \beta)$  is strictly decreasing in  $\beta$  for  $\beta \leq g(\alpha)$ .

Now suppose  $\alpha_n \uparrow \alpha$  as  $n \rightarrow \infty$  and let  $K = \lim_{n \rightarrow \infty} g(\alpha_n) \leq g(\alpha)$ . Suppose  $K < g(\alpha)$  and let  $\beta_0$  be such that  $K < \beta_0 < g(\alpha)$ . Then both  $(\alpha, \beta_0)$  and  $(\alpha, g(\alpha))$  are on the boundary of  $\mathcal{C}$  and  $\mathcal{A}V(\alpha, g(\alpha)) = \mathcal{A}V(\alpha, \beta_0) = 0$ . By Lemma 5,  $\mathcal{A}V(\alpha, g(\alpha)) - \mathcal{A}L(\alpha, g(\alpha)) < \mathcal{A}V(\alpha, \beta_0) - \mathcal{A}L(\alpha, \beta_0)$ , which implies that  $\mathcal{A}L(\alpha, g(\alpha)) > \mathcal{A}L(\alpha, \beta_0)$ . Because  $L$  satisfies Conditions A,  $\mathcal{A}L(\alpha, g(\alpha)) \leq \mathcal{A}L(\alpha, \beta_0)$ , which is a contradiction. Therefore,  $g$  is a continuous function of  $\alpha$ . This proves part (3) of Theorem 4 and completes the proof of that theorem.

Theorem 5 describes a uniqueness property of  $\mathcal{C}$ . The proof follows from the fact that  $\mathcal{A}V(\alpha, \beta) > 0$  for  $(\alpha, \beta) \in \sim \mathcal{C}$ . For proof, see Bartold (1976).

**THEOREM 5.** *The optimal continuation region  $\mathcal{C}$  defined by (3.1) is unique in the sense that  $\mathcal{C}^*$  is an open set satisfying (3.3) and if for every  $(\alpha, \beta) \in E$ ,  $V(\alpha, \beta) = E_{\alpha, \beta}[L(\alpha_\tau, \beta_\tau)]$ , where  $\tau = \tau_{\alpha, \beta}$  is the first exit time from  $\mathcal{C}^*$  starting at  $(\alpha, \beta)$ , then  $\mathcal{C}^* = \mathcal{C}$ .*

**3.3. Necessary and sufficient conditions for optimality.** In this section we prove Theorem 6, which gives a set of necessary and sufficient conditions for a function  $U$  and a region  $\mathcal{C}^*$  to be the value function  $V$  and the optimal continuation region  $\mathcal{C}$ . Lemma 2 gave one set of such conditions. Theorem 6 plays an important role in Section 4.

Given a measurable real valued function  $U$  defined on  $E = R^+ \times R^+$  and a subset  $\mathcal{C}^*$  of  $E$ , consider the following set of conditions on  $U$  and  $\mathcal{C}^*$ :

(1)  $\mathcal{C}^*$  is open set of the form  $\mathcal{C}^* = \{(\alpha, \beta) : \beta > f(\alpha)\}$  where  $f$  is a continuous nondecreasing function of  $\alpha$ , and the first exit time  $\tau = \tau_{\alpha, \beta}$  from  $\mathcal{C}^*$  starting at the point  $(\alpha, \beta)$  is finite with probability one  $P_{\alpha, \beta}$  for every  $(\alpha, \beta) \in E$ .

(2)  $U(\alpha, \beta) < L(\alpha, \beta)$  and  $\mathcal{A}U(\alpha, \beta) = 0$  for  $(\alpha, \beta) \in \mathcal{C}^*$ .

(3)  $U(\alpha, \beta) = L(\alpha, \beta)$  for  $(\alpha, \beta) \notin \mathcal{C}^*$ .

(4)  $U_\alpha(\alpha, \beta) = L_\alpha(\alpha, \beta)$  for  $(\alpha, \beta) \notin \mathcal{C}^*$ .

If we replace condition (4) by condition (4'):

(4')  $\mathcal{A}U(\alpha, \beta) \geq 0$  for  $(\alpha, \beta) \notin \mathcal{C}^*$

then by Lemma 2 conditions (1)–(3) and (4') imply that  $U = V$  and  $\mathcal{C}^* = \mathcal{C}$ . Therefore, to prove Theorem 6, we show that conditions (1)–(4) imply condition (4'). From previous sections we know that  $V$  and  $\mathcal{C}$  satisfy conditions (1)–(4). Therefore, conditions (1)–(4) are necessary conditions for optimality. Theorem 6 states that these conditions are also sufficient.

**THEOREM 6.** *Suppose  $U = U(\alpha, \beta)$  is a measurable, real valued function defined on  $E = R^+ \times R^+$  and  $\mathcal{C}^* \subseteq E$ . Then  $U = V$  and  $\mathcal{C}^* = \mathcal{C}$  if and only if conditions (1)–(4) are satisfied.*

**PROOF.** Let  $(\alpha, \beta)$  be on the boundary of  $\mathcal{C}^*$ . Then  $(\alpha - h, \beta)$  is in the closure of  $\mathcal{C}^*$  for  $h > 0$ . Using Dynkin's formula of Theorem 1,

$$E_{\alpha-h, \beta}[U(\alpha, \beta + Z_h)] - U(\alpha - h, \beta) = E_{\alpha-h, \beta} \left[ \int_0^h \mathcal{A}U(\alpha - h + t, \beta + Z_t) dt \right] = 0$$

because  $\mathcal{A}U = 0$  inside  $\mathcal{C}^*$  by condition (2). Therefore,

$$\begin{aligned} U_\alpha(\alpha, \beta) &= \lim_{h \downarrow 0} \frac{U(\alpha, \beta) - U(\alpha - h, \beta)}{h} \\ &= \lim_{h \downarrow 0} \frac{E_{\alpha-h, \beta}[U(\alpha, \beta) - U(\alpha, \beta + Z_h)]}{h} \\ &= -GU(\alpha, \beta). \end{aligned}$$

Therefore,  $\mathcal{A}U(\alpha, \beta) = U_\alpha(\alpha, \beta) + GU(\alpha, \beta) = 0$  for  $(\alpha, \beta)$  on the boundary of  $\mathcal{C}^*$ .

By condition (4),  $U_\alpha = L_\alpha$  outside  $\mathcal{C}^*$ . We can show as in Lemma 5 that  $\mathcal{A}U(\alpha, \beta) - \mathcal{A}L(\alpha, \beta)$  is strictly decreasing in  $\beta$  for fixed  $\alpha$ , for  $\beta \leq f(\alpha)$ .

Suppose there is a point  $(\alpha, \beta)$  not in the closure of  $\mathcal{C}^*$  such that  $\mathcal{A}U(\alpha, \beta) < 0$ . Then  $\mathcal{A}U(\alpha, \beta) - \mathcal{A}L(\alpha, \beta) < -\mathcal{A}L(\alpha, \beta)$ . Also,  $\beta < f(\alpha)$  implies that  $\mathcal{A}U(\alpha, f(\alpha)) - \mathcal{A}L(\alpha, f(\alpha)) < \mathcal{A}U(\alpha, \beta) - \mathcal{A}L(\alpha, \beta)$ . Since  $\mathcal{A}U(\alpha, f(\alpha)) = 0$ , then  $-\mathcal{A}L(\alpha, f(\alpha)) < -\mathcal{A}L(\alpha, \beta)$ . By Conditions A,  $\mathcal{A}L(\alpha, f(\alpha)) \leq \mathcal{A}L(\alpha, \beta)$ , which is a contradiction, so it must be the case that  $\mathcal{A}U(\alpha, \beta) \geq 0$  for each  $(\alpha, \beta)$  not in the closure of  $\mathcal{C}^*$ . Therefore,  $\mathcal{A}U(\alpha, \beta) \geq 0$  for every  $(\alpha, \beta) \in E$ . That is, condition (4') is satisfied. Applying Lemma 2, we see that  $U = V$  and  $\mathcal{C}^* = \mathcal{C}$ .

For all results whose proofs required that  $\mathcal{A}L(\alpha, \beta)$  be decreasing in  $\beta$  for each fixed  $\alpha$ , corresponding results hold for  $\alpha \geq \alpha_0$  if  $\mathcal{A}L(\alpha, \beta)$  is decreasing in  $\beta$  for each fixed  $\alpha \geq \alpha_0$ .

**4. Asymptotic expansions.**

4.1. *Introduction.* Let  $L$  be a loss function of the form  $L(\alpha, \beta) = \beta^\gamma h(\alpha) + c\alpha$  where  $\gamma \geq 1$ ,  $c > 0$ ,  $h$  is nonnegative and twice continuously differentiable, and  $h$  has an asymptotic expansion in  $\alpha$  for large  $\alpha$ , say  $h(\alpha) \sim k_{-p}\alpha^{-p} + k_{-p-1}\alpha^{-p-1} + \dots$ , where  $k_{-p} > 0$ ,  $p > \gamma$  and, if  $\gamma > 1$ , then  $\gamma > (1 + p)/2$ . Also,  $\mathcal{A}L(\alpha, \beta)$  is decreasing in  $\beta$  for each fixed  $\alpha$ . Such a loss function will be said to satisfy Conditions B. In this section we consider loss functions which satisfy Conditions B.

We will find asymptotic expansions for the boundary  $g$  of the optimal continuation region  $\mathcal{C}$  and the value function  $V(\alpha, \beta)$  for large  $\alpha$ , when  $c = 1$ ; and for  $V(\alpha, \beta)$  when  $c$  is small. The basis of the technique used was developed by Chernoff (1961, 1965) in the context of problems involving the Wiener process. The technique has also been applied to the sequential test of a normal mean when time  $\rightarrow \infty$  by Chernoff and Breakwell (1964) and to the one-armed bandit problem in the sampling inspection context by Chernoff and Ray (1965). Related results for the discrete time case are presented in Starr and Woodroffe (1969).

In Theorem 6 we used the conditions  $U < L$ ,  $\mathcal{A}U = 0$  inside  $\mathcal{C}^*$  and  $U = L$ ,  $U_\alpha = L_\alpha$  on  $\partial\mathcal{C}^*$  to prove that  $U$  and  $\mathcal{C}^*$  were the value function  $V$  and the optimal continuation region  $\mathcal{C}$ . We will find functions  $w = w(\alpha, \beta)$  such that  $\mathcal{A}w(\alpha, \beta) = 0$  and approximate the boundary conditions  $V = L$  and  $V_\alpha = L_\alpha$ . We will then investigate how well this technique has worked to approximate the value function  $V$  and the boundary  $g$  of the optimal continuation region  $\mathcal{C}$ .

If in Theorem 6  $U$  and  $L$  and the boundary  $f$  of  $\mathcal{C}^*$  are all differentiable and  $f'(\alpha) > 0$  for every  $\alpha$ , then the boundary condition  $U_\alpha(\alpha, f(\alpha)) = L_\alpha(\alpha, f(\alpha))$  is equivalent to the boundary condition  $U_\beta(\alpha, f(\alpha)) = L_\beta(\alpha, f(\alpha))$  by the chain rule. It is the boundary condition  $U_\beta = L_\beta$  which we will use to find the asymptotic expansions for  $V$  and  $g$ .

In 4.2 we illustrate the technique using a loss function  $L$  for which  $V$  and  $g$  can be found exactly. In 4.3 we derive the asymptotic expansions for  $V$  and  $g$  for a more general case when  $V$  and  $g$  cannot be determined exactly. In 4.4 we give the asymptotic expansions for  $V$  and  $g$  as the cost  $c$  per observation tends to zero. We look at some related problems to which these techniques apply in 4.5.

Define a function  $w = w(\alpha, \beta)$  by  $w(\alpha, \beta) = (\beta/2)^\nu \Gamma(\alpha/2 - \nu) / \Gamma(\alpha/2)$ . Since  $w(\alpha_t, \beta_t) = E[\theta^{-\nu} | \alpha_t, \beta_t]$ ,  $\{w(\alpha_t, \beta_t), t \geq 0\}$  is a martingale. Therefore,  $E_{\alpha, \beta}[w(\alpha_t, \beta_t)] = w(\alpha, \beta)$  for each  $t \geq 0$ , so that  $\mathcal{A}w(\alpha, \beta) = 0$  for every  $(\alpha, \beta)$ . It is also possible to show  $\mathcal{A}w = 0$  by computing  $\mathcal{A}w$  directly.

We will approximate the value function  $V = V(\alpha, \beta)$  by series expansions of the form

$$V^n(\alpha, \beta) = a_1 \beta^{\delta_1} \frac{\Gamma\left(\frac{\alpha}{2} - \delta_1\right)}{\Gamma\left(\frac{\alpha}{2}\right)} + \dots + a_n \beta^{\delta_n} \frac{\Gamma\left(\frac{\alpha}{2} - \delta_n\right)}{\Gamma\left(\frac{\alpha}{2}\right)}$$

and the boundary function  $g = g(\alpha)$  of the optimal continuation region  $\mathcal{C}$  by series expansions of the form  $\beta_n(\alpha) = c_1 \alpha^{\nu_1} + \dots + c_n \alpha^{\nu_n}$ .

4.2. *A special case.* Let  $L(\alpha, \beta) = \beta^2 \alpha^{-1} (\alpha - 2)^{-2} + c\alpha$  where  $c > 0$  and  $\alpha > 2$ . Then  $\gamma = 2$  and  $h(\alpha) = \alpha^{-1} (\alpha - 2)^{-2}$ . From Section 2.3,  $\mathcal{A}L(\alpha, \beta) = -\beta^2 (\alpha^2 - 8\alpha + 8) \alpha^{-2} (\alpha - 2)^{-3} (\alpha - 4)^{-1} + c$ , which is defined and decreasing in  $\beta$  for  $\alpha > 4$ .  $L$  satisfies Conditions B. Therefore, if there is a function  $U = U(\alpha, \beta)$  and a region  $\mathcal{C}^*$  which satisfy the conditions of Theorem 6, then for  $\alpha > 4$ ,  $U = V$  and  $\mathcal{C}^* = \mathcal{C}$ .

Let  $V^1(\alpha, \beta) = a_1 \beta^{\delta_1} \Gamma((\alpha/2) - \delta_1) / \Gamma(\alpha/2)$  and  $\beta_1(\alpha) = c_1 \alpha^{\nu_1}$ . We will approximate the boundary conditions  $V(\alpha, g(\alpha)) = L(\alpha, g(\alpha))$  and  $V_\beta(\alpha, g(\alpha)) = L_\beta(\alpha, g(\alpha))$ . Since  $\Gamma((\alpha/2) - \delta_1) / \Gamma(\alpha/2) \sim (\alpha/2)^{-\delta_1}$  for large  $\alpha$ , we have  $V^1(\alpha, \beta_1(\alpha)) \sim a_1 c_1^{\delta_1} 2^{\delta_1} \alpha^{\nu_1 \delta_1 - \delta_1}$  and  $V_\beta^1(\alpha, \beta_1(\alpha)) \sim a_1 \delta_1 c_1^{\delta_1 - 1} 2^{\delta_1} \alpha^{\nu_1 \delta_1 - \delta_1}$ . Also,  $L(\alpha, \beta_1(\alpha)) \sim c_1^2 \alpha^{2\nu_1 - 3} + c\alpha$  and  $L_\beta(\alpha, \beta) \sim 2c_1 \alpha^{\nu_1 - 3}$  for large  $\alpha$ .

Since  $\mathcal{A}L(\alpha, \beta) > 0$  in  $\sim \mathcal{C}$ , setting  $\mathcal{A}L(\alpha, \beta) = 0$  and solving for  $\beta$  gives an upper bound for  $\partial \mathcal{C}$ . This calculation gives  $\beta \sim c^{1/2} \alpha^2$ , suggesting that  $c_1 = c^{1/2}$  and  $\nu_1 = 2$ . Then the relation  $V = L$  on  $\partial \mathcal{C}$  suggests  $\delta_1 = 1$  and  $a_1 = c^{1/2}$ . When  $a_1 = c_1 = c^{1/2}$ ,  $\nu_1 = 2$  and  $\delta_1 = 1$ , the largest order terms in the expansions for  $V^1(\alpha, \beta_1(\alpha))$  and  $L(\alpha, \beta_1(\alpha))$  are equal, as are the largest order terms in the expansions for  $V_\beta^1(\alpha, \beta_1(\alpha))$  and  $L_\beta(\alpha, \beta_1(\alpha))$ .

If we equate the largest order terms in the expansions for  $V^2(\alpha, \beta_2(\alpha))$  and  $L(\alpha, \beta_2(\alpha))$  and in the expansions for  $V_\beta^2(\alpha, \beta_2(\alpha))$  and  $L_\beta(\alpha, \beta_2(\alpha))$ , we find that  $\delta_2 = a_2 = 0$ ,  $\nu_2 = 1$  and  $c_2 = -2c^{1/2}$ . Therefore,  $V^2(\alpha, \beta) = 2c^{1/2} \beta (\alpha - 2)^{-1}$  and  $\beta_2(\alpha) = c^{1/2} \alpha^2 - 2c^{1/2} \alpha$ .

Let  $\mathcal{C}^* = \{(\alpha, \beta) : \beta > \beta_2(\alpha)\}$  and define a function  $U = U(\alpha, \beta)$  by

$$\begin{aligned} U(\alpha, \beta) &= V^2(\alpha, \beta) & \text{if } (\alpha, \beta) \in \mathcal{C}^*; \\ &= L(\alpha, \beta) & \text{if } (\alpha, \beta) \notin \mathcal{C}^*. \end{aligned}$$

Then by Theorem 6,  $U = V$  and  $\mathcal{C}^* = \mathcal{C}$ , for  $\alpha > 4$ .

For this special case, the solutions  $V^2$  and  $\beta_2$  are exact solutions rather than asymptotic approximations.

4.3. *The general case.* Let the loss function  $L(\alpha, \beta) = \beta^\gamma h(\alpha) + \alpha$  satisfy Conditions B. We will show that the value function  $V$  and the boundary function  $g$  of the optimal continuation region  $\mathcal{C}$  can be approximated expressions of the form

$$V^n(\alpha, \beta) = a_1 \beta^{\delta_1} \frac{\Gamma\left(\frac{\alpha}{2} - \delta_1\right)}{\Gamma\left(\frac{\alpha}{2}\right)} + \dots + a_n \beta^{\delta_n} \frac{\Gamma\left(\frac{\alpha}{2} - \delta_n\right)}{\Gamma\left(\frac{\alpha}{2}\right)}$$

$$\beta_n(\alpha) = c_1 \alpha^{\nu_1} + \dots + c_n \alpha^{\nu_n}$$

where  $\delta_k = (-k + 2) \gamma (1 + p - \gamma)^{-1}$  and  $\nu_k = (1 + p) \gamma^{-1} - k + 1$  and  $a_k, c_k$  are appropriate constants, for  $k = 1, \dots, n$ .

**THEOREM 7.** *Suppose the loss function  $L(\alpha, \beta) = \beta^\gamma h(\alpha) + \alpha$  satisfies Conditions B. Let  $V^n$  and  $\beta_n$  be defined as above. Then there are constants  $a_1, \dots, a_n$  and  $c_1, \dots, c_n$  for which*

$$V^n(\alpha, \beta_n(\alpha)) - L(\alpha, \beta_n(\alpha)) = O(\alpha^{-n+1}) \quad \text{and}$$

$$V_\beta^n(\alpha, \beta_n(\alpha)) - L_\beta(\alpha, \beta_n(\alpha)) = O(\alpha^{((p+1)/\gamma) - n+1})$$

as  $\alpha \rightarrow \infty$ . The constants  $a_1, \dots, a_n, c_1, \dots, c_n$  may be computed by the algorithm

$$a_k = -m_{-k+2} 2^{-\delta_k} c_1^{-\delta_k}$$

$$c_k = (-w_{-k+2} - a_k \delta_k 2^{\delta_k} c_1^{\delta_k-1}) D^{-1}$$

where

$$D = a_1 \delta_1 (\delta_1 - 1) 2^{\delta_1} c_1^{\delta_1-2} - k_{-p} \gamma (\gamma - 1) c_1^{\gamma-2}$$

$m_{-k+2}$  is the coefficient of the order  $\alpha^{-k+2}$  term not involving  $a_k$  in the asymptotic expansion for  $V^k(\alpha, \beta_k(\alpha)) - L(\alpha, \beta_k(\alpha))$  and  $w_{-k+2}$  is the coefficient of the order  $\alpha^{-(p+1)/\gamma - k+2}$  term not involving  $a_k$  or  $c_k$  in the asymptotic expansion for  $V_{\beta}^k(\alpha, \beta_k(\alpha)) - L_{\beta}(\alpha, \beta_k(\alpha))$ .

The functions  $\beta_n$  and  $V^n$  are asymptotic expansions for  $g$  and  $V$  in the sense that  $g(\alpha) = \beta_n(\alpha) + O(\alpha^{n+1})$  as  $\alpha \rightarrow \infty$  and  $V(\alpha, \beta) = V^n(\alpha, \beta) + O(\alpha^{-n})$  for  $(\alpha, \beta) \in C$  as  $\alpha \rightarrow \infty$ .

PROOF. The proof will proceed in three parts. We first find the expansions  $V^n$  and  $\beta_n$ , then prove that the  $\beta_n$  form an asymptotic expansion for  $g$ , and finally prove that the  $V^n$  form an asymptotic expansion for  $V$  inside  $\mathcal{C}$ .

PART 1. *Finding the expansions  $V^n$  and  $\beta_n$ .* We assume that  $\delta_k$  and  $\nu_k$  are known and proceed to find the coefficients  $a_k$  and  $c_k$  for  $k = 1, \dots, n$ . This approach is taken only to simplify the presentation. The expressions for  $\delta_k$  and  $\nu_k$  can be found simultaneously with those for  $a_k$  and  $c_k$  by a procedure similar to that used in the special case of 4.2.

Let the asymptotic expansion of  $\Gamma(\alpha/2 + \delta)/\Gamma(\alpha/2)$  for large  $\alpha$  be denoted by  $(\alpha/2)^{\delta} [1 + k_1(\delta) \alpha^{-1} + k_2(\delta) \alpha^{-2} + \dots]$ .

For the case when  $n = 1$ ,  $c_1$  and  $a_1$  may be found as in 4.2, and it may be verified that  $V^1(\alpha, \beta_1(\alpha)) - L(\alpha, \beta_1(\alpha)) = O(1)$  and  $V_{\beta}^1(\alpha, \beta_1(\alpha)) - L_{\beta}(\alpha, \beta_1(\alpha)) = O(\alpha^{-(p+1)/\gamma})$  as  $\alpha \rightarrow \infty$ . For the induction step, suppose that there exist  $a_1, \dots, a_n$  and  $c_1, \dots, c_n$  such that  $V^n(\alpha, \beta_n(\alpha)) - L(\alpha, \beta_n(\alpha)) = O(\alpha^{-n+1})$  and  $V_{\beta}^n(\alpha, \beta_n(\alpha)) - L_{\beta}(\alpha, \beta_n(\alpha)) = O(\alpha^{-((p+1)/\gamma - n+1)})$  as  $\alpha \rightarrow \infty$ . Let  $U^n = V^n - L$ . Then

$$U^{n+1}(\alpha, \beta_{n+1}(\alpha)) = U^n(\alpha, \beta_{n+1}(\alpha)) + a_{n+1} \beta_{n+1}(\alpha)^{\delta_{n+1}}$$

$$\Gamma(\alpha/2 - \delta_{n+1})/\Gamma(\alpha/2)$$

$$= U^n(\alpha, \beta_n(\alpha)) + a_{n+1} \beta_{n+1}(\alpha)^{\delta_{n+1}}$$

$$\Gamma(\alpha/2 - \delta_{n+1})/\Gamma(\alpha/2) + O(\alpha^{-n}).$$

The relation  $U^n(\alpha, \beta_{n+1}(\alpha)) = U^n(\alpha, \beta_n(\alpha)) + O(\alpha^{-n})$  follows from the equality of the terms in the expansions of  $V^{n+1}(\alpha, \beta_{n+1}(\alpha))$  and  $L(\alpha, \beta_{n+1}(\alpha))$  which involve  $c_{n+1}$  and are of order  $\alpha^{-n+1}$ ; there are no terms in these expansions of order greater than  $\alpha^{-n+1}$  which involve  $c_{n+1}$ . By the induction hypothesis,  $U^n(\alpha, \beta_n(\alpha)) = O(\alpha^{-n+1})$ , and since  $U^n(\alpha, \beta_n(\alpha))$  has an asymptotic expansion in powers of  $\alpha$ ,  $m_{-n+1} = \lim_{\alpha \rightarrow \infty} \alpha^{n-1} U^n(\alpha, \beta_n(\alpha))$  exists. Also,  $a_{n+1} \beta_{n+1}(\alpha)^{\delta_{n+1}} \Gamma(\alpha/2 - \delta_{n+1})/\Gamma(\alpha/2) \sim a_{n+1} c_1^{\delta_{n+1}} 2^{\delta_{n+1}} \alpha^{-n+1}$  as  $\alpha \rightarrow \infty$ . Therefore, if  $a_{n+1} = -m_{-n+1} 2^{-\delta_{n+1}} c_1^{-\delta_{n+1}}$ , then  $U^{n+1}(\alpha, \beta_{n+1}(\alpha)) = O(\alpha^{-n})$ . Similarly,

$$U_{\beta}^{n+1}(\alpha, \beta_{n+1}(\alpha)) = U_{\beta}^n(\alpha, \beta_{n+1}(\alpha)) + a_{n+1} \delta_{n+1} \beta_{n+1}(\alpha)^{\delta_{n+1}-1}$$

$$\Gamma(\alpha/2 - \delta_{n+1})/\Gamma(\alpha/2)$$

$$= U_{\beta}^n(\alpha, \beta_n(\alpha)) + U_{\beta\beta}^n(\alpha, \beta_n(\alpha)) c_{n+1} \alpha^{\nu_{n+1}}$$

$$+ a_{n+1} \delta_{n+1} \beta_{n+1}(\alpha)^{\delta_{n+1}-1} \Gamma(\alpha/2 - \delta_{n+1})/\Gamma(\alpha/2)$$

$$+ O(\alpha^{-((p+1)/\gamma - n)}).$$

Now,  $U_{\beta\beta}^n(\alpha, \beta_n(\alpha)) c_{n+1} \alpha^{\nu_{n+1}} \sim D c_{n+1} \alpha^{-((p+1)/\gamma - n+1)}$ ,  $U_{\beta}^n(\alpha, \beta_n(\alpha)) \sim w_{-n+1} \alpha^{-((p+1)/\gamma - n+1)}$  and  $a_{n+1} \delta_{n+1} \beta_{n+1}(\alpha)^{\delta_{n+1}-1} \Gamma(\alpha/2 - \delta_{n+1})/\Gamma(\alpha/2) \sim a_{n+1} \delta_{n+1} 2^{\delta_{n+1}} c_1^{\delta_{n+1}-1} \alpha^{-((p+1)/\gamma - n+1)}$ . Therefore, if  $c_{n+1} = (-w_{-n+1} - a_{n+1} \delta_{n+1} 2^{\delta_{n+1}} c_1^{\delta_{n+1}-1}) D^{-1}$ , then  $U_{\beta}^{n+1}(\alpha, \beta_{n+1}(\alpha)) = O(\alpha^{-((p+1)/\gamma - n)})$  as  $\alpha \rightarrow \infty$ .

PART 2. *Proof that the  $\beta_n(\alpha)$  form an asymptotic expansion for  $g(\alpha)$  as  $\alpha \rightarrow \infty$ .* Suppose  $n \geq 3$  and define a function  $U^n = U^n(\alpha, \beta)$  by  $U^n(\alpha, \beta) = V^{n-1}(\alpha, \beta) + k\beta^{\delta_n}\Gamma(\alpha/2 - \delta_n)/\Gamma(\alpha/2)$  where  $k$  is an arbitrary real number. We are suppressing the dependence on  $k$  in the notation for  $U^n(\alpha, \beta)$  and for  $f_n$  and  $g_n$  defined below. If  $k = a_n$ , then  $U^n = V^n$ .

Define the function  $f_n$  by  $f_n(\alpha) = \beta_{n-1}(\alpha) + e_n(k)\alpha^n$  where  $e_n(k) = (-w_{-n+2} - k\delta_n 2^{\delta_n} c_1^{\delta_n-1})D^{-1}$ , and  $e_n(a_n) = c_n$ . As in the proof of the induction step in Part 1, we can show that for any  $k$ ,

$$U^n_\beta(\alpha, f_n(\alpha)) - L_\beta(\alpha, f_n(\alpha)) = O(\alpha^{-((p+1)/\gamma)-n+1})$$

as  $\alpha \rightarrow \infty$ . Since  $\delta_n < 0$  for  $n \geq 3$  and  $D < 0$ ,  $e_n(k)$  is decreasing in  $k$ . Thus,  $f_n(\alpha) < \beta_n(\alpha)$  if  $k > a_n$ ,  $f_n(\alpha) = \beta_n(\alpha)$  if  $k = a_n$  and  $f_n(\alpha) > \beta_n(\alpha)$  if  $k < a_n$ .

It may be shown (Bartold, (1976), Section 4.8) that for  $\alpha$  sufficiently large, say  $\alpha \geq \alpha_k$ , there is a differentiable function  $g_n = g_n(\alpha)$  satisfying  $U^n_\beta(\alpha, g_n(\alpha)) - L_\beta(\alpha, g_n(\alpha)) = 0$  and  $g_n(\alpha) = f_n(\alpha) + O(\alpha^{n+1})$  as  $\alpha \rightarrow \infty$ . Then for  $\alpha$  sufficiently large,  $g_n(\alpha) < \beta_n(\alpha)$  if  $k > a_n$ ,  $g_n(\alpha) = \beta_n(\alpha) + O(\alpha^{n+1})$  if  $k = a_n$  and  $g_n(\alpha) > \beta_n(\alpha)$  if  $k < a_n$ .

Define the function  $\theta_n = \theta_n(\alpha)$  by  $\theta_n(\alpha) = U^n(\alpha, g_n(\alpha)) - L(\alpha, g_n(\alpha))$  for  $\alpha \geq \alpha_k$ . By an argument similar to the discussion in Part 1, it may be shown that  $\theta_n(\alpha) \sim h_n(k)\alpha^{-n+2} + O(\alpha^{-n+1})$  as  $\alpha \rightarrow \infty$ , where  $h_n(k) = m_{-n+2} + k2^{\delta_n}c_1^{\delta_n}$ . Then  $h_n(a_n) = 0$  and  $h_n(k)$  is increasing in  $k$ . If  $k > a_n$ , then  $h_n(k) > 0$  and  $\theta_n(\alpha)$  is positive and decreasing in  $\alpha$  for  $\alpha$  large, say  $\alpha \geq \alpha_k$ . If  $k < a_n$ , then  $h_n(k) < 0$  and  $\theta_n(\alpha)$  is negative and increasing in  $\alpha$  for  $\alpha$  large, say  $\alpha \geq \alpha_k$ .

Let  $L^n(\alpha, \beta) = L(\alpha, \beta) + \theta_n(\alpha)$  and  $\mathcal{C}^n = \{(\alpha, \beta) : \beta > g_n(\alpha)\}$ . It may be shown (Bartold, (1976), Section 4.8) that  $U^n(\alpha, \beta) < L^n(\alpha, \beta)$  for  $(\alpha, \beta) \in \mathcal{C}^n$  and  $\alpha$  large, say  $\alpha \geq \alpha_k$ . Also, for  $\alpha \geq \alpha_k$ ,  $U^n(\alpha, g_n(\alpha)) = L^n(\alpha, g_n(\alpha))$ ,  $U^n_\beta(\alpha, g_n(\alpha)) = L^n_\beta(\alpha, g_n(\alpha))$  and by the chain rule  $U^n_\alpha(\alpha, g_n(\alpha)) = L^n_\alpha(\alpha, g_n(\alpha))$ . Define a function  $\hat{U}^n$  by

$$\begin{aligned} \hat{U}^n(\alpha, \beta) &= U^n(\alpha, \beta) & \text{if } (\alpha, \beta) \in \mathcal{C}^n \\ &= L^n(\alpha, \beta) & \text{if } (\alpha, \beta) \notin \mathcal{C}^n. \end{aligned}$$

By Theorem 6,  $\hat{U}^n$  is the value function and  $\mathcal{C}^n$  the optimal continuation region for  $L^n$ , for  $\alpha \geq \alpha_k$ . The infinitesimal operator of  $L^n$  is given by  $\mathcal{A}L^n(\alpha, \beta) = \mathcal{A}L(\alpha, \beta) + \theta'_n(\alpha)$ . If  $k > a_n$  and  $\alpha$  is large, say  $\alpha \geq \alpha_k$ , then  $\theta'_n(\alpha) < 0$  and  $\mathcal{A}L_n(\alpha, \beta) < \mathcal{A}L(\alpha, \beta)$ . This implies that  $\mathcal{C} \subseteq \mathcal{C}^n$  for  $\alpha \geq \alpha_k$ , by Lemma 3. If  $k < a_n$  and  $\alpha \geq \alpha_k$ , then  $\theta'_n(\alpha) > 0$  and  $\mathcal{A}L^n(\alpha, \beta) > \mathcal{A}L(\alpha, \beta)$ . This implies that  $\mathcal{C}^n \subseteq \mathcal{C}$  for  $\alpha \geq \alpha_k$ , again by Lemma 3. Then for  $\alpha$  sufficiently large,  $g(\alpha) \geq \beta_{n-1}(\alpha) + e_n(k)\alpha^n$  if  $k > a_n$  and  $g(\alpha) \leq \beta_{n-1}(\alpha) + e_n(k)\alpha^n$  if  $k < a_n$ . This implies that  $g(\alpha) = \beta_{n-1}(\alpha) + e_n(a_n)\alpha^n + O(\alpha^{n+1}) = \beta_n(\alpha) + O(\alpha^{n+1})$  as  $\alpha \rightarrow \infty$ .

Therefore, the  $\beta_n(\alpha)$  form an asymptotic expansion for the boundary function  $g(\alpha)$  of the optimal continuation region  $\mathcal{C}$  as  $\alpha \rightarrow \infty$ .

PART 3. *Proof that the  $V^n$  form an asymptotic expansion for  $V$  inside  $\mathcal{C}$  for large  $\alpha$ .* We have proved that  $g(\alpha) = \beta_n(\alpha) + O(\alpha^{n+1})$  as  $\alpha \rightarrow \infty$ , where  $g$  is the boundary of  $\mathcal{C}$ . Let  $\phi_n(\alpha) = L(\alpha, g(\alpha)) - V^n(\alpha, g(\alpha))$ . As with  $\theta_n(\alpha)$ , it may be shown that  $\phi_n(\alpha) = O(\alpha^{-n+1})$  as  $\alpha \rightarrow \infty$ .

Let  $(\alpha, \beta) \in \mathcal{C}$ , so that  $\beta > g(\alpha)$ . Let  $\sigma = \sigma_{\alpha, \beta}$  be the first exit time from  $\mathcal{C}$ , the optimal stopping rule, starting at the point  $(\alpha, \beta)$ . Because  $\mathcal{A}V^n = 0$ , by Dynkin's formula of Theorem 1,

$$E_{\alpha, \beta}[V^n(\alpha_\sigma, \beta_\sigma)] - V^n(\alpha, \beta) = E_{\alpha, \beta} \left[ \int_0^\sigma \mathcal{A}V^n(\alpha_t, \beta_t) dt \right] = 0.$$

Therefore,  $V(\alpha, \beta) = E_{\alpha, \beta}[L(\alpha_\sigma, \beta_\sigma)] = E_{\alpha, \beta}[L(\alpha_\sigma, g(\alpha_\sigma))] = E_{\alpha, \beta}[V^n(\alpha_\sigma, g(\alpha_\sigma))] + E_{\alpha, \beta}[\phi_n(\alpha_\sigma)] = V^n(\alpha, \beta) + E_{\alpha, \beta}[\phi_n(\alpha_\sigma)]$ . Because  $\phi_n(\alpha) = L(\alpha, g(\alpha)) - V^n(\alpha, g(\alpha)) = O(\alpha^{-n+1})$  as  $\alpha \rightarrow \infty$ , there is an  $\alpha_n$  such that  $|\phi_n(\alpha)| \leq |\phi_n(\alpha_n)|$  for  $\alpha \geq \alpha_n$ . Let  $\alpha_0 > 0$  be given so that we consider only  $\alpha \geq \alpha_0$ . Since  $|\phi_n(\alpha)|$  is bounded for  $\alpha_0 \leq \alpha \leq \alpha_n$ ,  $|\phi_n(\alpha)|$  is bounded for every  $\alpha \geq \alpha_0$ . In a similar fashion, we can show that  $|\alpha_\sigma^{n-2}\phi_n(\alpha_\sigma)|$ ,  $|\alpha^{n-2}\phi_n(\alpha_\sigma)|$  and  $|\alpha^{n-1}\phi_n(\alpha_\sigma)|$  are bounded for  $\alpha \geq \alpha_0$ . Since  $\alpha^{n-2}\phi_n(\alpha) \rightarrow 0$  as  $\alpha \rightarrow \infty$  we have  $\alpha^{n-2}\phi_n(\alpha_\sigma) \rightarrow 0$  as  $\alpha \rightarrow \infty$ . Therefore, by the dominated convergence theorem,  $\alpha^{n-2}E_{\alpha, \beta}[\phi_n(\alpha_\sigma)] \rightarrow 0$  as  $\alpha \rightarrow \infty$ .

$\alpha \rightarrow \infty$  and  $\alpha^{n-1} E_{\alpha, \beta}[\phi_n(\alpha_\sigma)]$  is bounded. Therefore,  $V(\alpha, \beta) = V^n(\alpha, \beta) + O(\alpha^{-n+1})$  for  $(\alpha, \beta) \in \mathcal{C}$  and  $\alpha$  large. This completes the proof of Theorem 7.

As an example, note that a continuous time analogue of squared error loss when estimating the mean  $\Delta$  of the normally distributed random variables  $X_1, X_2, \dots$  can be given by  $L(\alpha, \beta) = \beta(\alpha - 2)^{-2} + \alpha - 2$  for  $\alpha > 2$ , assuming unit cost  $c = 1$  per observation. Using this loss function  $L$  and going through four steps of the procedure described in Theorem 7, we find that

$$V^4(\alpha, \beta) = 2^{1/2} \beta^{1/2} \frac{\Gamma\left(\frac{\alpha}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} + \frac{1}{2} - \frac{2^{1/2}}{8} \beta^{-1/2} \frac{\Gamma\left(\frac{\alpha}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} + \frac{1}{16} \frac{\alpha}{\beta}$$

and

$$\beta_4(\alpha) = \alpha^3 - \frac{13}{2} \alpha^2 + \frac{57}{4} \alpha - \frac{165}{16}.$$

The techniques of Theorem 7 can be applied in the case of slightly more general loss functions  $L$ . For instance, suppose  $L(\alpha, \beta) = \beta^\gamma h(\alpha) + \phi(\alpha)$  satisfies Conditions B, but with  $\phi(\alpha)$  replacing  $\alpha$  as the cost of observation, where  $\phi$  is nonnegative, differentiable, increasing,  $\phi(\alpha) \rightarrow \infty$  as  $\alpha \rightarrow \infty$ , and  $\phi(\alpha)$  has an asymptotic expansion in  $\alpha$ . For example,  $\phi(\alpha)$  could be  $c\alpha$ , where  $c \neq 1$ . Then we replace the cost of observation  $\alpha$  by the asymptotic expansion for  $\phi(\alpha)$  in finding the asymptotic expansions for  $V$  and  $g$ .

We can also apply the same techniques to loss functions of the form  $L(\alpha, \beta) = \beta^{\gamma_1} h_1(\alpha) + \dots + \beta^{\gamma_n} h_n(\alpha) + \phi(\alpha)$  where  $\gamma_i$  and  $h_i$  satisfy the conditions of Section 4.1 for  $i = 1, \dots, n$ ,  $\phi$  is as described above and  $\gamma_1 > \gamma_2 > \dots > \gamma_n > 0$ . This sort of loss function appears when, for instance, we try to estimate at the same time both the mean  $\Delta$  and the variance  $\theta^{-1}$  of the normally distributed random variables  $X_1, X_2, \dots$ .

**4.4. Expansions as  $c$  tends to zero.** Consider the loss function  $L^c(\alpha, \beta) = \beta^\gamma h(\alpha) + c\alpha$  where  $L^c$  satisfies Conditions B. Let  $L(\alpha, \beta) = L^1(\alpha, \beta)$ . Then  $L^c(\alpha, \beta) = cL(\alpha, \beta/c)$  where  $\beta_c = \beta c^{-1/\gamma}$ . Let  $V^c$  be the value function for  $L^c$  and let  $V = V^1$ . Then  $V^c(\alpha, \beta) = cV(\alpha, \beta/c)$ .

Let  $V_n^c(\alpha, \beta) = cV^n(\alpha, \beta/c)$ . It may be shown (Bartold, (1976), Section 4.5) that  $V^c(\alpha, \beta) - V_n^c(\alpha, \beta) = c[V(\alpha, \beta/c) - V^n(\alpha, \beta/c)] = O(c^{(n-1)/(1+\gamma)+1})$ . Also, the optimal boundary function  $g^c(\alpha)$  for  $L^c(\alpha, \beta)$  is approximated by  $\beta_n^c(\alpha) = c^{1/\gamma} \beta_n(\alpha) = c^{1/\gamma} \sum_{i=1}^n c_i \alpha^{\nu_i}$  where  $c_i$  and  $\nu_i, i = 1, \dots, n$ , are as determined in Theorem 7.

An example involving arbitrary cost  $c > 0$  per observation was discussed in 4.2.

**4.5. Some other problems in sequential estimation.** In this section we describe some additional problems to which the techniques of 4.3 apply. For each of these problems, the statement of Theorem 7 is valid, except that the expansions  $V^n$  of 4.3 are replaced by expansions appropriate to the particular problem. The exponents  $\delta_1, \dots, \delta_n$  and  $\nu_1, \dots, \nu_n$  remain as indicated in Theorem 7, but the coefficients  $a_1, \dots, a_n$  and  $c_1, \dots, c_n$  will be different for different problems.

Suppose that the random variables  $X_1, X_2, \dots$  are i.i.d. exponential ( $\theta$ ) with density  $f(x | \theta) = \theta e^{-\theta x} I_{[0, \infty)}(x)$ . If the prior distribution of  $\theta$  is  $\Gamma(\alpha, \beta)$ , then the posterior distribution of  $\theta$  given  $X_1, \dots, X_n$  is  $\Gamma(\alpha_n, \beta_n)$  where  $\alpha_n = \alpha + n, \beta_n = \beta + S_n$  and  $S_n = X_1 + \dots + X_n$  is the sum of  $n$  i.i.d.  $\Gamma(1, \theta)$  random variables.

The analogous problem in continuous time is to consider the process  $(\alpha_t, \beta_t)$  where  $\alpha_t = \alpha + t$  and  $\beta_t = \beta + Z_t$  for  $t \geq 0, Z_t$  has a  $\Gamma(t, \theta)$  distribution conditionally given  $\theta$  and  $\theta$  has a  $\Gamma(\alpha, \beta)$  distribution. Then  $\{(\alpha_t, \beta_t), t \geq 0\}$  forms a strongly measurable standard Markov process with infinitesimal operator given by

$$\mathcal{A}f(\alpha, \beta) = f_\alpha^+(\alpha, \beta) + \int_0^\infty \frac{f(\alpha, \beta + \beta y) - f(\alpha, \beta)}{y(1+y)^\alpha} dy.$$

The equation  $\mathcal{A}f(\alpha, \beta) = 0$  is solved by functions of the form

$$f(\alpha, \beta) = \beta^\nu \frac{\Gamma(\alpha - \nu)}{\Gamma(\alpha)}.$$

The case of gamma  $\Gamma(\Delta, \theta)$  random variables where  $\Delta$  is known is a simple extension of the exponential case.

The methods of Section 4 can also be applied to three additional normal cases, the first involving the univariate normal distribution with known mean, unknown variance. The second case is that of the  $k$ -variate normal distribution,  $N_k(\Delta, 1/\theta I_k)$ , with mean vector  $\Delta$  and covariance matrix  $1/\theta I_k$ , where  $I_k$  is the  $k \times k$  identity matrix,  $\Delta$  and  $\theta$  both unknown. The third case involves the  $k$ -variate normal distribution,  $N_k(\Delta, 1/\theta W)$ , having mean vector  $\Delta$  and covariance matrix  $1/\theta W$ , where  $\Delta$  and the positive definite matrix  $W$  are known and  $\theta > 0$  is unknown.

**Acknowledgments.** I am grateful to Professor Michael Woodroffe for advice and encouragement during the course of this research. I am also grateful to Professor A. N. Sirjaev for pointing out via personal communication the new results in the 1976 edition of his book.

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