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OPTIMALITY OF SOME WEIGHING AND 2^n FRACTIONAL FACTORIAL DESIGNS[1]

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Some asymmetrical weighing and 2^n fractional factorial designs are proved to be optimal over all possible designs with respect to a very general class of criteria. This strengthens and unifies many previously published results in this area. An easy method to prove E-optimality is also presented.

1. Introduction. In this paper, using a simple method, we are able to improve some known results on optimum weighing designs significantly. Because weighing designs and 2^n fractional factorial designs are closely related, the method can also be applied to the latter.

Suppose we want to estimate the weights of n objects by weighing them N times on a chemical balance. Let $x_{ij} = 1, -1$, or 0 depending on whether the jth object is on the left or right scale, or is not present in the ith weighing. Then the $N \times n$ matrix $X = (x_{ij})$ is called the design matrix. For clarity, we denote the design matrix of a weighing design d by X_d. Let $y_1, \cdots, y_N$ be the readings in the N weighings, and $w_1, \cdots, w_n$ be the actual weights of the n objects. Then we have the following model:

$$\mathbf{y} = X_d\mathbf{w} + \mathbf{e}, \tag{1.1}$$

where $\mathbf{y} = (y_1, \cdots, y_N)'$, $\mathbf{w} = (w_1, \cdots, w_n)'$, and $\mathbf{e}$ is an $N \times 1$ random vector such that $E(\mathbf{e}) = \mathbf{0}$, and $\text{Cov}(\mathbf{e}) = \sigma^2 I_N$.

If $X_d'X_d$ is nonsingular, then the covariance matrix of the least squares estimate of $\mathbf{w}$ is $\sigma^2(X_d'X_d)^{-1}$. A weighing design d^* is called Φ-optimal if it minimizes some functional Φ of the information matrix $X_d'X_d$. Φ is called an *optimality criterion*. For convenience, $X_d'X_d$ will be denoted by M_d hereafter. In this paper, we consider the following two types of criteria:

(a) *Criteria of type* 1. Let $\mathfrak{M}_{\mathfrak{D}} = \max_{d \in \mathfrak{D}} \text{tr } M_d$, where $\mathfrak{D}$ is the class of all designs under consideration. Then a criterion of type 1 Φ_f is defined by $\Phi_f(M_d) = \sum_{i=1}^n f(\lambda_{di})$, where $\lambda_{d1}, \cdots, \lambda_{dn}$ are the eigenvalues of M_d, and f is a real-valued function defined on $[0, \mathfrak{M}_{\mathfrak{D}}]$ such that

(1) f is continuous, strictly convex, and strictly decreasing on $[0\ \mathfrak{M}_{\mathfrak{D}}]$. We include here the possibility that $\lim_{x\to 0^+} f(x) = f(0) = +\infty$.

(2) f is continuously differentiable on $(0, \mathfrak{M}_{\mathfrak{D}})$, and f' is strictly concave on $(0, \mathfrak{M}_{\mathfrak{D}})$.

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That is, $f' < 0, f'' > 0$, and $f''' < 0$ on $(0, \mathfrak{M}_{\mathfrak{D}})$.

(b) *Criteria of type* 2. Same as (a) except that the strict concavity of f' is replaced by strict convexity, i.e., $f''' > 0$ on $(0, \mathfrak{M}_{\mathfrak{D}})$.

We also define a *generalized criterion of type i* ($i = 1, 2$) to be the pointwise limit of a sequence of type i criteria.

One can easily see that the well-known D- and A-criteria are of type 1 by taking $f(x) = -\log x$ and $f(x) = x^{-1}$, respectively. Also, the E-criterion is a generalized criterion of type 1.

It is well-known that if there is a design d such that

$$X_d'X_d = NI_n, \tag{1.2}$$

then it is D-, A-, and E-optimal. By Proposition 1′ of Kiefer (1975), it is optimal with respect to a very general class of criteria. But this kind of design does not always exist. For example, when $n = N$, it (a *Hadamard matrix*) exists only if $n = 2$ or n is a multiple of 4. Therefore, it is important to investigate the case where there is no such design. (The readers may consult Hedayat and Wallis (1978) for a review on the subject of Hadamard matrices.)

Raghavarao (1959, 1960) and Bhaskararao (1966) had studied this problem, but they only considered designs d such that $X_d'X_d$ is of the form $aI_n + bJ_n$, where I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ matrix consisting entirely of 1's. This is a very stringent restriction. Let $\mathfrak{D}_{N,n}^S$ be the set of all such designs, and $\mathfrak{D}_{N,n}$ be the set of all possible designs. Then there is no guarantee that the best design in $\mathfrak{D}_{N,n}^S$ is really optimal over $\mathfrak{D}_{N,n}$. Actually, counterexamples exist (see Section 5). Furthermore, they only considered A-, D-, and E-criteria.

The first purpose of the present work is to develop a theory which embraces the results of Raghavarao and Bhaskararao as immediate consequences and can be used to prove the optimality of some weighing designs *over all possible designs with respect to a very general class of criteria.*

The second purpose of this paper is to apply the results on weighing designs to the setting of 2^n fractional factorial designs of odd resolution. Suppose we have n factors each with 2 levels 0 and 1. A fractional factorial design of resolution $2t + 1$ is a design which allows the estimation of all the main effects and interactions up to order $t - 1$ when all the interactions with order higher than $t - 1$ are assumed zero. In this set-up, the unknown parameters are ϕ^0 (grand mean), $\{\phi_i^1\}_{1 \leq i \leq n}$ (main effects), $\{\phi_{ij}^2\}_{1 \leq i < j \leq n}$ (first order or 2-factor interactions), $\cdots$, and $\{\phi_{i_1 i_2 \cdots i_t}^t\}_{1 \leq i_1 < i_2 < \cdots < i_t \leq n}$ (t-factor interactions). There are many ways of parametrization. We will use the one described in Section 7.2 of John (1971). For any observation $y_{j_1 j_2 \cdots j_n}$ on the j_1th level of factor 1, j_2th level of factor 2, $\cdots$, etc., $j_l = 0$ or 1, $E(y_{j_1 j_2 \cdots j_n})$ can be written as

$$E(y_{j_1 j_2 \cdots j_n}) = \phi^0 + \Sigma_{1 \leq i \leq n} x_i \phi_i^1 + \Sigma_{1 \leq i < j \leq n} x_i x_j \phi_{ij}^2 + \cdots + \Sigma_{1 \leq i_1 < i_2 < \cdots < i_t \leq n} x_{i_1} x_{i_2} \cdots x_{i_t} \phi_{i_1 i_2 \cdots i_t}^t, \tag{1.3}$$

where

$$x_i = 1, \quad \text{if } j_i = 1 \tag{1.4}$$
$$= -1, \quad \text{if } j_i = 0.$$

In this setting, each design matrix consists of $+1$ and -1 entries. We will still denote the design matrix of a design d by X_d and write $X_d'X_d$ as M_d. Then optimality criteria of both types are defined similarly. The problem now is to find a best design for fixed number of observations N. In this case, the equality in (1.2) is achieved by an orthogonal array of size N, n constraints, 2 levels, and strength $2t$, abbreviated as $0A(N, n, 2, 2t)$. For a definition of orthogonal arrays, see Raghavarao (1971, page 10).

In a series of papers (see, e.g., Srivastava and Chopra (1971), Chopra (1975), Chopra and Srivastava (1973a, 1973b)), Srivastava and Chopra obtained best 2^4, 2^5, 2^6, 2^7, and 2^8 resolution V designs under the A-criterion among *balanced designs* for some practical ranges of N. A balanced 2^n design of resolution $2t+1$ with N observations is obtained by observing the treatment combinations specified by a balanced array with size N, n constraints, 2 levels and strength $2t$, denoted by $\mathrm{BA}(N, n, 2, 2t)$. Balanced arrays were first studied by Chakravarti (1956) under the name of *partially balanced arrays.* The readers are referred to Srivastava and Chopra (1971) for a definition.

Shirakura (1976) worked on the same problem for 2^6, 2^7, and 2^8 resolution VII designs. Again, there is no guarantee that the best balanced design is really optimal. From our theory, it will be shown later that some special types of balanced designs are indeed optimal over all possible designs.

For convenience, we will denote the set of all $N \times n$ matrices of $+1$, -1, or 0 by $\mathcal{F}_{N,n}$, and denote that of all $N \times n$ matrices of $+1$ or -1 by $\mathcal{F}'_{N,n}$.

In Section 2, some general results on the optimality of certain asymmetrical designs are obtained. These results are applied to establish the optimality of some weighing and 2^n fractional factorial designs in Section 3 and Section 4. We will also indicate how the results of earlier papers follow from our theory.

2. General results on the optimality of certain asymmetrical designs. Theorem 2.2 of Cheng (1978) provided a tool for proving the optimality of some asymmetrical designs in the settings where the information matrices have zero row and column sums. This result is rephrased in the following form which is suitable for the present settings of weighing and 2^n fractional factorial designs.

THEOREM 2.1. *Let $\mathcal{C} = \{M_d\}_{d \in \mathfrak{D}}$ be a class of $n \times n$ symmetric nonnegative definite matrices.*

(a) *Suppose $M_{d^*} \in \mathcal{C}$ is either a multiple of I_n or has two distinct nonzero eigenvalues $\lambda > \lambda'$ such that the multiplicity of λ' is $n-1$, and*

$$M_{d^*} \textit{ maximizes } \operatorname{tr} M_d \textit{ over } \mathcal{C}. \tag{2.1}$$

$$\operatorname{tr}(M_{d^*}^2) < (\operatorname{tr} M_{d^*})^2/(n-1). \tag{2.2}$$

(2.3) M_{d^*} *maximizes* $\operatorname{tr} M_d - [n/(n-1)]^{\frac{1}{2}}[\operatorname{tr}(M_d^2) - (\operatorname{tr} M_d)^2/n]^{\frac{1}{2}}$ *over* $\mathcal{C}$.

Then M_{d^*} *is optimal over* $\mathcal{C}$ *with respect to any generalized criterion of type* 1.

(b) *Suppose* $M_{d^*} \in \mathcal{C}$ *is either a multiple of* I_n *or has two distinct nonzero eigenvalues* $\lambda > \lambda'$ *such that* λ *has multiplicity* $n-1$ *and*

(2.4) $$M_{d^*} \text{ maximizes } \operatorname{tr} M_d \text{ over } \mathcal{C}.$$

(2.5) M_{d^*} *maximizes* $\operatorname{tr} M_d - \{n(n-1)[\operatorname{tr}(M_d^2) - (\operatorname{tr} M_d)^2/n]\}^{\frac{1}{2}}$ *over* $\mathcal{C}$.

Then M_{d^*} *is optimal over* $\mathcal{C}$ *with respect to any generalized criterion of type* 2.

In settings where $\operatorname{tr} M_d$ *is a constant, for all* $d \in \mathfrak{D}$, (2.1) *and* (2.3) (*or* (2.4) *and* (2.5)) *can be replaced by*

(2.6) $$M_{d^*} \text{ minimizes } \operatorname{tr}(M_d^2) \text{ over } \mathcal{C},$$

and the condition "$f' < 0$*" in the definitions of type* 1 *and type* 2 *criteria can be dropped.*

The following theorems are the main results of this paper.

THEOREM 2.2. *Let* $\mathcal{C}_{N,n} = \{X'X\}_{X \in \mathfrak{F}_{N,n}}$. *If* $N \geqslant n$ *and* $(N-1)I_n + J_n \in \mathcal{C}_{N,n}$, *then this matrix is optimal over* $\mathcal{C}_{N,n}$ *with respect to any generalized type* 1 *criterion.*

PROOF. Suppose $X^{*\prime}X^* = (N-1)I_n + J_n$ for some $X^* \in \mathfrak{F}_{N,n}$. Then $X^{*\prime}X^*$ has two distinct nonzero eigenvalues $N-1$ and $N+n-1$. Since $N+n-1 > N-1$, and $N+n-1$ has multiplicity 1, it suffices to verify (2.1), (2.2) and (2.3).

(2.1) is trivial. (2.2) follows from the assumption $N \geqslant n$ and noting that $\operatorname{tr}[(X^{*\prime}X^*)^2] = nN^2 + n(n-1)$.

(2.3) is the most difficult condition to verify. We have to show that for any $X \in \mathfrak{F}_{N,n}$,

(2.7) $$\operatorname{tr} X^{*\prime}X^* - \operatorname{tr} X'X \geqslant [n/(n-1)]^{\frac{1}{2}}(P^* - P),$$

where $P^* = [\operatorname{tr}(X^{*\prime}X^*)^2 - (\operatorname{tr} X^{*\prime}X^*)^2/\mathrm{n}]^{\frac{1}{2}}$, and P is defined similarly.

Now we have $[n/(n-1)]^{\frac{1}{2}}P^* = n$. Therefore $\operatorname{tr} X^{*\prime}X^* - \operatorname{tr} X'X \geqslant n \Rightarrow$ (2.7). Thus we may assume

(2.8) $$\operatorname{tr} X'X > \operatorname{tr} X^{*\prime}X^* - n.$$

The existence of X^* implies that N is odd. Therefore, if $\operatorname{tr} X'X = \operatorname{tr} X^{*\prime}X^* = nN$, then all the entries of X are $+1$ or -1, and hence the absolute value of any off-diagonal element of $X'X$ is at least one. This implies $\operatorname{tr}(X'X)^2 \geqslant \operatorname{tr}(X^{*\prime}X^*)^2$, and hence (2.7) holds.

So we may assume $\operatorname{tr} X'X = \operatorname{tr} X^{*\prime}X^* - \alpha$ for some positive integer $\alpha < n$. Then at least $n-\alpha$ of the diagonal elements of $X'X$ are N. The entries of the corresponding rows of X' are $+1$ or -1. Hence at least $(n-\alpha)(n-\alpha-1)$ of the off-diagonal elements of $X'X$ have absolute value $\geqslant 1$. On the other hand, we have

$$\Sigma_{i=1}^n[(X'X)_{ii}]^2 \geqslant (n-\alpha)N^2 + \alpha(N-1)^2.$$

Therefore

(2.9) $P^2 \geqslant (n-\alpha)N^2 + \alpha(N-1)^2 + (n-\alpha)(n-\alpha-1) - (nN-\alpha)^2/n.$

Now, (2.7) is equivalent to

$$\alpha - n \geqslant -[n/(n-1)]^{\frac{1}{2}}P, \tag{2.10}$$

i.e.,

$$(\alpha - n)^2 \leqslant nP^2/(n-1). \tag{2.11}$$

By (2.9), a sufficient condition for (2.11) is

$$\begin{aligned}(\alpha - n)^2 \leqslant\ & n(n-\alpha)N^2/(n-1) + n\alpha(N-1)^2/(n-1)\\ & + n(n-\alpha)(n-\alpha-1)/(n-1) - (nN-\alpha)^2/(n-1),\end{aligned} \tag{2.12}$$

or equivalently,

(2.13)

$$(n-\alpha)[(n-1)(n-\alpha) - nN^2 - n(n-\alpha-1)] \leqslant n\alpha(N-1)^2 - (nN-\alpha)^2.$$

Both sides of (2.13) are equal to $(n-\alpha)(\alpha - nN^2)$. This proves (2.11). □

It is difficult to prove a type 2 analogue of Theorem 2.2. A reasonable candidate for an optimum M_d is $(N+1)I_n - J_n$. But condition (2.5) is too stringent to be verified. Let $M^* = (N+1)I_n - J_n$. Then $\operatorname{tr} M^* - \{n(n-1)[\operatorname{tr}(M^*)^2 - (\operatorname{tr} M^*)^2/n]\}^{\frac{1}{2}} = nN - n(n-1)$, with a deficit of $n(n-1)$ from the ideal maximum.

However, we can prove the following weaker result.

THEOREM 2.3. *Let $\mathcal{C}'_{N,n} = \{X'X\}_{X \in \mathcal{F}'_{N,n}}$. If $N \geqslant n$ and $(N+1)I_n - J_n \in \mathcal{C}'_{N,n}$, then this matrix is optimal over $\mathcal{C}'_{N,n}$ with respect to any generalized type 2 criterion.*

PROOF. For any $X \in \mathcal{F}'_{N,n}$, we have $\operatorname{tr} X'X = nN$. Therefore we only have to verify (2.6). This is a trivial consequence of the fact that N must be odd for the existence of an $X^* \in \mathcal{F}'_{N,n}$ such that $X^{*\prime}X^* = (N+1)I_n - J_n$. □

This theorem is not too weak for the application to 2^n fractional factorial designs, since the design matrix of a 2^n fractional factorial design with N runs belongs to $\mathcal{F}'_{N,n}$.

Using a delicate computation, Ehlich (1964) proved that if there is a matrix $X^* \in \mathcal{F}'_{n,n}$ such that $X^{*\prime}X^* = (n-1)I_n + J_n$, then X^* has maximal determinant over $\mathcal{F}'_{n,n}$. This is a very special case of our Theorem 2.2. Actually, in order to get Ehlich's result, we need not even go through the somewhat complicated computations in the proof of Theorem 2.2. For any $X \in \mathcal{F}'_{n,n}$, we have $\operatorname{tr} X'X = n^2$. Therefore, as in the proof of Theorem 2.3, X^* minimizes $\operatorname{tr}(X'X)^2$ over $\mathcal{F}'_{n,n}$. Thus, this result of Ehlich follows easily from our theory. Furthermore, his method does not work for other criteria.

Usually, among the various optimality criteria, the E-criterion is the easiest one to verify. We will give an alternative proof for the E-optimality of $(N-1)I_n + J_n$ over $\mathcal{C}_{N,n}$. Similar argument can also be applied to other kinds of matrices.

Let $X^* \in \mathcal{F}_{N,n}$ be such that $X^{*\prime}X^* = (N-1)I_n + J_n$, and $\mathcal{Q} = \{X'X - (N-1)I_n : X \in \mathcal{F}_{N,n}\}$. In order to prove the E-optimality of $X^{*\prime}X^*$ over $\mathcal{C}_{N,n}$, it suffices to prove that

(2.14) $\quad X^{*\prime}X^* - (N-1)I_n = J_n$ is non-negative definite,

and

(2.15) $\quad$ each matrix in $\mathcal{Q}$ is not positive definite.

(2.14) is trivial. To prove (2.15), we note that for any matrix M in $\mathcal{Q}$, either some diagonal element is $\leqslant 0$ or all the diagonal elements are equal to 1 and the absolute value of any off-diagonal element is $\geqslant 1$. This clearly implies that M is not positive definite. Thus, the E-optimality of $X^{*\prime}X^*$ over $\mathcal{C}_{N,n}$ is proved. This kind of technique is due to Takeuchi (1961).

Using similar arguments, we can establish the following results:

THEOREM 2.4. *If $NI_n \notin \mathcal{C}'_{N,n}$, and there is an $M^* \in \mathcal{C}'_{N,n}$ such that*

$$M^* = \operatorname{diag}\big((N-2)I_{n_1} + 2J_{n_1}, (N-2)I_{n_2} + 2J_{n_2}, \cdots, (N-2)I_{n_k} + 2J_{n_k}\big),$$

where $n = n_1 + n_2 + \cdots + n_k$, then M^* is E-*optimal over $\mathcal{C}'_{N,n}$.*

THEOREM 2.5. *If $(N+1)I_n - J_n \in \mathcal{C}'_{N,n}$, then this matrix minimizes the maximum eigenvalue over the matrices in $\mathcal{C}'_{N,n}$.*

PROOF. Multiply all the matrices in $\mathcal{C}'_{N,n}$ by -1, and then the result follows from the above alternative proof for the E-optimality of $(N-1)I_n + J_n$ over $\mathcal{C}_{N,n}$. □

THEOREM 2.6. *If $NI_n \notin \mathcal{C}'_{N,n}$, and there is an $M^* \in \mathcal{C}'_{N,n}$ such that*

$$M^* = \operatorname{diag}\big((N+2)I_{n_1} - 2J_{n_1}, (N+2)I_{n_2} - 2J_{n_2}, \cdots, (N+2)I_{n_k} - 2J_{n_k}\big),$$

where $n = n_1 + n_2 + \cdots + n_k$. Then M^* minimizes the maximum eigenvalue over $\mathcal{C}'_{N,n}$.

Theorem 2.5 and Theorem 2.6 are not too interesting from the optimum design theoretic point of view.

THEOREM 2.7. *If $N \equiv 3 \pmod 4$, and there is an $M^* \in \mathcal{C}'_{N,n}$ such that $M^* = (N-3)I_n + 3J_n$, then M^* is E-optimal over $\mathcal{C}'_{N,n}$.*

PROOF. Let $M = (m_{ij})_{n\times n} = X'X$ for some $X \in \mathcal{F}'_{N,n}$. Multiplying some columns of X by -1 does not change the eigenvalues of $X'X$. So by Lemma 3.1 of Ehlich (1964), we may assume that $m_{ij} \equiv 3 \pmod 4$, for all i, j.

If $m_{ij} = -1$ for all i, j, then certainly M is E-worse than M^*. If some $m_{ij} \neq -1$, then $|m_{ij}| \geqslant 3$. The E-optimality of M^* follows from the same argument used in the proofs of the previous theorems. □

3. Weighing designs. Modifying the notation in Bhaskararao (1966), we denote a design d in $\mathfrak{D}_{N,n}$ such that $X_d'X_d = (N - s - \lambda)I_n + \lambda J_n$ by $[N, s, \lambda]_n$.

Theorems 2.2, 2.3, 2.4, 2.5, 2.6, and 2.7 can be translated into the context of weighing designs. For example, Theorem 2.2 says that if there is a $[N, 0, 1]_n$ design in $\mathfrak{D}_{N,n}$, then it is optimal over $\mathfrak{D}_{N,n}$ with respect to any generalized type 1 criterion. When specialized to the D-criterion, this actually solves a conjecture of Mood (1946). (When $N = n$, the same design was shown by Raghavarao (1959, 1960) to be D-, A-, and E-optimal over $\mathfrak{D}^S_{n,n}$.) The other theorems are weaker in the sense that we only claim the optimality over those designs in which all the objects are present in each weighing.

It can easily be seen that if $n > 2$ and there is a $[N, 0, 1]_n$ design in $\mathfrak{D}_{N,n}$, then $N \equiv 1 \pmod 4$. Similarly, when $n > 2$, a necessary condition for the existence of a $[N, 0, -1]_n$ design is that $N \equiv 3 \pmod 4$. It was shown in Raghavarao (1959) that a $[n, 0, 1]_n$ exists only if $n = (1 + \alpha^2)/2$ for some odd integer α. This necessary condition is very stringent. Therefore, there are not too many $[n, 0, 1]_n$ designs. Some examples are given in Raghavarao's paper.

However, there are abundant $[n, 0, -1]_n$ designs. If the conjecture that a $4t \times 4t$ Hadamard matrix exists for any positive integer t is true, then "$n \equiv 3 \pmod 4$" is also a sufficient condition for the existence of a $[n, 0, -1]_n$ design, Actually, we have

THEOREM 3.1. *If there exists a $4t \times 4t$ Hadamard matrix, then there is a $[4t - 1, 0, -1]_{4t-1}$ design.*

PROOF. Pick any $4t \times 4t$ Hadamard matrix in standard form, i.e., all the entries in the first row and first column are 1. Delete the first row and first column. Then the remaining submatrix gives a $[4t - 1, 0, -1]_{4t-1}$ design. □

In general, if there is a $[N, 0, 0]_n$ design, or equivalently, an orthogonal array of size N, $n - 1$ constraints, 2 levels, and strength 2, then there exist $[N + 1, 0, 1]_n$ and $[N - 1, 0, -1]_n$ designs.

When $N = n$ and $n_1 = n_2 = n/2$, the matrix in Theorem 2.4 was also shown to be D-optimal over $\mathcal{C}'_{n,n}$ by Ehlich (1964). The problem of constructing such designs was considered by Ehlich (1964) and Yang (1966, 1968).

The following is a design whose information matrix has the form described in Theorem 2.6 with $n_1 = n_2 = 3$.

$$X_d = \begin{bmatrix} 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

We now indicate how the results of Raghavarao and Bhaskararao can be derived from our Theorem 2.1.

For comparison among the designs $[N, s, \lambda]_n$ in $\mathfrak{D}^S_{N,n}$ with $\lambda \geq 0$ under type 1 criteria, we can firstly compare the two values $\operatorname{tr} M_d$ and $\operatorname{tr} M_d - [n/(n-1)]^{\frac{1}{2}}[\operatorname{tr}(M_d^2) - (\operatorname{tr} M_d)^2/n]^{\frac{1}{2}}$, since for any such design d, M_d is either a multiple of the identity matrix or has two distinct nonzero eigenvalues such that the smaller one has multiplicity $n-1$. If d is a $[N, s, \lambda]_n$ with $\lambda \geq 0$, then $\operatorname{tr} M_d = n(N-s)$, and $\operatorname{tr} M_d - [n/(n-1)]^{\frac{1}{2}}. [\operatorname{tr}(M_d^2) - (\operatorname{tr} M_d)^2/n]^{\frac{1}{2}} = n(N-s) - \lambda n = nN - n(s + \lambda)$. Therefore by Theorem 2.1 we have to make both of s and $s + \lambda$ as small as possible. In other words, we have

THEOREM 3.2. *Assume $\lambda \geq 0$ and $\lambda' \geq 0$. If $s \geq s'$ and $s + \lambda \geq s' + \lambda'$, and one of them is a strict inequality, then $[N, s', \lambda']_n$ is strictly better than $[N, s, \lambda]_n$ with respect to any type 1 criterion.*

Similarly, we have

THEOREM 3.3. *Assume $\lambda < 0$ and $\lambda' \geq 0$. If $s \geq s'$ and $s - \lambda \geq s' + \lambda'$, then $[N, s', \lambda']_n$ is strictly better than $[N, s, \lambda]_n$ with respect to any type 1 criterion.*

Theorems 3.2 and 3.3 can be used to eliminate most of the designs in $\mathfrak{D}^S_{N,n}$ except a few competitors. Then we can compare these remaining designs according to various optimality criteria of interest directly. For example, if $n \equiv 2 \pmod 4$ and $n \neq 2$, then $[n, 0, 2]_n$ and $[n, 1, 0]_n$ are better than the other designs in $\mathfrak{D}^S_{n,n}$ under any type 1 criterion if they exist. Comparing these two designs, we conclude that $[n, 0, 2]_n$ is D-optimal and $[n, 1, 0]_n$ is A- and E-optimal over $\mathfrak{D}^S_{n,n}$. The other results of Raghavarao and Bhaskararao can also be obtained in this way. Actually, this method enables us to order the designs in $\mathfrak{D}^S_{N,n}$ with respect to any type 1 and type 2 criteria.

4. 2^n fractional factorial designs of odd resolution. Although from the viewpoint of weighing designs, Theorems 2.3, 2.4, 2.5, 2.6 and 2.7 are incomplete in the sense that only the optimality over $\mathcal{C}'_{N,n}$ instead of $\mathcal{C}_{N,n}$ is proved, they are perfect for the problem of 2^n fractional factorial designs since the design matrix of a 2^n design consists of $+1$ and -1 entries. Also, the condition "$f' < 0$" in the definitions of type 1 and type 2 criteria can be dropped in the present section.

It can easily be seen that if there is a balanced array BA$(N, n, 2, 2t)$ such that $\mu_0 = \mu_1 = \cdots = \mu_{2t-1} = \mu$, and $\mu_{2t} = \mu + a$, then it defines a balanced 2^n fractional factorial design d^* of resolution $2t+1$ such that $M_{d^*} = X_{d^*}'X_{d^*} = (N-a)I_k + aJ_k$, where $k = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{t}$. Therefore, by Theorems 2.2 and 2.3, we have

THEOREM 4.1. *If there is a* BA$(N, n, 2, 2t)$ *with index set $(\mu_0, \mu_1, \cdots, \mu_{2t})$ such that $\mu_0 = \mu_1 = \cdots = \mu_{2t-1} = \mu$, and $\mu_{2t} = \mu + 1$ (or $\mu - 1$), then it defines a* *balanced design which is optimal with respect to any generalized type* 1 (*or type* 2) *criterion over all the* 2^n *designs of resolution* $2t + 1$ *with* N *runs.*

Thus, some of the designs obtained by Srivastava, Chopra, and Shirakura are really optimal over all possible designs with respect to a very general class of criteria, not just A-optimal over the balanced designs. It is worthwhile pointing out that all the balanced designs with index sets of the form $(\mu, \mu, \cdots, \mu + 1)$ do not appear in the lists of designs in the papers by Srivastava, Chopra, and Shirakura. Instead, they list balanced designs with index sets of the form $(\mu + 1, \mu, \cdots, \mu)$ for the same values of N. This kind of design is obtained by interchanging the two symbols 0 and 1 in the design with index set $(\mu, \mu, \cdots, \mu, \mu + 1)$. The information matrices of these two designs have the same eigenvalues, although that of the one with index set $(\mu + 1, \mu, \cdots, \mu)$ is no longer of the form $aI + bJ$. This shows that the optimum information matrices are not unique.

A BA$(N, n, 2, 2t)$ with index set $(\mu_0, \mu_1, \cdots, \mu_{2t})$ such that $\mu_0 = \mu_1 = \cdots = \mu_{2t-1} = \mu$ and $\mu_{2t} = \mu + 1$ (or $\mu - 1$) can be obtained by adding (or deleting) one column consisting entirely of 1's to (or from) an orthogonal array $0A(N - 1, n, 2, 2t)$ (or $0A(N + 1, n, 2, 2t)$). Therefore, the construction of the designs in Theorem 4.1 is essentially the same as that of orthogonal arrays. There has been an enormous literature on the construction of orthogonal arrays.

It is worthwhile pointing out that the conclusion in Theorem 4.1 is still true if the design is obtained by adding (or deleting) *any* observation to (or from) an orthogonal array. Optimal designs obtained in this way are not necessarily balanced according to the definition of Srivastava and Chopra. The recipe of adding one observation to an orthogonal design was suggested by Mood (1946) and was supported by the computer search of Mitchell (1974). The present paper gives a proof of the optimality of this procedure.

Similarly, we have

THEOREM 4.2. *A design obtained by adding any two* (*or three*) *observations to an* $0A(N, n, 2, 2t)$ *is E-optimal over all the* 2^n *designs of resolution* $2t + 1$ *with* $N + 2$ (*or* $N + 3$) *observations.*

So there are quite a few E-optimal designs. For example, let d and $\bar{d}$ be BA$(N, n, 2, 2t)$ with index sets $(\mu, \mu, \cdots, \mu, \mu + 2)$ and $(\mu + 1, \mu, \cdots, \mu, \mu + 1)$, respectively. Then both of them are E-optimal. But the eigenvalues of $X_d'X_d$ majorize those of $X_{\bar{d}}'X_{\bar{d}}$. Thus, $\bar{d}$ is Φ_f-better than d for any strictly convex function f. A similar conclusion holds for BA$(N, n, 2, 2t)$ with index sets $(\mu, \mu, \cdots, \mu, \mu + 3)$ and $(\mu + 1, \mu, \cdots, \mu, \mu + 2)$.

5. Remarks.

(1) We will give an example to show that, in finding an optimal design over $\mathfrak{D}_{N,n}$, it is not enough to search among the designs in $\mathfrak{D}_{N,n}^S$ only.

Let

$$X_{d_1} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix},$$

$$X_{d_2} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

Then d_2 is a $[6, 0, 2]_6$, and $X'_{d_1}X_{d_1} = \mathrm{diag}(4I_3 + 2J_3, 4I_3 + 2J_3)$.

It is obvious that d_1 is D-better than d_2. On the other hand, d_2 is D-optimal over $\mathfrak{D}^S_{6,6}$. This shows that an optimal design over $\mathfrak{D}^S_{N,n}$ need not be optimal over $\mathfrak{D}_{N,n}$. Actually, since the eigenvalues of $X'_{d_2}X_{d_2}$ majorize those of $X'_{d_1}X_{d_1}$, d_1 is Φ_f-better than d_2 for any strictly convex function f.

(2) The observation in (1) also shows that the best balanced 2^n design of resolution III might not be optimal. The author believes, although he has not yet found, that counterexamples also exist for resolution V designs.

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