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VARIANCE AND DISTRIBUTION OF THE GRAYBILL-DEAL ESTIMATOR OF THE COMMON MEAN OF TWO NORMAL POPULATIONS

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Let x and y be two independent normal variables with mean μ and variances σ_1^2 and σ_2^2 respectively. Also let S_1^2 and S_2^2 be two independent estimators of σ_1^2 and σ_2^2 such that $mS_1^2\sigma_1^{-2}$ and $nS_2^2\sigma_2^{-2}$ are chi-squares with m and n degrees of freedom respectively. The Graybill-Deal estimator of μ is $\hat{\mu} = (S_1^{-2}x + S_2^{-2}y)/(S_1^{-2} + S_2^{-2})$. In this paper an expression for the variance of $\hat{\mu}$ is given. Also bounds for the distribution of $\hat{\mu}$ are studied.

1. Introduction. In this paper we consider some properties of an estimator of a parameter μ obtained by combining two independent estimators of μ. Let x and y be two independent normally distributed unbiased estimators of μ with variances σ_1^2 and σ_2^2 respectively. Also let S_1^2 and S_2^2 be two independent estimators of σ_1^2 and σ_2^2 such that $mS_1^2\sigma_1^{-2}$ and $nS_2^2\sigma_2^{-2}$ are chi-squares with m and n degrees of freedom. The Graybill-Deal estimator of μ is $\hat{\mu} = (S_1^{-2}x + S_2^{-2}y)/(S_1^{-2} + S_2^{-2})$. Note that $\hat{\mu}$ is unbiased. Expressions and approximations for the variance of this type of estimator in more general cases are given by Meier (1953), Zacks (1966), Williams (1967), and Bement and Williams (1969). The efficiency of $\hat{\mu}$ has been studied by Graybill and Deal (1959). For a recent bibliography in this area see Norwood and Hinkelmann (1977). An expression for the variance of $\hat{\mu}$ is given by Cohen and Sackrowitz (1974) page 1277. It can also be calculated using a result in Khatri and Shah (1974) page 653 (4.1). Brodsky (1977) indicates that sharp bounds for the variance of $\hat{\mu}$ are available. In this paper we give an expression similar to that of Cohen-Sackrowitz and Khatri-Shah. Our method can also be used to study the variance of the estimator $T_a(1)$ proposed by Brown and Cohen (1974) page 969.

Also bounds for the distribution of $\hat{\mu}$ are given. This is a new result.

2. Variance of $\hat{\mu}$. Let $\alpha = n\sigma_1^2/m\sigma_2^2$. In this and the next section we assume that $0 < \alpha \leqslant 1$. Note that there is no loss of generality in making this assumption since this can always be achieved by naming the estimators x and y appropriately.

THEOREM 2.1.

(2.1)

$$V(\hat{\mu}) = \sigma_1^2\Sigma_{i=0}^{\infty}\frac{(i+1)(1-\alpha)^i}{B(m/2, n/2)}\left[B\left(\frac{m}{2}+i, \frac{n}{2}+2\right) + \frac{n}{m}\alpha B\left(\frac{m}{2}+i+2, \frac{n}{2}\right)\right].$$

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PROOF. Since $mS_1^2\sigma_1^{-2}$ and $nS_2^2\sigma_2^{-2}$ are independent chi-squares, $S_1^2/S_2^2 = \alpha w/(1-w)$ where w is a $B(m/2, n/2)$ variable. Hence

$$\hat{\mu} = \frac{(1-w)x + \alpha w y}{1-(1-\alpha)w}$$

Using the facts that x, y, and w are independent and $E(\hat{\mu}/w) = \mu$ we get

$$V(\hat{\mu}) = EV(\hat{\mu}/w) + VE(\hat{\mu}/w) = E\left[(1-w)^2\sigma_1^2 + \alpha^2\sigma_2^2 w^2\right]\left[1-(1-\alpha)w\right]^{-2}$$
$$= \sigma_1^2 E\left[(1-w)^2 + nm^{-1}\alpha w^2\right]\Sigma_{i=0}^{\infty}(i+1)(1-\alpha)^i w^i.$$

Since $0 \leq 1-\alpha < 1$ and $0 \leq w \leq 1$, term-by-term expectations can be taken in the above. Using the fact that w is a $B(m/2, n/2)$ variable we get (2.1). This proves the theorem.

If σ_1^2 and σ_2^2 are known then the minimum variance estimator of μ is $(\sigma_1^{-2}x + \sigma_2^{-2}y)/(\sigma_1^{-2} + \sigma_2^{-2})$ with variance $V = \sigma_1^2(1 + \alpha m/n)^{-1}$. The efficiency of $\hat{\mu}$ is $V/V(\hat{\mu})$. An explicit expression for the case $m = n = 2$ is given by Zacks (1966) page 473. Taking $\alpha = \frac{1}{2}$, the efficiency $E = .733$. The first seven terms of (2.1) gives $E = .738$ approximately. An upper bound for the error of approximation can be found using the following theorem.

THEOREM 2.2. *The remainder after r terms in* (2.1) *is bounded above by*

$$\sigma_1^2\left[B(m/2, n/2)\right]^{-1}\left[B\left(\frac{m}{2}+r, \frac{n}{2}+2\right) + \frac{n}{m}\alpha B\left(\frac{m}{2}+r+2, \frac{n}{2}\right)\right]$$
$$\times(1-\alpha)^r(1+r\alpha)\alpha^{-2}.$$

PROOF. Denote the remainder by R_r. Since $B(x, y)$ is a decreasing function of x and y,

$$R_r \leq \sigma_1^2\left[B(m/2, n/2)\right]^{-1}\left[B\left(\frac{m}{2}+r, \frac{n}{2}+2\right) + \frac{n}{m}\alpha B\left(\frac{m}{2}+r+2, \frac{n}{2}\right)\right]$$
$$\times\Sigma_{i=r}^{\infty}(i+1)(1-\alpha)^i.$$

The result of the theorem follows from the fact that the sum in the above expression is equal to $(1-\alpha)^r(1+r\alpha)\alpha^{-2}$.

Next we consider the asymptotic nature of $V(\hat{\mu})$.

THEOREM 2.3. *Let $m = n$. Then*

$$0 < \frac{V(\mu)}{\sigma_1^2} - \frac{1}{1+\alpha} \leq \frac{1+\alpha}{4\alpha^2(m+1)} + \frac{(1+3\alpha)\left[1+\alpha-a(1-\alpha)\right](1-\alpha)}{8\alpha^2\left[1-a(1-\alpha)\right]^2(m+3)}$$

where $a = 2^{-1+(e\ln 2)^{-1}}$.

PROOF. The first inequality is true since $\sigma_1^2(1+\alpha)^{-1}$ is the minimum variance. To prove the second inequality note that

$$\Sigma_{i=0}^{\infty}(i+1)(1-\alpha)^i(2^{-i-2} + \alpha 2^{-i-2}) = (1+\alpha)^{-1}$$

so that from (2.1) we get

$$\frac{V(\hat{\mu})}{\sigma_1^2} - \frac{1}{1+\alpha} = \Sigma(i+1)(1-\alpha)^i[P_i(m) + \alpha Q_i(m)] \tag{2.2}$$

where

$$P_i(m) = [B(m/2, m/2)]^{-1} B\left(\frac{m}{2} + i, \frac{m}{2} + 2\right) - 2^{-i-2}$$

and

$$Q_i(m) = [B(m/2, m/2)]^{-1} B\left(\frac{m}{2} + i + 2, \frac{m}{2}\right) - 2^{-i-2}$$

Using a property of the beta integral we get

$$\begin{aligned} P_{i+1}(m) &= \frac{m/2 + i}{m + i + 2}[P_i(m) + 2^{-i-2}] - 2^{-i-3} \\ &= \frac{m + 2i}{2(m + i + 2)} P_i(m) + \frac{i - 2}{m + i + 2} 2^{-i-3}. \end{aligned}$$

Taking absolute values

$$|P_{i+1}(m)| \leqslant |P_i(m) + \frac{|i-2|}{m+i+2} 2^{-i-3}.$$

Using the inequality $i < 2^{i(e \ln 2)^{-1}}$ we get, for $i > 0$,

$$|P_{i+1}(m)| \leqslant |P_i(m)| + \frac{|i-2|}{i} \frac{a^i}{8(m+i+2)}$$

where $a = 2^{-1+(e \ln 2)^{-1}}$. Hence

$$|P_{i+1}(m)| \leqslant |P_i(m)| + \frac{a^i}{8(m+3)}.$$

Since

$$P_0(m) = \frac{1}{4(m+1)} \text{ and } P_1(m) = -\frac{1}{8(M+1)}$$

the above inequality holds for $i = 0$ also. Therefore

$$|P_{i+1}(m)| \leqslant P_0(m) + \frac{1 - a^{i+1}}{8(m+3)(1-a)}$$

which is true for $i = -1$ also. Thus for $i \geqslant 0$

$$|P_i(m)| \leqslant \frac{1}{4(m+1)} + \frac{1 - a^i}{8(m+3)(1-a)} \tag{2.3}$$

Similarly

$$|Q_i(m)| \leqslant \frac{1}{4(m+1)} + \frac{3(1 - a^i)}{8(m+3)(1-a)}. \tag{2.4}$$

Using (2.3) and (2.4) in (2.2) we get, after some simplifications, the upper bound given in the theorem. This completes the proof of the theorem.

The bound given in the theorem is not very useful for small values of α. Calculations given in the table below indicate that for $\alpha \geqslant \frac{1}{2}$ the bound is good even for small values of m. Denote the bound by $A(\alpha, m)$.

TABLE 1

Values of $A(\alpha, m)$

$\alpha \backslash m$	5	10	20	50
.1	11.2942	6.6298	3.6437	1.5630
.5	0.3217	0.1805	0.0964	0.0402
.9	0.1129	0.0626	0.0332	0.0138

3. Distribution of $\hat{\mu}$.

THEOREM 3.1. *If $m = n$,*

$$0 \leqslant \Phi(z) - P\left[\sigma_1^{-1}(1+\alpha)^{\frac{1}{2}}(\hat{\mu}-\mu) < z\right]$$

$$\leqslant \frac{(1+\alpha)^{\frac{3}{2}}}{(1+\alpha)^{\frac{1}{2}} + \alpha^{\frac{1}{2}}} A(\alpha, m) z\phi\left[z\alpha(1+\alpha)^{-\frac{1}{2}}\right], \qquad z > 0$$

and the inequalities are reversed for $z < 0$ where Φ is the standard normal distribution, ϕ is the standard normal density and $A(\alpha, m)$ is the upper bound given by Theorem 2.3.

PROOF. The conditional distribution of $\hat{\mu}$ given w is normal with mean μ and variance $\sigma_1^2[f(w)]^{-2}$ where

$$f(w) = \frac{1-(1-\alpha)w}{\left[(1-w)^2 + \alpha w^2\right]^{\frac{1}{2}}}.$$

Therefore $P[\sigma_1^{-1}(1+\alpha)^{\frac{1}{2}}(\hat{\mu}-\mu) < z] = E\Phi[zf(w)(1+\alpha)^{-\frac{1}{2}}]$. Since $0 \leqslant w \leqslant 1$, $\alpha^{\frac{1}{2}} \leqslant f(w) \leqslant (1+\alpha)^{\frac{1}{2}}$, and if $z > 0$,

$$0 \leqslant \Phi(z) - \Phi\left[zf(w)(1+\alpha)^{-\frac{1}{2}}\right] \leqslant zE\left[1 - f(w)(1+\alpha)^{-\frac{1}{2}}\right]\phi\left[z\alpha^{\frac{1}{2}}(1+\alpha)^{-\frac{1}{2}}\right]$$

$$= zE\left\{\frac{f^2(w)}{1+f(w)(1+\alpha)^{-\frac{1}{2}}}\left[f^{-2}(w) - (1+\alpha)^{-1}\right]\right\}\phi\left[z\alpha^{\frac{1}{2}}(1+\alpha)^{-\frac{1}{2}}\right]$$

$$\leqslant \frac{(1+\alpha)^{\frac{3}{2}}}{(1+\alpha)^{\frac{1}{2}} + \alpha^{\frac{1}{2}}} A(\alpha, m) z\phi\left[z\alpha^{\frac{1}{2}}(1+\alpha)^{-\frac{1}{2}}\right].$$

The bound for $z < 0$ can be established similarly. The proof of the theorem is complete.

The bound given in Theorem 3.1, as in Theorem 2.3, is good for $\alpha \geqslant \frac{1}{2}$. Some idea for the sharpness or nonsharpness of the bound can be obtained from the following table. The bound is denoted by $B(\alpha, m, z)$. Calculations are given for $z = 1$. For $z = 2$ the calculations indicate the same pattern.

TABLE 2

Values of $B(\alpha, m, 1)$

$\alpha \backslash m$	5	10	20	50
.1				0.5038
.5	0.1032	0.0579	0.0309	0.0129
.9	0.0400	0.0222	0.0117	0.0049

REFERENCES

[1] Bement, T. E. and Williams, J. S. (1969). Variance of weighted regression estimators when sampling errors are independent and heteroscedastic. *J. Amer. Statist. Assoc.* **64** 1369–1382.
[2] Brodsky, J. (1977). Some efficiency results for estimating the common mean of two normal populations. Abs. Inst. Math. Statist. *Bull.* **34** 207–208.
[3] Brown, L. D. and Cohen, A. (1974). Point and confidence estimation of a common mean of two normal distributions. *Ann. Statist.* **2** 963–976.
[4] Cohen, A. and Sackrowiyz, H. B. (1974). On estimating the common mean of two normal populations. *Ann. Statist.* **2** 1274–1282.
[5] Graybill, F. A. and Deal, R. B. (1959). Combining unbiased estimators. *Biometrics* **15** 543–550.
[6] Khatri, C. C. and Shah, K. R. (1974). Estimation of location parameters from two linear models under normality. *Comm. Statist.* **3** 647–663.
[7] Meier, P. (1953). Variance of a weighted mean. *Biometrics* **9** 59–73.
[8] Norwood, Jr., T. E. and Hinkelmann, K. (1977). Estimating the common mean of several normal populations. *Ann. Statist.* **5** 1047–1050.
[9] Williams, J. S. (1967). The variance of weighted regression estimators. *J. Amer. Statist. Assoc.* **62** 1290–1301.
[10] Zacks, S. (1966). Unbiased estimation of the common mean. *J. Amer. Statist. Assoc.* **61** 467–476.

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