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RATIOS OF OFF-DIAGONAL C-WISHART[1]

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In this paper, the density function for the real and imaginary parts of the quotient of some off-diagonal elements of the three-dimensional complex Wishart matrix is derived. Connection to closed-loop transfer function determination is shown.

1. Introduction. In this note, we will derive the density function for the real and imaginary parts of the quotient of some off-diagonal elements of the three-dimensional complex Wishart matrix. The motivation for this study arose from an attempt to estimate the transfer function of a subsystem of a closed-loop system contaminated by noise. The statistical properties of estimators of open-loop transfer functions are fairly well known [1], [5]. However, much less is known about the closed-loop counterpart. Several attempts in this direction have appeared in the engineering literature, notably [6], [7] and [8], but these tend to be rough estimates. In Section 2, we will formally derive the closed-loop transfer function estimator, and in Sections 3 and 4 we will derive the density function for it.

2. A closed-loop transfer function estimator. Suppose an input-output system with impulse response function $h(t)$ is modelled as

$$c(t) = \int_0^\infty h(u)e(t-u)\,du + n(t), \tag{1}$$

where the input $e(t)$, the output $c(t)$, and the error (noise) $n(t)$ are zero mean stationary time series. A fundamental problem in system identification is to estimate h or its Fourier transform H (transfer function) given only finite samples of c and e. If we assume that n is uncorrelated with e, i.e., (in view of stationarity)

$$E(n(t)e(0)) = 0, \quad -\infty < t < \infty, \tag{2}$$

then an estimate of h, or equivalently, an estimate of its Fourier transform $H(jw)$ (assume $h(t) = 0$ for $t < 0$) is given by

$$\hat{H}(jw) = \bar{C}_{ce}(jw)/\bar{C}_{ee}(jw), \tag{3}$$

where $\bar{C}_{ee}[\bar{C}_{ce}]$ is a smoothed [cross] spectral estimate of the theoretical [cross] spectrum $\Gamma_{ee}[\Gamma_{ce}]$. The choice of the estimate (3) is motivated by the fact that because of (2),

$$\gamma_{ce}(\tau) = \int_0^\infty h(u)\gamma_{ee}(\tau-u)\,du, \tag{4}$$

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where γ_{ce} and γ_{ee} are the appropriate covariance functions, and hence that

(5) $$H(jw) = \Gamma_{ce}(jw)/\Gamma_{ee}(jw).$$

In the engineering literature, the system described by (1) and satisfying (2) is sometimes called an open-loop system; this is because there is no "feedback", i.e., the input $e(t)$ is not influenced by the output $c(\tau)$, $\tau \leqslant t$.

Now consider a system (Figure 1) consisting of a subsystem described by (1) and a second subsystem described by

(6) $$e(t) = i(t) - \int_0^\infty g(u)c(t-u)\,du;$$

here $i(t)$ is the input to the system (whereas e is now an input to a subsystem) and is also assumed to be a zero mean stationary time series; the function g is the impulse response of the second subsystem and is not assumed to be known. The system identification problem here is again to estimate H given, however, not only finite samples of c and e but also of i. This closed-loop (feedback) case differs from the previous open-loop case in that the assumption (2) is no longer supportable in general; otherwise, we would have

(7) $$\gamma_{in}(\tau) = \int_0^\infty g(u)\gamma_{nn}(\tau-u)\,du,$$

and this would imply (except for some extreme cases) that the input i to the system is correlated in a predetermined manner (through g) with the noise n in the system; however, for most systems encountered in practice, this is not a valid assumption; in fact, one often makes the assumption, as we will do here, that i and n are uncorrelated;

(8) $$\gamma_{in}(\tau) = 0, \quad -\infty < \tau < \infty.$$

Just as the estimate $\hat{H}(jw)$ in (3) was motivated by the theoretical calculation of $H(jw)$ based on the assumption (2), here we can also derive a similar estimate $\hat{H}(jw)$ of $H(jw)$ based on the assumption (8); in fact by using i as an instrumental series, we can derive analogously:

(9) $$H(jw) = \Gamma_{ci}(jw)/\Gamma_{ei}(jw);$$

an estimate $\hat{H}$ of H is then obtained by replacing the theoretical cross spectra by their corresponding smoothed cross spectral estimates. Hence if $i(k)$, $c(k)$ and $e(k)$,

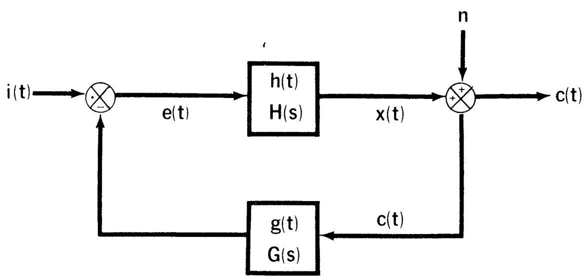

FIG. 1. Closed-loop System

$k = 0, \cdots, N-1$, are given, an estimate of $H(jw)$ is given by

(10) $$H^{(N)}(jw) = F_{ci}^{(N)}(jw)/F_{ei}^{(N)}(jw),$$

where, for series $x(k)$ and $y(k)$, $k = 1, \cdots, N-1$, $F_{xy}^{(N)}(jw)$ is a smoothed cross spectral estimate of the cross spectrum $\Gamma_{xy}(jw)$, and is given by

(11) $$F_{xy}^{(N)}(jw) = (2m+1)^{-1}\Sigma_{k=-m}^{m} I_{xy}^{(N)}(w + 2\pi k/N);$$

here $I_{xy}^{(N)}$ is the cross-periodogram of the series x and y and is given by

$$I_{xy}^{(N)}(w) = (2\pi N)^{-1} d_x^{(N)}(w)\, \overline{d_y^{(N)}(w)}\ ;$$

m is a nonnegative integer which is fixed throughout the following discussion. Finally, the finite Fourier transform $d_x^{(N)}$ of the series x (and similarly for y) is

$$d_x^{(N)}(w) = \Sigma_{k=0}^{N-1} x(k) \exp\{-jwk\}.$$

3. Relationship between closed-loop transfer function and the complex Wishart distribution. Let $X(k) = (i(k), c(k), e(k))$, $k = 0, 1, \cdots, N-1$ be given. The finite Fourier transform of X is defined as

$$d_X^{(N)}(w) = \Sigma_{k=0}^{N-1} X(k) \exp\{-jwk\}, \quad -\infty < w < \infty,$$

and the (second-order) periodogram of X is defined as ("*" denotes conjugate transpose)

$$I_{xx}^{(N)}(w) = \frac{1}{2\pi N} d_x^{(N)}(w) d_x^{(N)}(w)^*$$

Suppose $\{j_n\}_{n=1}^{\infty}$ is a sequence of integers such that

$$2\pi j_n/n \rightarrow w_0, \quad w_0 \neq 0 (\text{mod } \pi)$$

as $n \rightarrow \infty$; then, under some regularity assumptions (Assumption 2.5.1, Brillinger),

$$f_{xx}^{(N)}(w_0) = \frac{1}{(2m+1)} \Sigma_{k=-m}^{m} I_{xx}^{(N)}(2\pi(j_N + k)/N)$$

is an asymptotically unbiased estimate of $f_{xx}(w_0)$, the spectral density matrix at frequency w_0 of the series, i.e.,

$$f_{xx}(w_0) = (2\pi)^{-1}\Sigma_{k=-\infty}^{\infty} \exp\{-jw_0 k\} c_{xx}(k),$$

where, under the assumption of stationarity,

$$c_{xx}(k) = E[X(j+k) - c_x][X(j) - c_x]^T$$
$$c_x = E[X(j)] = E[X(j+k)]$$

are respectively the covariance matrix and mean vector of X. Here m is a nonnegative integer. Under slightly stronger regularity conditions, the asymptotic variability of the estimate $f_{xx}^{(N)}(w_0)$ can be shown to be inversely proportional to $2m+1$. [See Brillinger Theorem 7.3.3 and Corollary 7.3.1]. In this paper we will assume that m is nonzero, and hence $2m + 1 \geqslant 3$. Moreover, $f_{xx}^{(N)}(w_0)$ is asymptotically distributed as $(2m + 1)^{-1}W_3^c(2m + 1, f_{xx}(w_0))$, where $W_r^c(k, V)$, the complex Wishart matrix of dimension r and degrees of freedom k, denotes the distribution of the $r \times r$ matrix-valued random variable

$$W = \Sigma_{j=1}^{k} X_j X_j^*$$

where the r-dimension random variables $X_1, \cdots, X_k$ are identically independently complex-normally distributed with mean zero and covariance V. [See Brillinger, Theorem 7.3.3]. For simplicity, we will only consider the case where $w_0 \neq 0 \pmod{\pi}$.

From the definitions of $F_{ci}(jw)$ and $F_{ei}(jw)$, we see that, for large N, $H^{(N)}(jw)$ is approximately distributed as the quotient of two off-diagonal elements of a complex Wishart matrix. We will derive this distribution exactly in the next section.

4. Derivation of density function. The question we want to answer is the following: given that the 3×3 complex matrix-valued random variable

$$W = (w_{ij}), \quad w_{ij} = \bar{w}_{ji} \quad (1 \leqslant i, j \leqslant 3)$$

is distributed as a complex Wishart $W_3^c(n, \Sigma)$, how is w_{21}/w_{31} distributed? We have the following

THEOREM. *Let the* 3 *by* 3 *random (self-adjoint) matrix* $W = (w_{ij})$ *be distributed as a complex Wishart* $W_3^c(n, \Sigma)$ *with* n *degrees of freedom* ($n \geqslant 3$). *Assume* Σ *is positive definite. Then the distribution of the real and imaginary parts* (s_1 *and* s_2 *respectively*) *of* w_{21}/w_{31} ($= s_1 + js_2$) *has density* $f(s_1, s_2)$ *given by*

$$f(s_1, s_2) = h(s_1, s_2)/[g(s_1, s_2)]^2$$

where

$$g(s_1, s_2) = t^T \Sigma_{22}^{-1} \bar{t};$$

$$t = (s, 1)^T, \quad \text{where } s = s_1 - js_2;$$

$$\Sigma^{-1} = (\sigma^{ij})_{1 \leqslant i,j \leqslant 3} = \begin{pmatrix} \Sigma_{11}^{-1} & \Sigma_{12}^{-1} \\ \Sigma_{21}^{-1} & \Sigma_{22}^{-1} \end{pmatrix};$$

the latter is the partition of the inverse of $\Sigma = (\sigma_{ij})_{1 \leqslant i,j \leqslant 3}$ *in which*

Σ_{22}^{-1} *is* 2 *by* 2;

$$h(s_1, s_2) = A_n \Sigma_{m=0}^{n-1} B_{m,n} \beta^{2m+1}(2m + 3 - \beta^2)(1 - \beta^2)^{-m-2}$$

$$+ C_n \Sigma_{k=0}^{\infty} D_{k,n} \Sigma_{m=0}^{E[k/2]} P_{k,m}(1 - \beta^2/2)^{k-2m}(1 - \beta^2)^m;$$

$$\beta^2 = |t^T \Sigma_{21}^{-1}|^2 / [\sigma^{11} g(s_1, s_2)];$$

A_n, $B_{m,n}$, C_n, $D_{k,n}$ and $P_{k,m}$ are constants:

$$A_n = (2n-1)!|\Sigma_{22}^{-1}| \big/ \left[\pi 4^n (n-1)! \left(\sigma^{11}\sigma_{11}\right)^n\right];$$

$$B_{m,n} = B\left(\tfrac{1}{2}, m+1\right) \big/ \left[m!\,(n-m-1)!\right],$$

where B is the Beta function;

$$C_n = 3|\Sigma_{22}^{-1}| \big/ \left[\pi(2n+3)(2n+1)\left(\sigma^{11}\sigma_{11}\right)^n\right];$$

$$D_{k,n} = (n)_k (k+1) \big/ \left[\left(n+\tfrac{5}{2}\right)_k 2^k\right],$$

where

$$(n)_k = n(n+1)\dots(n+k-1), \qquad (n)_0 = 1;$$

$$P_{k,m} = (-1)^m (2k-2m)!/\left[m!\,(k-m)!\,(k-2m)!\right].$$

Finally,

$$E[k/2] = j \quad \text{if } k = 2j \quad \text{or if } k = 2j+1.$$

PROOF. (outline). The density f_w of W is [Goodman, 3], for $W \geqslant 0$, proportional to

$$(\det W)^{n-3} \exp\{-tr\Sigma^{-1}W\}.$$

By using the well-known trick of decomposing W as the unique product of a 3×3 complex lower triangular matrix T^* with its conjugate transpose T, i.e.,

$$W = T^*T$$

where

$$T = \begin{bmatrix} t_1 & t_2 & t_3 \\ 0 & t_4 & t_5 \\ 0 & 0 & t_6 \end{bmatrix}$$

with t_1, t_4, $t_6 > 0$ and t_2, t_3, t_5 are complex, it can be shown [Goodman, 2] that the density f_T of T is proportional to

$$t_1^{2n-1} t_4^{2n-3} t_6^{2n-5} \exp\{-tr\Sigma^{-1}T^*T\}.$$

It can easily be shown that $w_{12}/w_{13} = t_2/t_3$. Since w_{21}/w_{31} is just the complex conjugate of w_{12}/w_{13}, it suffices to calculate the distribution of t_2/t_3. This is accomplished through a series of integrations and transformations. In particular, t_4, t_5, t_6, and t_1 are integrated out to obtain the density function of t_2 and t_3; then by using the transformation $s_1 = \mathrm{Re}(t_2/t_3)$, $s_2 = -\mathrm{Im}(t_2/t_3)$, $s_3 = \mathrm{Re}(t_3)$ and $s_4 = \mathrm{Im}(t_3)$, and integrating out s_3 and s_4, one obtains, after some long calculations and relying heavily on [4], the density function of the real and imaginary parts of w_{21}/w_{31} as given in the statement of the theorem. The details of the proof will not be given here, but will appear elsewhere.

COROLLARY. *The variances of the real and imaginary parts of w_{21}/w_{31} are infinite.*

PROOF. It can be shown that there exists a positive constant δ such that

$$f(s_1, s_2) \geqslant \delta / g(s_1, s_2)^2.$$

If the variances of s_1, s_2 were finite, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s_i^2 / g(s_1, s_2)^2 \, ds_1, ds_2 < \infty, \qquad i = 1, 2.$$

But

$$g(s_1, s_2) = (s, 1)\Sigma_{22}^{-1}\overline{(s, 1)^T}$$
$$\leqslant \delta_1(1 + |s|^2) \qquad (s = s_1 - js_2)$$

for some positive δ_1, since Σ_{22}^{-1} is Hermitian and positive definite. Hence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s_i^2 / g(s_1, s_2)^2 \, ds_1 \, ds_2 \geqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s_i^2 / \left[\delta_1(1 + s_1^2 + s_2^2)\right]^2 ds_1 \, ds_2$$
$$= \infty.$$

This is a contradiction.

5. Conclusion. In this short note, we have outlined the derivation of the distribution of the real and imaginary parts of ratios of some off-diagonal elements of the three dimensional complex Wishart matrix. We have indicated that ratios of this type appear naturally in the determination of transfer functions of subsystems in closed-loop (feedback) engineering control systems. We also observed that these ratios have infinite variances.

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