The Annals of Statistics
1980, Vol. 8, No. 1, 116–131

ON EFFICIENCY AND OPTIMALITY OF QUADRATIC TESTS

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Let $Z_1, Z_2, \cdots$ be i.i.d. standard normal variables. Results are obtained which relate to the tail behavior as $x \to \infty$ of distributions of the form $F(x) = P\{\Sigma_{k=1}^{\infty}\lambda_k[(Z_k + a_k)^2 - 1] \leqslant x\}$. For test statistics which have such limiting distributions F, asymptotic relative efficiency measures are discussed. One of these is the limiting approximate Bahadur efficiency. Applications are to tests of fit and tests of symmetry.

1. Introduction. Let $Z_1, Z_2, \cdots$ be i.i.d. standard normal variables. This paper is concerned with test statistics which have a limiting distribution

$$G(x) = P\{\Sigma_{k=1}^{\infty}\lambda_k(Z_k^2 - 1) \leqslant x\}$$

under some null hypothesis (H_0) and a limiting distribution

$$F(x) = P\{\Sigma_{k=1}^{\infty}\lambda_k[(Z_k + a_k)^2 - 1] \leqslant x\}$$

under alternatives converging to H_0. In Section 2 several results are obtained relating to the tail behavior as $x \to \infty$ of G and F. These results are used in Section 3 to obtain asymptotic relative efficiency measures for a pair of competing test statistics. One of these measures is the limiting approximate Bahadur efficiency which is the ratio of efficacies of the form

$$\Sigma_{k=1}^{\infty}\lambda_k a_k^2/\max\{\lambda_k\}.$$

The condition of Wieand (1976) which allows this to be equated to a limiting Pitman efficiency is investigated. Applications to goodness of fit tests and tests of symmetry are considered in Section 4.

2. Asymptotics for infinite weighted sums of independent chi-square variables. Let $Z_1, Z_2, \cdots$ be i.i.d. standard normal variables and $\{\lambda_k\}_{k=1}^{\infty}$ and $\{a_k\}_{k=1}^{\infty}$ be two sequences of constants satisfying

$$\text{(2.1)} \qquad \begin{aligned} &\text{(i)} \quad \lambda_1 \geqslant \lambda_2 \geqslant \cdots > 0 \\ &\text{(ii)} \quad \Sigma_{k=1}^{\infty}\lambda_k^2 < \infty \\ &\text{(iii)} \quad \Sigma_{k=1}^{\infty}\lambda_k a_k^2 < \infty. \end{aligned}$$

Let n_i be the multiplicity of the ith largest value in $\{\lambda_k\}$. Thus $\lambda_1 = \lambda_2 = \cdots = \lambda_{n_1} > \lambda_{n_1+1} = \lambda_{n_1+2} = \cdots = \lambda_{n_1+n_2}$ and so forth. The results of Gregory (1977a)

Received January 1978; revised August 1978.
AMS 1970 *subject classifications.* Primary 62E15, 62E20; secondary 62G10.
Key words and phrases. Bahadur efficiency, U-statistic, Cramér-von Mises test, chi-square test.

Section 2 show the existence, under the conditions (2.1), of the following distributions:

$$\begin{aligned} G(x) &= P\{\Sigma_{k=1}^{\infty}\lambda_k(Z_k^2 - 1) \leqslant x\} \\ G_1(x) &= P\{\lambda_1\Sigma_{k=1}^{n_1}(Z_k^2 - 1) \leqslant x\} \\ F(x) &= P\{\Sigma_{k=1}^{\infty}\lambda_k[(Z_k + a_k)^2 - 1] \leqslant x\} \\ F_1(x) &= P\{\lambda_1\Sigma_{k=1}^{n_1}[(Z_k + a_k)^2 - 1] \leqslant x\}. \end{aligned} \tag{2.2}$$

When asymptotic centering is not effected we would drop from (2.2) the components generated from the (-1) but would require $\Sigma\lambda_k < \infty$. Specific changes in the results for this case are noted at the end of the section.

The following theorem is the analogue for centered distributions of Theorem 2 of Beran (1975b). Beran's result generalizes that of Zolotarev (1961). Our proof follows along the lines of Hoeffding's (1964) proof of Zolotarev's result.

THEOREM 2.1. *Under condition* (2.1)

$$\lim_{x\to\infty}\frac{1 - F(x)}{1 - F_1(x)}$$

$$= \Pi_{k=n_1+1}^{\infty}(1 - \lambda_k/\lambda_1)^{-\frac{1}{2}}\exp\{(a_k^2\lambda_k/2\lambda_1)/(1 - \lambda_k/\lambda_1)\}\exp\{-\lambda_k/2\lambda_1\}.$$

PROOF. Let $f(f_1)$ be the density of $F(F_1)$ and let f_2 be the density of $\Sigma_{k=n_1+1}^{\infty}\lambda_k[(Z_k + a_k)^2 - 1]$. It is shown that $\lim_{x\to\infty}f(x)/f_1(x)$ equals the right-hand side in the statement of the theorem.

Based on the convolution formula we have

$$\frac{f(x - \lambda_1 n_1)}{f_1(x - \lambda_1 n_1)} = \int_{-\infty}^{x}\frac{f_1(x - \lambda_1 n_1 - y)}{f_1(x - \lambda_1 n_1)}f_2(y)\,dy. \tag{2.3}$$

We first show that

$$\lim_{x\to\infty}\frac{f_1(x - y)}{f_1(x)} = \exp\{y/2\lambda_1\}. \tag{2.4}$$

Let $\chi^2(\nu, \rho^2)$ represent a noncentral chi-square variable with ν degrees of freedom and noncentrality parameter ρ^2. One may think of f_1 as the density of $\lambda_1\{\chi^2(1, \delta^2) + \chi^2(n_1 - 1, 0)\} - \lambda_1 n_1$ where the chi-square variables are independent and $\delta^2 = \Sigma_{k=1}^{n_1}a_k^2$. Writing $\gamma = (n_1 - 1)/2$ it is found for $x > 0$ that

$$f_1(x - \lambda_1 n_1) = x^{\gamma-\frac{1}{2}}e^{-\frac{x}{2\lambda_1}}e^{-\frac{\delta^2}{2}}(2\pi\lambda_1)^{-\frac{1}{2}}\lambda_1^{-\gamma}b_\gamma(x/\lambda_1), \tag{2.5}$$

where

$$\begin{aligned} b_\gamma(x) &= [2^\Gamma(\gamma)]^{-1}\int_0^1\frac{(e^{\delta(xz)^{\frac{1}{2}}} + e^{-\delta(xz)^{\frac{1}{2}}})/2}{z^{\frac{1}{2}}(1 - z)^{1-\gamma}}\,dz, \qquad \gamma > 0 \\ &= (e^{\delta x^{\frac{1}{2}}} + e^{-\delta x^{\frac{1}{2}}})/2, \qquad \gamma = 0. \end{aligned}$$

When $\gamma > 0$ asymptotic behavior ($x \to \infty$) for $b_\gamma(x)$ may be ascertained by making the change of variable to $w = (xz)^{\frac{1}{2}}$ in the integral. It is found that

$$\lim_{x\to\infty} b_\gamma(x) / \left\{2^{-1}\delta^{-\gamma}x^{-\gamma/2}\exp(\delta x^{\frac{1}{2}})\right\} = 1, \qquad \gamma \geqslant 0.$$

Also $b_\gamma(x)$ is increasing in (positive) x. Now (2.4) may be established by using these properties of b_γ and the inequality

$$x^{\frac{1}{2}} - (x-y)^{\frac{1}{2}} < 2^{-1}y(x-y)^{-\frac{1}{2}}, \; x > y > 0, \tag{2.6}$$

which shows that $\lim_{x\to\infty}(x^{\frac{1}{2}} - (x-y)^{\frac{1}{2}}) = 0$.

Suppose $n_1 \geqslant 2$. It is not difficult to establish that there are positive constants d_1 and d_2 so that

$$I[y \leqslant x] f_1(x - \lambda_1 n_1 - y)/f_1(x - \lambda_1 n_1) \leqslant d_1 \exp\{d_2 y\}, \; -\infty < y < \infty$$

for all x sufficiently large and independent of y. Then we have

$$\lim_{x\to\infty} f(x)/f_1(x) = \textstyle\int_{-\infty}^{\infty} e^{y/2\lambda_1} f_2(y)\, dy \tag{2.7}$$

which is calculated to be the right-hand side in the statement of Theorem 2.1. Thus the proof is complete if $n_1 \geqslant 2$.

Suppose $n_1 = 1$ so that $\gamma = 0$ in (2.5). In this case sufficient for (2.7) is

$$\lim_{x\to\infty} \textstyle\int_{\rho x}^{x} (1 - y/x)^{-\frac{1}{2}} e^{y/2\lambda_1} f_2(y)\, dy = 0 \qquad \text{for some } 0 < \rho < 1. \tag{2.8}$$

Sufficient for (2.8) is

$$\lim_{x\to\infty} x e^{x/2\lambda_1} f_2(x) = 0. \tag{2.9}$$

Let $\tilde{f}_2$ be the density of the variable

$$\lambda_{n_1+1}\Sigma_{k=n_1+1}^{n_1+n_2}\left[(Z_k + a_k)^2 - 1\right] + \lambda_{n_1+1}\Sigma_{k=n_1+n_2+1}^{n_1+n_2+n_3}\left[(Z_k + a_k)^2 - 1\right] + \Sigma_{k=n_1+n_2+n_3+1}^{\infty}\lambda_k\left[(Z_k + a_k)^2 - 1\right]. \tag{2.10}$$

This variable differs from that for density f_2 in that the coefficient for the second summation is λ_{n_1+1} instead of $\lambda_{n_1+n_2+1}$. The density f_1(2.5) is a location and scale change on a noncentral chi-square variable. If the scale factor is increased then the original density is bounded by a constant multiple of the new density for all large arguments. From this observation we can deduce that

$$f_2(x) \leqslant C_1 \tilde{f}_2(x) \tag{2.11}$$

for large x where C_1 is some constant. In (2.10) the multiplicity of the largest λ is $\geqslant 2$. The results proved so far show

$$\lim_{x\to\infty} \tilde{f}_2(x)/p(x) = C_2 \tag{2.12}$$

where C_2 is a constant and $p(x)$ is the density of the sum of the first two components of (2.10). Recall that $\lambda_1 > \lambda_{n_1+1}$. The density $p(x)$ is obtained from

(2.5) with suitable changes in notation. One can see that

(2.13) $$\lim_{x\to\infty} xe^{x/2\lambda_1}p(x) = 0.$$

Now (2.9) is implied by (2.11), (2.12) and (2.13). □

COROLLARY 2.2. *Under condition* (2.1)

$$\lim_{x\to\infty}\frac{1-G(x)}{1-G_1(x)} = \Pi_{k=n_1+1}^{\infty}(1-\lambda_k/\lambda_1)^{-\frac{1}{2}}\exp\{-\lambda_k/2\lambda_1\}.$$

LEMMA 2.3.

$$\lim_{x\to\infty}\frac{1-G_1(x)}{x^{(n_1/2)-1}e^{-x/2\lambda_1}} = e^{-n_1/2}/\,(2\lambda_1)^{(n_1/2)-1}\Gamma(n_1/2).$$

Together the above two results give

LEMMA 2.4. *Under condition* (2.1)

$$\lim_{x\to\infty}\frac{-\log(1-G(x))}{x/2\lambda_1} = 1.$$

In Section 3, G will be the limiting null hypothesis distribution of a test statistic and F will be the limiting distribution under alternatives. The limiting power of the size α test will be $p(\alpha) = 1 - F(G^{-1}(1-\alpha))$, $0 < \alpha < 1$. This is to be compared to the limiting power of a second test statistic. Let all previously defined quantities be given superscript * to denote the limiting form of this second test statistic.

THEOREM 2.5. *Let* $\delta = \{\Sigma_{k=1}^{n_1}a_k^2\}^{\frac{1}{2}}$, $L = \exp\{(1/2)\,\Sigma_{k=n_1+1}^{\infty}a_k^2\lambda_k/(\lambda_1-\lambda_k)\}$ *and similarly for* δ^* *and* L^*. *Under condition* (2.1) *and the similar condition for* $\{\lambda_k^*\}$ *and* $\{a_k^*\}$,

(2.14)
$$\begin{aligned}\lim_{\alpha\to 0}p(\alpha)/p^*(\alpha) &= 0 && \text{if } \delta^* > \delta && \text{or}\\ & && \delta^* = \delta && \text{and } n_1^* < n_1\\ &= \infty && \text{if } \delta > \delta^* && \text{or}\\ & && \delta^* = \delta && \text{and } n_1 < n_1^*\\ &= L/L^* && \text{if } \delta = \delta^* && \text{and}\\ & && n_1 = n_1^*. && \end{aligned}$$

PROOF. Assume without loss of generality that $\lambda_1 = \lambda_1^* = 1$. Let $f(f^*)$ and $g(g^*)$ be the densities of $F(F^*)$ and $G(G^*)$ respectively. We prove the theorem by showing

(2.15) $$\lim_{u\to 1} g(G^{-1}(u))/g^*(G^{*-1}(u)) = 1$$

and

(2.16) $$\lim_{u\to 1} f(G^{-1}(u))/f^*(G^{*-1}(u)) = \text{right-hand side of (2.14)}.$$

Lemma 2.4 yields (2.15) without difficulty. In order to show (2.16) several pieces of information are needed.

First we look at asymptotic behavior for $G^{-1}(u)$ and $G^{*-1}(u)$ as $u \to 1$. Note that Lemma 2.4 applies with the obvious transformation. From Corollary 2.2 and Lemma 2.3 (recall $\lambda_1 = 1$) we get

(2.17)

$$\lim_{u\to 1}(1-u)/\left\{\left[G^{-1}(u)\right]^{\gamma-\frac{1}{2}}\exp(-G^{-1}(u)/2)\,2^{\gamma-\frac{1}{2}}\Gamma(\gamma+1/2)\exp(n_1/2)\right\}$$

$$= \Pi_{k=n_1+1}^{\infty}(1-\lambda_k)^{-\frac{1}{2}}\exp\{-\lambda_k/2\}, \qquad \text{where } \gamma = (n_1-1)/2.$$

Comparison of $G^{-1}(u)$ to $G^{*-1}(u)$ is made by taking the ratio of terms as in the left-hand side of (2.17). Taking the logarithm of the ratio and using Lemma 2.4 yield

$$\text{(2.18)}\quad \lim_{u\to 1}\left\{(n_1-n_1^*)\log(-2\log(1-u)) - \left[G^{-1}(u) - G^{*-1}(u)\right]\right\} = c,$$

where c is a finite constant. Using (2.18), the inequality (2.6) and the rates implied by Lemma 2.4 we have

$$\text{(2.19)}\qquad \lim_{u\to 1}\left\{\left[G^{-1}(u)\right]^{\frac{1}{2}} - \left[G^{*-1}(u)\right]^{\frac{1}{2}}\right\} = 0.$$

From the proof of Theorem 2.1 (with $\lambda_1 = 1$) we have

$$\text{(2.20)}\quad \lim_{u\to 1} f(G^{-1}(u))/f_1(G^{-1}(u))$$

$$= \Pi_{k=n_1+1}^{\infty}(1-\lambda_k)^{-\frac{1}{2}}\exp\left\{\left(a_k^2/(1-\lambda_k)-1\right)\lambda_k/2\right\},$$

where $f_1(x)$ is the density of $F_1(x)$ given by (2.5) with $\lambda_1 = 1$. Substituting f_1 into (2.20) and using (2.17) gives

$$\text{(2.21)}\quad \lim_{u\to 1}\frac{f(G^{-1}(u))(2\pi)^{\frac{1}{2}}2^{\gamma-1/2}\Gamma(\gamma+1/2)\exp\left(n_1+\delta^2/2\right)}{(1-u)b_\gamma\left(G^{-1}(u)+n_1\right)}$$

$$= \Pi_{k=n_1+1}^{\infty}\exp\left\{a_k^2\lambda_k/2(1-\lambda_k)\right\},$$

where $b_\gamma(\cdot)$ is given by (2.5). The asymptotic form for b_γ together with (2.21) implies

$$\lim_{u\to 1}\frac{f(G^{-1}(u))\left[G^{-1}(u)\right]^{\gamma/2}}{(1-u)\exp\left\{\delta(G^{-1}(u))^{\frac{1}{2}}\right\}}\cdot\left(\pi^{\frac{1}{2}}2^{\gamma+1}\Gamma(\gamma+1/2)\right.$$

$$\left.\cdot\exp\left(n_1+\delta^2/2\right)\delta^{\gamma}\right) = \Pi_{k=n_1+1}^{\infty}\exp\left\{a_k^2\lambda_k/2(1-\lambda_k)\right\}.$$

Now using (2.19) and Lemma 2.4 one obtains (2.16) by examining the cases. □

Suppose it is desired to consider instead of $G(x)$ and $F(x)$ in (2.2) the distributions $P\{\Sigma_{k=1}^{\infty}\lambda_k Z_k^2 \leqslant x\}$ and $P\{\Sigma_{k=1}^{\infty}\lambda_k(Z_k+a_k)^2 \leqslant x\}$ respectively. Define distributions analogous to G_1 and F_1 by dropping terms in the summation as in (2.1).

These distributions exist if $\Sigma\lambda_k < \infty$ in addition to (2.1). Similar results to the ones proved hold in this case also.

(1) Theorem 2.1 and Corollary 2.2 hold with the factor $\exp\{-\lambda_k/2\lambda_1\}$ removed.
(2) Lemma 2.3 holds with the factor $\exp\{-n_1/2\}$ removed.
(3) Lemma 2.4 and Theorem 2.5 hold as stated.

The results for efficiency calculations in the next section are Lemma 2.4 and Theorem 2.5 which have the same form regardless of whether or not asymptotic centering is effected.

We conclude this section by noting the possibility of using Corollary 2.2 to approximate the tail probability of the null distribution G. If $n_1 = 1$ then $G_1(x)$ can be found from chi-square (one degree of freedom) tables or normal tables. Consider the example $\lambda_k = 1/k$, $k = 1, 2, \cdots$. The right-hand side in Corollary 2.2 is found to be $\exp\{(\gamma - 1)/2\}$ where γ is Euler's constant. The distribution G is tabulated in de Wet and Venter (1972) page 147. They find the upper tail probability $1 - G(3.5) = .048$. The approximation suggested by Corollary 2.2 is $1 - G(3.5) \doteq 1.23539\,(1 - G_1(3.5)) = .042$. This holds out the possibility of having in general a useful and simple approximation to the distribution G of an infinite sum, in the region where hypotheses would be rejected.

3. Efficiency and optimality of quadratic tests. Suppose χ is a random variable on a measurable space (κ, B) with unknown probability law $\mathcal{L}(\chi)$. Relative to this unknown law suppose one would like to test a null hypothesis H_0 against an alternative hypothesis H_1, based on n independent copies of χ. In some applications (Section 4) H_0 is simple and in others composite. In all cases H_1 is composite. For two suitable tests of H_0 vs. H_1 we are interested in asymptotic relative efficiency measures for a *particular* sequence of alternatives converging to *one* member of the null hypothesis. Accordingly in this section we will consider a sequence of simple alternatives converging to a simple null hypothesis which specifies $\mathcal{L}(\chi)$. We will also call these hypotheses H_0 and H_1 but no confusion will result.

For each $n = 1, 2, \cdots$ let χ_{ni}, $i = 1, \cdots, n$, be random variables which are independent under the null hypothesis $H_0 : \mathcal{L}(\chi_{ni}) = P_{01}$, $i = 1, \cdots, n$, for some fixed probability measure P_{01} on (κ, B) as well as under the alternatives $H_1 : \mathcal{L}(\chi_{ni}) = P_{n1}$, $i = 1, \cdots, n$. Let P_{n1} be dominated by P_{01} with $dP_{n1}/dP_{01} = 1 + n^{-\frac{1}{2}}h_{no}$ for some sequence $\{h_{no}\}$ in $L_2 = L_2(\kappa, B, P_{01})$ converging to $h_0 \in L_2$, say. Note that $\int h_0 dP_{01} = 0$. The sequence $\{P_n\}$ of product measures $P_n = P_{n1} \times \cdots \times P_{n1}$, n times, is contiguous to the sequence $\{P_0\}$ of product measures $P_0 = P_{01} \times \cdots \times P_{01}$, n times. We are concerned with tests which reject H_0 for large values of a test statistic $T_n(Q)$ satisfying

$$T_n(Q) - n^{-1}\Sigma_{i \neq j} Q(\chi_{ni}, \chi_{nj}) \to_{P_0} 0, \tag{3.1}$$

where Q is in the following class:

$$\mathfrak{Q} = \{Q(\cdot,\cdot) \mid Q(s,t),\ s, t \in \kappa, \text{ is a symmetric non-zero kernel on } \kappa \times \kappa \text{ with } \int Q^2(\cdot,\cdot)\,dP_{01} \times P_{01} < \infty, \tag{3.2}$$
$$\int Q(\cdot, t) P_{01}(dt) = 0 \text{ a.e. } (P_{01}) \text{ and } \int\int Q(s,t)h(s)h(t)\,dP_{01} \times P_{01} \geqslant 0,\ \forall h \in L_2 \text{ with } \int h\,dP_{01} = 0\}.$$

The last two conditions for Q to be in $\mathfrak{Q}$ can be described in terms of the eigenvalues $\{\lambda_k\}_{k=0}^{\infty}$ and corresponding orthonormal eigenfunctions $\{f_k\}_{k=0}^{\infty}$ of $Q(\int Q(\cdot, t) f_k(t) P_{01}(dt) = \lambda_k f_k$ a.e. (P_{01}), $\int f_k f_{k'} dP_{01} = 0$ if $k \neq k'$ and $\int f_k^2 dP_{01} = 1)$. The next to last condition on Q is equivalent to requiring that $\lambda_0 = 0$ corresponds to the constant eigenfunction $f_0 \equiv 1$. Since

$$\int\int Q(s,t)h(s)h(t)\,dP_{01} \times P_{01} = \Sigma\lambda_k(\int f_k h\, dP_{01})^2,$$

the last condition on Q is equivalent to requiring that all eigenvalues be nonnegative.

We will obtain in this section asymptotic efficiency measures of a test sequence $\{T_n(Q)\}$ relative to another $\{T_n(Q^*)\}$, $Q, Q^* \in \mathfrak{Q}$. We will also obtain optimal tests in $\mathfrak{Q}$ and in the class consistent against *all* alternatives, $h \in L_2$, $\int h\, dP_{01} = 0$. This class is

$$\underline{\mathfrak{Q}} = \{Q(\cdot,\cdot) \mid Q \in \mathfrak{Q}; \lambda_0 = 0, f_0 \equiv 1; \lambda_1, \lambda_2, \cdots > 0; \{f_k\}_{k=0}^{\infty} \text{ is a complete system in } L_2\}.$$

In a particular application a family of tests will correspond to a subset $\mathfrak{Q}_0$ say of $\mathfrak{Q}$ (or $\underline{\mathfrak{Q}}$). In Section 3 will be examples (Cramér-von Mises tests) where $\mathfrak{Q}_0 \subset \underline{\mathfrak{Q}}$ and the optimal test in $\underline{\mathfrak{Q}}$ lies in $\mathfrak{Q}_0$.

Consider two test sequences $\{T_n(Q)\}$ and $\{T_n(Q^*)\}$ where $Q, Q^* \in \mathfrak{Q}$. We drop the dependency on Q and Q^* and write $\{T_n\}$ and $\{T_n{}^*\}$ and similarly for all quantities that depend on Q or Q^*. Theorem 2.1 of Gregory (1977a) shows that the limiting distribution of T_n under $H_0(H_1)$ is of the form $G(F)$ of (2.2) with $a_k = \int h_0 f_k\, dP_{01}$. The limiting power of the size α test is

$$p_{h_0}(\alpha) = 1 - F(G^{-1}(1 - \alpha)).$$

The notion of Pitman asymptotic relative efficiency of $\{T_n\}$ with respect to $\{T_n^*\}$ is the limiting ratio of sample sizes for the tests of common size α to achieve the same limiting power β at the same alternative. This depends on α and β. It is not difficult to see that the Pitman efficiency of $\{T_n\}$ to $\{T_n^*\}$ would be

$$e^*/e \quad \text{where } p_{e^{1/2}h_0}(\alpha) = \beta = p^*_{(e^*)^{1/2}h_0}(\alpha). \tag{3.3}$$

(The existence of (3.3) would follow if the power functions were continuous monotone increasing functions of e and e^*.)

Wieand (1976) has a general treatment of Pitman efficiency and has conditions under which a limiting $(\alpha \to 0)$ Pitman efficiency is the limiting approximate

Bahadur efficiency. We will study this. Now however consider the attempt at a related measure of efficiency,

$$e^*/e \quad \text{where } \lim_{\alpha\to 0} p_{e^{1/2}h_0}(\alpha)/p^*_{(e^*)^{1/2}h_0}(\alpha) = 1.$$

One might anticipate that this measure would exist and be the same as the limit ($\alpha \to 0$) of (3.3). This is not true. From Theorem 2.5 it is seen that there is not in general such a choice for e and e^*. The cleanest comparison of the tests is the following which does not involve the limiting ratio of sample sizes. Recall δ and L from Theorem 2.5. Here $a_k = \int h_0 f_k \, dP_{01}$, $k = 1, 2, \cdots$.

(3.4)

(a) The test with the larger δ is superefficient.

(b) If $\delta = \delta^*$ the test with the smaller multiplicity for the largest eigenvalue is superefficient.

(c) If $\delta = \delta^*$ and $n_1 = n_1^*$ the test with the larger L is more efficient (but not superefficient).

When $n_1 = n_1^*$ it is possible to obtain from Theorem 2.5 a type of asymptotic relative efficiency involving the limiting ratio of sample sizes. If $n_1 = n_1^*$ then

$$e^*/e = \delta^2/(\delta^*)^2 \tag{3.5}$$

is the unique choice giving $0 < \lim_{\alpha\to 0} p_{e^{1/2}h_0}(\alpha)/p_{(e^*)^{1/2}h_0}(\alpha) < \infty$. There is some difficulty with the efficiency measure (3.5) however since the limit above cannot be made equal to one. A related concept is that of local asymptotic efficiency and its limit as $\alpha \to 0$ (see Theorem 4 of Beran (1975a)). If Beran's Theorem 4 can be generalized to our distributions, this limit ($\alpha \to 0$) of the local asymptotic efficiency of $\{T_n\}$ with respect to $\{T_n^*\}$ will be $[\delta^2/n_1]/[(\delta^*)^2/n_1^*]$. This quantity can also be thought of as the approximate value of $\lim_{\alpha\to 0} p(\alpha)/p^*(\alpha)$ obtained by using first order approximations to the asymptotic power functions. We have not developed this since we prefer the inferences from Theorem 2.5 where no approximation was made to the asymptotic power function.

Consider now the relation of Bahadur to Pitman efficiency and the work of Wieand (1976). Some notational variance must occur here to accomodate indexing of null and alternative laws by a parameter θ. Write $P_{\theta_0 1}$ (instead of P_{01}) for a null hypothesis law and $P_{\theta 1}$ (instead of P_{n1}) for an alternative hypothesis law. For the product laws write P_{θ_0} and P_θ instead of P_0 and P_n. Since a particular *sequence* of alternatives is not considered write the sample as $\chi_1, \cdots, \chi_n$ instead of $\chi_{n1}, \cdots, \chi_{nn}$. Assume all expressions considered retain their meaning with these substitutions. Contiguous alternatives considered earlier were characterized by a function h_0; in the present context those same alternatives could be expressed for example by

$$dP_{\theta 1}/dP_{\theta_0 1} = 1 + (\theta - \theta_0)h_0. \tag{3.6}$$

Instead of having test statistics $T_n(Q)$ satisfying (3.1) we want (for Bahadur efficiency considerations) a so-called standard sequence of tests. Let

(3.7) $$S_n(Q) \equiv S_n = \{\max(T_n, 0)\}^{\frac{1}{2}}$$

where T_n satisfies (3.1). Under the null hypothesis we have

$$\begin{aligned} \lim_n P(S_n(Q) \leqslant x) &= 0 && \text{if } x < 0 \\ &= G(0) && \text{if } x = 0 \\ &= G(x^2) && \text{if } x > 0 \end{aligned}$$

where G is given by (2.2). This limiting distribution has a mass point at zero but is otherwise continuous, which essentially satisfies one of Bahadur's conditions for $\{S_n\}$ to be a standard sequence. Using Lemma 2.4 the other Bahadur conditions are satisfied with approximate Bahadur slope $c(\theta) = \int\int Q(s, t)dP_{\theta 1} \times P_{\theta 1}/\max\{\lambda_k\}$ if $c(\theta) > 0$ when $\theta \neq \theta_0$. With (3.6) this becomes

(3.8) $$c(\theta) = (\theta - \theta_0)^2 \int\int Q(s, t)h_0(s)h_0(t)dP_{\theta_0 1} \times P_{\theta_0 1}/\max\{\lambda_k\}.$$

Let $c^*(\theta)$ be the approximate slope of another such test. Then $c(\theta)/c^*(\theta)$ is the approximate Bahadur efficiency. Wieand's result equates the limit ($\alpha \to 0$) of Pitman efficiency to the limiting approximate Bahadur efficiency

(3.9) $$\begin{aligned} \lim_{\theta \to \theta_0} \frac{c(\theta)}{c^*(\theta)} &= \frac{\int\int Q(s, t)h_0(s)h_0(t)dP_{\theta_0 1} \times P_{\theta_0 1}/\max\{\lambda_k\}}{\int\int Q^*(s, t)h_0(s)h_0(t)dP_{\theta_0 1} \times P_{\theta_0 1}/\max\{\lambda_k^*\}} \\ &= \frac{\Sigma_k \lambda_k (\int h_0 f_k dP_{\theta_0 1})^2/\max\{\lambda_k\}}{\Sigma_k \lambda_k^* (\int h_0 f_k^* dP_{\theta_0 1})^2/\max\{\lambda_k^*\}}. \end{aligned}$$

The notion of Pitman efficiency used there is general and we think would correspond to (3.3) for our applications but we need not be concerned with details because of the nature of Wieand's results: whatever is true as regards Pitman efficiency considerations, any limiting ($\alpha \to 0$) efficiency notion is captured by the limiting Bahadur efficiency $\lim_{\theta \to \theta_0} c(\theta)/c^*(\theta)$ if the limit exists. Wieand's principal condition implying this result is called Condition III*. Below is a theorem concerning Condition III* when the approximation in (3.1) is exact. Recall the notation changes we have in the definition of $\mathcal{Q}$. Here P_θ does not have to be determined through (3.6).

THEOREM 3.1. *For θ near θ_0 let $\chi_1, \cdots, \chi_n$ represent independent random variables with $\mathcal{L}(\chi_i) = P_{\theta 1}$, $i = 1, \cdots, n$ where $P_{\theta 1}$ is a probability measure on some space (κ, B). Write the product measure $P_\theta = P_{\theta 1} \times \cdots \times P_{\theta 1}$, n times, and let E_θ be expectation under P_θ. Suppose the test statistics are S_n, $n = 1, 2, \cdots$ given by (3.7) where $T_n = n^{-1}\Sigma_{i \neq j} Q(\chi_i, \chi_j)$ for some $Q \in \mathcal{Q}$. Assume $E_\theta Q(\chi_1, \chi_2) > 0$ for all θ near $\theta_0 (\theta \neq \theta_0)$. If*

(3.10) $$E_\theta Q^2(\chi_1, \chi_2) \leqslant B \quad \text{and} \quad \left| \frac{E_\theta Q(\chi_1, \chi_2) Q(\chi_2, \chi_3)}{E_\theta Q(\chi_1, \chi_2)} \right| \leqslant B$$

for all θ near θ_0, where B is finite then $\{S_n\}$ satisfies Condition III of Wieand* (1976) *with* $b(\theta) = \{\int\int Q(s, t)dP_{\theta 1} \times P_{\theta 1}\}^{\frac{1}{2}}$.

PROOF. The probability in Condition III* is

$$P_\theta\{(1-\varepsilon)^2 < S_n^2/nb^2(\theta) < (1+\varepsilon)^2\}$$
$$\geqslant P_\theta\{(1-\varepsilon) < S_n^2/nb^2(\theta) < (1+\varepsilon)\}$$
$$= 1 - P_\theta\{|S_n^2/n - b^2(\theta)| > \varepsilon b^2(\theta)\}$$
$$\geqslant 1 - P_\theta\{|T_n^2/n - b^2(\theta)| > \varepsilon b^2(\theta)\}$$
$$- P_\theta\{T_n < 0\} \qquad > (\text{if } \varepsilon < 1)$$
$$1 - 2P_\theta\{|T_n^2/n - b^2(\theta)| > \varepsilon b^2(\theta)\}$$
$$> 1 - 2E_\theta|n^{-2}\Sigma_{i \neq j} Q(\chi_i, \chi_j) - b^2(\theta)|^2/\varepsilon^2 b^4(\theta).$$

The continuation is routine. □

The recommendation (3.4) and the limiting approximate Bahadur efficiency measure (3.9) give us two ways of comparing tests. Consider now the problem of finding an optimal test in a class of tests. Let

$$\phi(Q, h) = \int\int Q(s, t)h(s)h(t)dP_{\theta_0 1} \times P_{\theta_0 1} / \int h^2 dP_{\theta_0 1},$$

$$\text{for } Q \in \mathcal{Q} \quad \text{and} \quad h \in L_2(\kappa, B, P_{\theta_0 1}) \quad \text{with } \int h dP_{\theta_0 1} = 0.$$

It is well known that

$$\max\{\lambda_k\} = \sup_{h \in L_2} \phi(Q, h).$$

It is easy to see that for $Q \in \mathcal{Q}$

$$\max\{\lambda_k\} = \sup_{h \in L_2;\ \int h dP_{\theta_0 1} = 0} \phi(Q, h).$$

Given an alternative $h_0 \in L_2$ with $\int h_0 dP_{\theta_0 1} = 0$ and a class of tests associated with some $\mathcal{Q}_0 \subset \mathcal{Q}$, we have

(3.11)

(*a*) With respect to the recommendation (3.4) an optimal test is one which (in order) (1) maximizes δ^2, (2) minimizes n_1 and (3) maximizes L.

(*b*) With respect to the Bahadur efficiency measure (3.9) an optimal test is one which maximizes

$$\phi(Q, h_0)/\sup_{h \in L_2;\ \int h dP_{\theta_0 1} = 0} \phi(Q, h).$$

It is not difficult to show that for the class of all quadratic tests ($\mathcal{Q}_0 = \mathcal{Q}$) we have

(3.12)
(*a*) With respect to the recommendation (3.4) an optimal quadratic test has $n_1 = 1$ and f_1 proportional to h_0.

(*b*) With respect to the Bahadur efficiency measure (3.9) an optimal quadratic test is one which has an eigenfunction proportional to h_0 and corresponding to the largest eigenvalue.

Observe that in (3.12b) no advantage is given to tests with small n_1. However we will see an example where the optimality question as regards a subclass $\mathcal{Q}_0$ (Cramér-von Mises tests) can have the same answer using (3.11a) or (3.11b). In this example specifying *any* eigenfunction determines the test in $\mathcal{Q}_0$.

The general impact of (3.12) for power considerations is to focus attention on the so-called first component of a quadratic test statistic, i.e., the part associated with the largest eigenvalue. This same general conclusion was stated in Beran (1975a) relative to a discussion of certain quadratic rank test statistics.

4. Applications. Limiting approximate Bahadur efficiencies will be called simply Bahadur efficiencies and limiting ($\alpha \to 0$) Pitman efficiencies will be called simply Pitman efficiencies.

4.1 *Tests of Fit: The Simple Hypothesis.* Suppose H_0 specifies a uniform distribution on the unit interval (0, 1), i.e., $\kappa = (0, 1)$ and P_{01} is Lebesgue measure. Refer to Gregory (1977a) Sections 3 and 6 for a discussion of chi-square and Cramér-von Mises (CVM) tests. The expressions developed there allow the computation of Bahadur efficiencies (3.9). Notice that the efficiency measure proposed there would be the Bahadur measure if $\{2\Sigma\lambda_k^2\}^{\frac{1}{2}}$ were replaced by $\max\{\lambda_k\}$. To apply the comparison of (3.4) it may be necessary to know for a given test the eigenfunction(s) associated with the largest eigenvalue.

Concentrate on the class of CVM kernels (call it $\mathcal{Q}_0$) generated by (3.3) of Gregory (1977a) for all acceptable choices of a positive weight function $w(x)$, $0 < x < 1$. "Acceptable" weight functions would be the largest class so that $\mathcal{Q}_0 \subset \mathcal{Q}$.

Given a kernel with positive weight function $w(x)$, $0 < x < 1$, any eigenfunction f and corresponding positive eigenvalue λ satisfy

$$w(x) = \lambda^{-1}[-f'(x)/\textstyle\int_0^x f], \quad 0 < x < 1 \tag{4.1}$$

(see de Wet and Venter (1973) equation (21)). Conversely, a given function f with

$$\begin{gathered} f \in L_2,\ \textstyle\int_0^1 f = 0, \\ f \text{ differentiable and} \\ f \text{ monotone near 0 and 1} \end{gathered} \tag{4.2}$$

is an eigenfunction corresponding to positive eigenvalue λ for the kernel generated by weight function w of (4.1) if $w(x)$, $0 < x < 1$, is positive.

Suppose there is an alternative sequence characterized by a function $h_0(s)$, $0 < s < 1$, satisfying (4.2) with $f = h_0$. If a CVM statistic can be found whose largest eigenvalue (λ_1) corresponds to an eigenfunction (f_1) proportional to h_0 (it will be given by $w(x) = -h_0'(x)/\int_0^x h_0$) this statistic will provide the optimum CVM test with respect to either (3.11a) or (3.11b). Such a test will also be optimum among all quadratic tests. If such a CVM statistic cannot be found one can still use the CVM kernel Q with $w(x) = -h_0'(x)/\int_0^x h_0$ to construct a test which is Bahadur optimum. Suppose for example that $f_2 = c_2 h_0$. Then the derived kernel $Q(s, t) - (\lambda_1 - \lambda_2 + \varepsilon) f_1(s) f_1(t)$ provides a Bahadur optimum test for any $0 < \varepsilon < \lambda_2$.

Consider the two examples $w(x) \equiv 1$ and $w(x) = 1/x(1-x)$. The kernels involved are treated in de Wet and Venter (1973) ($W_{-\frac{1}{2}, -\frac{1}{2}}(x)$ there should be π^2) with reference to Erdelyi et al. (1953). One finds the following largest eigenvalues and unique eigenfunctions:

$w(x) \equiv 1$	$w(x) = 1/x(1-x)$
$\lambda_1 = \pi^{-2}$	$\lambda_1 = 2^{-1}$
$f_1(s) = 2^{\frac{1}{2}} \cos \pi s,$	$f_1(s) = 3^{\frac{1}{2}}(2s - 1),$
$0 < s < 1$	$0 < s < 1.$

Thus if $h_0(s) = (\text{constant})(2s - 1)$, $0 < s < 1$, then the best CVM test uses $w(x) = 1/x(1-x)$. Two efficiency measures discussed in Section 3 are easy to calculate. The Bahadur efficiency (3.9) is $\pi^2/10 = .98696$. The measure (3.5) is $96/\pi^4 = .98553$. The measure proposed in Gregory (1977a) Section 6 assumes here the unrelated value .417.

The following theorem gives conditions which guarantee the equivalence of limiting Pitman and limiting approximate Bahadur efficiencies. The conditions below imply Q is of trace class, i.e., $\int_0^1 Q(s, s)\, ds < \infty$. More work will show the same result under broader conditions in the spirit of (3.6) of Gregory (1977a).

THEOREM 4.1. *Let* $\kappa = (0, 1)$ *and* F_θ *for* θ *near* θ_0 *represent the cumulative distributions on* (0, 1) *in Theorem* 3.1. *Let* $F_{\theta_0}(x) = x$, $0 < x < 1$. *Let*

$$Q(s, t) = \int_0^1 \{I[s \leq u] - u\}\{I[t \leq u] - u\} w(u) du.$$

If

(1) $F_\theta(u \wedge v) - F_\theta(u) F_\theta(v) \leq B\{u \wedge v - uv\}$ *for all* θ *near* θ_0, B *finite,*

and

$$(2) \quad \int_0^1 u(1-u) w(u) du < \infty$$

then (3.10) *is satisfied.*

Proof. Write $F_\theta = F$ and $E_\theta = E$. We have

$$\begin{aligned} &EQ(\chi_1, \chi_2)Q(\chi_2, \chi_3) \\ &\quad = E\{EQ(\chi_1, \chi_2)|\chi_2\}^2 \\ &\quad = E\{\textstyle\int_0^1 (F(u) - u)\{I[\chi_2 \le u] - u\}w(u)du\}^2 \\ &\quad = \textstyle\int_0^1\int_0^1 (F(u) - u)(F(v) - v)\{F(u \wedge v) - vF(u) \\ &\qquad - uF(v) + uv\}w(u)w(v)dudv \\ &\quad = \textstyle\int_0^1\int_0^1 (F(u) - u)(F(v) - v)\{F(u \wedge v) - F(u)F(v) \\ &\qquad + (F(u) - u)(F(v) - v)\}w(u)w(v)du\,dv \\ &\quad \le B\textstyle\int_0^1\int_0^1 |(F(u) - u)(F(v) - v)|(u \wedge v - uv)w(u)w(v)du\,dv \\ &\qquad + \{\textstyle\int_0^1 (F(u) - u)^2 w(u)du\}^2. \end{aligned}$$

Using $(u \wedge v - uv) \le [u(1-u)]^{\frac{1}{2}}[v(1-v)]^{\frac{1}{2}}$ and the Cauchy-Schwarz inequality the proof can be completed. □

Consider now the relative efficiency of chi-square tests to CVM tests. The δ for chi-square tests is given by the first factor of (6.2) of Gregory (1977a) and approaches, as the number of cells approaches infinity while the maximum cell width approaches zero, the δ for the optimal CVM test. However with respect to a CVM test which is not optimal, a chi-square test can be found (by taking a sufficiently large number of cells) which is superefficient according to the recommendation (3.4). With the aid of the formulas in Gregory (1977a) the Bahadur efficiency (3.9) can be found. For example, let the alternatives be the ones against which the CVM test with $w(x) = 1/x(1-x)$ provides the optimal quadratic test. Then the Bahadur efficiency of the chi-square test using c intervals of equal length, to this optimal test is $1 - c^{-2}$. It can be shown that condition (3.10) always holds for chi-square kernels.

In considering chi-square tests one should add the result noted in Gregory (1977a): for any sequence of alternatives the limiting power of the chi-square test approaches zero as c (number of cells) $\to \infty$. This confusing picture involving the comparison of chi-square tests to CVM tests points out the need for more work relating efficiency measures to realistic criteria one would encounter in practice.

4.2 *Tests of Fit: The Composite Hypothesis.* If H_0 specifies a parametric family, then a natural procedure is to estimate the parameters and apply a test of fit for a simple hypothesis. For example, to test that a sample $\chi_1, \cdots, \chi_n$ is from a normal distribution with unknown mean μ and variance σ^2, one can test that $\chi_1' = (\chi_1 - \hat{\mu})/\hat{\sigma}, \cdots, \chi_n' = (\chi_n - \hat{\mu})/\hat{\sigma}$ is a sample from a standard normal distribution using a CVM statistic. (Of course the percentage points will be different from the simple hypothesis case.) This latter CVM test is defined relative to the space $\kappa = (-\infty, \infty)$ and null hypothesis law P_{01} of the standard normal distribution. In Gregory (1977a) details are given for such a test applied to $\chi_1', \cdots, \chi_n'$. The test statistic adjusted by a constant is shown to satisfy (3.1) where the kernel Q has eigenvalues $\lambda_k = 1/k$ and corresponding eigenfunctions $f_k(x)$, $-\infty < x < \infty$, which are the normalized Hermitian polynomials, $k = 3, 4, \cdots$. The first two eigenfunctions are "lost" due to estimation.

The test is location and scale invariant so we can assume that the true distribution under H_0 is given by the standard normal law P_{01}. Let alternatives P_{n1} approach P_{01} according to $dP_{n1}/dP_{01} = 1 + n^{-\frac{1}{2}}h_{n0}$ where h_{n0} converges to h_0 in $L_2(\kappa, B, P_{01})$. Location (scale) alternatives correspond to $h_0(x) = f_1(x)(f_2(x))$, the first (second) Hermitian polynomial. All location and scale invariant test statistics satisfying (3.1) will have eigenfunctions orthogonal to the first two Hermitian polynomials.

For the test we are considering the largest eigenvalue corresponds to the third Hermitian polynomial. Therefore if $h_0(x)$ is proportional to $x^3 - 3x$, $-\infty < x < \infty$, this test is Bahadur optimum in the class

$\underline{\underline{\mathcal{Q}}} = \{Q(\cdot, \cdot) | Q \in \mathcal{Q}$; all eigenvalues are nonnegative; eigenfunctions corresponding to positive eigenvalues are a complete system for the subspace of L_2 orthogonal to the constant function and the first two Hermitian polynomials.$\}$

In Gregory (1977c) this CVM statistic for the composite hypothesis of normality is shown to be asymptotically equivalent to a statistic related to the Shapiro-Wilk W statistic. There a class of statistics called generalized W statistics is defined and depends on the choice of a weight function. (Actually generalized W statistics is a misnomer since the original Shapiro-Wilk statistic has *not* been treated.) This new class of quadratic tests does *not* coincide with the class of CVM tests and thus provides additional choices in any search for an optimal test.

4.3 *Tests for Symmetry*. Suppose H_0 is the hypothesis of symmetry about zero. Distribution free rank tests of the CVM type are discussed in Gregory (1977b). Without loss of generality assume that the null hypothesis law P_{01} is the uniform law on the interval $\kappa = (-1, 1)$. The CVM statistics adjusted by a constant satisfy (3.1), where the associated kernels $Q(s, t)$, $-1 < s, t < 1$, depend on a weight function w *on the interval* (0, 1). The kernels for testing symmetry may thus be made to correspond (through w) to the kernels for testing uniformly on (0, 1). The following conclusion is reached: $(4\lambda; f((1 + s)/2), -1 < s < 1)$ is an eigenpair for the kernel for testing symmetry if and only if $(\lambda; f(u), 0 < u < 1)$ is an eigenpair for the kernel for testing uniformity on (0, 1) and $f((1 + s)/2)$, $-1 < s < 1$ is an odd function. The other eigenfunctions f for testing uniformity on (0, 1) give $f((1 + s)/2)$, $-1 < s < 1$, an even function and are related to testing for uniformity on $(-1, 1)$ *given* symmetry.

Now the conditions of Gregory (1977b) under which a test statistic would satisfy (3.1), are restrictive; they would imply for example that w is a bounded function.

We anticipate the same results for a larger class of weight functions including in particular $w(x) = 1/x(1-x)$. In any case we are now dealing with kernels but will speak of the symmetry test using weight function w, for any w that could be used in Section 4.1.

In the previous manner let $h_0(s)$, $-1 < s < 1$, characterize alternatives approaching P_{01} (uniformity on $(-1, 1)$). Logistic location alternatives when transferred to $(-1, 1)$ give $h_0(s) = s$, $-1 < s < 1$. From the remarks above and Section 4.1 it can be seen, for the symmetry test using weight function $w(x) = 1/x(1-x)$, $0 < x < 1$, that $h_0(s) = s$, $-1 < s < 1$, is proportional to the eigenfunction corresponding to the largest eigenvalue. Thus $w(x) = 1/x(1-x)$ provides the optimum CVM symmetry test for logistic location alternatives. (This is of course a different conclusion from that reached in Gregory (1977b) since a different efficiency measure was used there.)

The logistic location alternatives to symmetry, $h_0(s) = s$, $-1 < s < 1$, correspond to the transformation to $(-1, 1)$ of the alternatives to uniformity on $(0, 1)$ used in the example in Section 4.1. We claim for these alternatives and for weight functions $w \equiv 1$ and $w(x) = 1/x(1-x)$, that the relative efficiency measures of Section 3 are the same for the uniformity test on $(0, 1)$ as for the symmetry test. The kernels for symmetry are obtained from those for uniformity on $(0, 1)$ by deleting eigenpairs $(\lambda; f)$. However in the full expressions of (3.9) the terms corresponding to deleted eigenpairs are seen to be zero using odd-even function considerations. Therefore using either efficiency measure (3.5) or (3.9) the symmetry test using $w \equiv 1$ has efficiency approximately .98 with respect to the best CVM test against logistic location alternatives.

Hajek and Sidak (1967) discuss the Wilcoxon signed rank test for symmetry. This test is the optimum linear rank test for logistic location alternatives. The square of the Wilcoxon statistic can be associated with the first component $\lambda_1 f_1(s) f_1(t)$ of the kernel $Q(s, t)$ of the best CVM test against logistic location alternatives.

5. Acknowledgments. The improved nature of this paper over its original form is a result of a very careful reading by the referees. Their assistance is much appreciated.

REFERENCES

[1] Beran, R. (1975a). Local asymptotic power of quadratic rank tests for trend. *Ann. Statist.* **3** 401–412.

[2] Beran, R. (1975b). Tail probabilities of noncentral quadratic forms. *Ann. Statist.* **3** 969–974.

[3] De Wet, T. and Venter, J. H. (1972). Asymptotic distributions of certain test criteria of normality. *South African Statist. J.* **6** 135–149.

[4] De Wet, T. and Venter, J. H. (1973). Asymptotic distributions for quadratic forms with applications to tests of fit. *Ann. Statist.* **1** 380–387.

[5] Erdélyi, A. et al. (1953). *Higher Transcendental Functions 2.* McGraw-Hill, New York.

[6] Gregory, G. G. (1977a). Large sample theory for *U*-statistics and tests of fit. *Ann. Statist.* **5** 110–123.

[7] GREGORY, G. G. (1977b). Cramer-von Mises type tests for symmetry. *South African Statist. J.* **11** 49–61.

[8] GREGORY, G. G. (1977c). Functions of order statistics and tests of fit. *South African Statist. J.* **11** 99-118.

[9] HÁJEK, J. AND ŠIDÁK, Z. (1967). *Theory of Rank Tests*. Academic Press, New York.

[10] HOEFFDING, W. (1964). On a theorem of V. M. Zolotarev. *Theory of Probability and Appl.* **9** 89–91.

[11] WIEAND, H. S. (1976). A condition under which the Pitman and Bahadur approaches to efficiency coincide. *Ann. Statist.* **4** 1003–1011.

[12] ZOLOTAREV, V. M. (1961). Concerning a certain probability problem. *Theor. Probability Appl.* **6** 201–204.

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