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THE TRADE-OFF METHOD IN THE CONSTRUCTION OF BIB DESIGNS WITH VARIABLE SUPPORT SIZES

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A balanced incomplete block (BIB) design with b blocks is said to have the support size b^* when exactly b^* of the b blocks are distinct. BIB designs with $b^* < b$ have interesting applications in design of experiments and controlled sampling as explained in details in Foody and Hedayat (1977) and Wynn (1977). A method called "trade-off" is introduced for the construction of BIB designs with repeated blocks. This method is utilized to study BIB designs with arbitrary v treatments in blocks of size $k = 3$ in general and with $v = 7$ and $k = 3$ in particular. It is shown that BIB designs with $v = 7$, $k = 3$, any b, and any b^* exist if and only if (i) b is divisible by 7, (ii) $7 < b^* < \min(b, 35)$, (iii) $b^* \neq 8$, 9, 10, 12, or 16, (iv) $(b, b^*) \neq (28, 24)$, (28, 27), (35, 30), (35, 32), (35, 33), (35, 34) or (42, 34).

1. Motivation. Suppose an experimenter wants to test and evaluate $v = 7$ treatments based on b blocks each of size $k = 3$. According to the usual homoscedastic linear additive model for measurements, the best possible design under any reasonable statistical criterion is a *balanced incomplete block design* (BIB design). This is a result due to Kiefer (1958, 1975). When b is not a multiple of 7, no BIB design exists and therefore the existing literature is not of much help to the experimenter. But if b is a multiple of 7, the designs do exist. Thus, label the treatments as $1, 2, \cdots, 7$. For $b = 7$, one example of BIB design is

$$\begin{matrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \\ 4 & 5 & 7 \end{matrix} \qquad \begin{matrix} 5 & 6 & 1 \\ 6 & 7 & 2 \\ 7 & 1 & 3 \end{matrix}.$$

If $b = 7t$, one can simply take t copies of the above design. The resulting design consists of only seven distinct blocks and is, therefore, said to have the *support size* 7, (See Definition 2.1 below). There are BIB designs with different support sizes. For example, if $b = 35$, the collection of all $\binom{7}{3} = 35$ possible blocks of size 3 forms a BIB design; and this design has the support size 35.

BIB designs with repeated blocks are useful as experimental designs. To the experimenter the implementation of designs with different support sizes may cost differently. On the other hand, certain mixtures of treatments may be more preferable than others. Besides, BIB designs with repeated blocks can be easily converted into survey designs for controlled sampling as explained in detail in

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Wynn (1977) and Foody and Hedayat (1977). These considerations lead to the search for BIB designs with various support sizes. It is then natural to ask the following question: for $v = 7$, $k = 3$, $b = 7t$, and a given number b^*, does there exist a BIB design consisting of b^* distinct blocks?

In our setting we may require that b^* satisfies the obvious inequalities $b^* \leqslant b$ and $7 \leqslant b^* \leqslant 35$. As we shall see in Section 4, the answer to the above question is basically yes, with a few exceptional cases. The construction of designs or proof of nonexistence of designs heavily rely upon a method called *trade-off*, which is introduced and studied in Section 3.

2. Definitions and notations. Let $V = \{1, 2, \cdots, v\}$ and let $v\Sigma k$ be the set of all distinct subsets of size k based on V. Elements of $v\Sigma k$ will be called blocks. A block of size 2 will be referred to as a pair. The notation for a block of size k consisting of the elements $x_1, x_2, \cdots, x_k$ will be $(x_1x_2 \cdots x_k)$, while the order among the k elements is immaterial.

A *balanced incomplete block design*, d, with parameters v, b, r, k, λ, written BIB(v, b, r, k, λ), is a collection of b elements of $v\Sigma k$ with properties that (i) each element of V occurs in exactly r blocks; (ii) each pair of distinct elements of V appears together in exactly λ blocks. We emphasize that this definition does *not* require that the blocks of a BIB design be distinct. Following Wynn (1977) and Foody and Hedayat (1977), we define

DEFINITION 2.1. The *support* of a BIB design, d, is the collection of distinct blocks in d, denoted by d^*. The number of elements in d^* is denoted by b^* and called the *support size* of d.

We will denote a BIB(v, b, r, k, λ) with support size b^* by BIB$(v, b, r, k, \lambda|b^*)$. A BIB design with $b = b^* = vCk$ is denoted by $d(v, k)$ and referred to as the *trivial* BIB *design* based on v and k. Note that $d(v, k) \equiv v\Sigma k$.

3. The method of trade-off. Any incomplete block design may be specified by the number of times that each element of $v\Sigma k$ is repeated in the design. Thus, order the blocks lexicographically. We write f_i for the frequency of the ith element of $v\Sigma k$ in the design. Identify an incomplete block design with a $\binom{v}{k}$-dimensional column vector $F = (f_1, f_2, \cdots)'$. Conversely a $\binom{v}{k}$-dimensional column vector F with nonnegative integer entries defines a BIB (v, b, r, k, λ) design if

$$PF = \lambda \mathbf{1}.$$

Here P is the incidence matrix of pairs versus blocks, i.e., P is a $\binom{v}{2}$ by $\binom{v}{k}$ matrix with $P_{ij} = 1$ if the ith element of $v\Sigma 2$ is contained in the jth element of $v\Sigma k$ and $P_{ij} = 0$ otherwise. The vector $\mathbf{1}$ is a $\binom{v}{2}$-dimensional vector with all entries equal to 1. The corresponding r and b are found from $r = \lambda(v-1)/(k-1)$ and $b = vr/k$.

DEFINITION 3.1. An integer vector T of dimension $\binom{v}{k}$ is called a (v, k) trade if $PT = 0$. The sum of all positive entries in a trade is called its volume, and v and k are called the parameters of the trade.

Foody and Hedayat (1977) showed that for given v and k the matrix P has rank $\binom{v}{2}$. Therefore, there are precisely $\binom{v}{k} - \binom{v}{2}$ independent (v, k) trades which form a basis of the kernel of P. Note that if F_1 and F_2 are two BIB designs with the same values of v, k, and λ then $F_1 - F_2$ is a (v, k) trade.

The problem of constructing trades is difficult but very important in the theory of BIB designs in general and BIB designs with repeated blocks in particular. Hedayat and Li (1978) presented techniques for constructing trades when $k = 3$. It is important to note that if T is a (v, k) trade then a corresponding (v', k) trade for $v' > v$ can be produced by inserting into T zeros corresponding to the blocks of $v'\Sigma k - v\Sigma k$.

Hereafter if we do not mention the parameters of a trade, either they are immaterial, or they can be deduced from the context.

It is easy to see that:

LEMMA 3.1. *Let F be a* BIB *design. For every trade T, the vector $F + T$ is another* BIB *design with the same parameters provided that all of its entries are nonnegative.*

Also, it is clear that any BIB design sharing the same parameters with F can be written in the form $F + T$ for some trade T. Therefore, in order to search for all BIB designs with the same parameters as F, it suffices to investigate the trades.

If a block $(x_1x_2 \cdots x_k)$ is the ith element of $v\Sigma k$ in the lexicographical order, then this block should be identified with the $\binom{v}{k}$-dimensional column vector whose entries are all zeros except that the ith entry is 1. Thus, if B_j are blocks and t_j are integers, then $\Sigma t_j B_j$ is also identified with a $\binom{v}{k}$-dimensional column vector; this vector is a trade if and only if, for every pair (xy),

$$\Sigma_{j\,:\,B_j \supset (xy)} t_j = 0. \tag{3.1}$$

Hereafter we shall restrict our attention to the case $k = 3$. So a block simply means a triplet.

Through the following examples of trades, we shall familiarize ourselves with the notations defined in the above. Example 3.1 is a trade with the smallest volume. Example 3.2 is a trade with repeated blocks. Example 3.3 exhibits trades of volumes 6, 7 and 9 which will be needed in proving Theorem 3.1.

EXAMPLE 3.1. Let $v = 7$. Then $(125) + (146) + (234) + (356) - (124) - (156) - (235) - (346)$ represents a trade of volume 4. When this trade is added to the BIB design $(124) + (137) + (156) + (235) + (267) + (346) + (457)$, we obtain another BIB design $(125) + (137) + (146) + (234) + (267) + (356) + (457)$. In other words, from the first design the four blocks (124), (156), (235), and (346) have been traded for the blocks (125), (146), (234), and (356) to obtain the second design.

EXAMPLE 3.2. Let $v = 7$. Then $2(123) + (145) + (146) + 2(247) + (357) + (367) - 2(124) - (135) - (136) - 2(237) - (457) - (467)$ represent a trade of volume 8 with repeated blocks.

EXAMPLE 3.3. Let $v = 9$.

(i) A trade of volume 6: $(137) + (145) + (247) + (235) + (346) + (126) - (125) - (147) - (237) - (345) - (246) - (136)$.

(ii) A trade of volume 7: $(127) + (457) + (246) + (367) + (235) + (134) + (156) - (247) - (236) - (137) - (125) - (345) - (146) - (456)$.

(iii) A trade of volume 9: $(124) + (235) + (136) + (457) + (568) + (469) + (178) + (289) + (379) - (245) - (356) - (146) - (578) - (689) - (479) - (128) - (239) - (137)$.

THEOREM 3.1. *For any integer i, there exists a $(v, 3)$ trade of volume i if and only if* $i \neq 1, 2, 3, 5$.

PROOF. Examples 3.1 and 3.3 show the existence of trades of volumes 4, 6, 7 and 9. Adding m copies of a trade of volume 4 to a trade of volume t based on unrelated symbols yields a trade of volume $4m + t$. Every positive integer $i \neq 1, 2, 3, 5$ can be written in the form of $4m + t$ with $t \in \{4, 6, 7, 9\}$. This proves the sufficiency part.

We now prove the necessity part. It is trivial that there are no trades of volumes 1, 2 or 3. Thus, all we have to show is the nonexistence of a trade of volume 5. Assume to the contrary and let

$$T = B_1 + B_2 + B_3 + B_4 + B_5 - \bar{B}_1 - \bar{B}_2 - \bar{B}_3 - \bar{B}_4 - \bar{B}_5$$

be a trade of volume 5 based on v varieties. We claim that (i) v is at most seven; and (ii) the five B blocks cannot be all distinct. (Recall that our definition of a trade does allow repeated blocks). To see (i), note that if a variety appears in some B_i then it must appear in some B_j, $j \neq i$. Thus each variety takes at least two positions out of $5 \times 3 = 15$ positions in the five B blocks. Therefore $v \leqslant 7$. To prove (ii), observe that if the five B blocks are distinct then adding the trade T to the trivial BIB design $d(8, 3)$ would yield a design with $v = 8$, $k = 3$, $b = 56$, and $b^* = 51$. But the latter design does not exist according to a footnote in Foody and Hedayat (1977).

Thus from now on, we assume that $B_1 = B_2$. By symmetry, we may also assume that $\bar{B}_1 = \bar{B}_2$. Let the varieties be represented by the integers $1, 2, \cdots, v$. Write $B_1 = (123)$. We claim that $\bar{B}_1$ must contain two varieties among 1, 2 and 3. Assume to the contrary and let $\bar{B}_1 = (x45)$. Then among the blocks B_3, B_4 and B_5, at least two contain the pair $(x4)$ and at least two contain the pair $(x5)$. This is a contradiction. Now we may assume $\bar{B}_1 = (124)$ without loss of generality. But then at least two blocks among B_3, B_4 and B_5 contain the pair (14), and at least two of them contain the pair (24). Again this is a contradiction. ☐

The following theorem is useful in studying BIB designs with blocks of size 3. In fact we shall repeatedly apply this theorem in the proof of Theorem 4.1.

THEOREM 3.2. *There is no* (7, 3) *trade* T *in the form*

$$T = \Sigma_{i=1}^{8} B_i - \Sigma_{i=1}^{8} \bar{B}_i$$

with properties: (i) *for all* i, B_i *and* $\bar{B}_i$ *are blocks of size* 3 *based on* $\{1, 2, \cdots, 7\}$; (ii) $B_1, B_2, \cdots, B_7$ *form a* BIB(7, 7, 3, 3, 1) *design*; (iii) $B_8 \neq \bar{B}_i$, *for all* i.

PROOF. Let $B_8 = (123)$. Assuming that $\bar{B}_i \neq (123)$ for all i, we shall derive a contradiction. $B_1, B_2, \cdots, B_7$ cover all the 21 possible pairs exactly once each. Thus $B_1, B_2, \cdots, B_7, B_8$ cover the three pairs (12), (13) and (23) doubly and all other pairs singly. By symmetry and (3.1), we may assume that $\bar{B}_1 = (12u)$, $\bar{B}_2 = (12v)$, $\bar{B}_3 = (13w)$, $\bar{B}_4 = (13x)$, $\bar{B}_5 = (23y)$, and $\bar{B}_6 = (23z)$. But the above covering properties imply that u, v, w, x, y, z are distinct elements of the 4-element set $\{4, 5, 6, 7\}$, which is impossible. □

It is tempting to generalize Theorem 3.2 as follows: there is no (7, 3) trade, T, of volume $7t + 1$ in the form

$$T = \Sigma_{i=1}^{7t+1} B_i - \Sigma_{i=1}^{7t+1} \bar{B}_i$$

with properties: (i) for all i, B_i and $\bar{B}_i$ are all blocks of size 3 based on $\{1, 2, \cdots, 7\}$; (ii) $B_1, B_2, \cdots, B_{7t}$ form a BIB($7, 7t, 3t, 3, t$) design; (iii) $B_{7t+1} \neq \bar{B}_i$, for all i. But this generalization fails as the following example shows in case $t = 2$.

EXAMPLE 3.4. Let $\{B_1, B_2, \cdots, B_{14}\}$ be any BIB(7, 14, 6, 3, 2) design, $B_{15} = (123)$, $\bar{B}_1 = (124)$, $\bar{B}_2 = (126)$, $\bar{B}_3 = (127)$, $\bar{B}_4 = (134)$, $\bar{B}_5 = (135)$, $\bar{B}_6 = (137)$, $\bar{B}_7 = (156)$, $\bar{B}_8 = (234)$, $\bar{B}_9 = (235)$, $\bar{B}_{10} = (236)$, $\bar{B}_{11} = (257)$, $\bar{B}_{12} = (367)$, $\bar{B}_{13} = (456)$, $\bar{B}_{14} = (457)$, and $\bar{B}_{15} = (467)$. Then these blocks form the required trade. In fact this example is unique up to isomorphism.

We shall need this example in Section 4 when we construct BIB design with support size 34 from the method of trade off.

4. An application of the method of trade off: BIB designs based on 7 treatments in blocks of size 3. All the designs in this section refer to BIB($7, b, r, 3, \lambda$) designs based on the set of symbols $\{1, 2, \cdots, 7\}$. From the relations $rv = bk$ and $\lambda(v - 1) = r(k - 1)$, one can see that b must be a multiple of 7. Also, we have $r = 3b/7$ and $\lambda = b/7$.

In order to cover all the $\binom{7}{2} = 21$ possible pairs, at least 7 distinct blocks are needed in a design. This means that $b^* \geq 7$. In particular, it is known that every design with $b = b^* = 7$ is isomorphic to a finite projective plane of order 2. The existence or nonexistence of designs with any b and b^* has been determined.

THEOREM 4.1. *For any b and any b^* there exists a* BIB$(7, b, r, 3, \lambda | b^*)$ *design if and only if the following are true*:

(i) b *is divisible by* 7,

(ii) $7 \leqslant b^* \leqslant \min(b, 35)$,

(iii) $b^* \neq 8, 9, 10, 12,$ *or* 16,

(iv) $(b, b^*) \neq (28, 24), (28, 27), (35, 30), (35, 32), (35, 33), (35, 34),$ *or* $(42, 34)$.

PROOF. First we show the necessity of these conditions. Conditions (i) and (ii) are clear from the discussion preceding the theorem. Theorem 3.2 in van Lint and Ryser (1972) shows b^* can never be 8. Pesotchinsky (1977) showed that $b^* \neq 9, 10,$ or 12. Hedayat and Li (1977) conjectured the nonexistence of BIB designs with $b^* = 16$. In response to this conjecture, Seiden (1977) proved that $b^* \neq 16$ if $b = 21$. Recently, Foody (1978) established the truth of the conjecture by showing that $b^* \neq 16$ for any b. These establish the necessity of condition (iii).

In order not to disrupt the continuity of the argument, the rather lengthy proof of the case when $b = 28$ and $b^* = 24$ is deferred to Section 5. We now prove that there is no BIB design, d, based on $b = 28$ blocks consisting of exactly 27 distinct blocks. If such a design exists we may assume, by symmetry, that the unique doubled block in the design is (123). Let the 8 blocks that are missing, in comparison to $d(7, 3)$, be denoted as $\bar{B}_1, \bar{B}_2, \cdots, \bar{B}_8$. Thus $\bar{B}_i \neq (123)$ for all i. Adding any BIB(7, 7, 3, 3, 1) designs to d and subtracting from the result the design $d(7, 3)$, we obtain a trade which does not exist by Theorem 3.2. This contradiction originates from the assumption of the existence of the design d.

When $b = 35$, there are no BIB designs based on exactly 30, 32, 33 or 34 distinct blocks. Because if there existed such a design, its difference from the complete design would be a trade of volume 5, 3, 2, or 1, which does not exist by Theorem 3.1. Thus $(b, b^*) \neq (35, i)$, $i = 30, 32, 33,$ or 34.

All that is left to prove for the necessity part is to show that there is no BIB design with $(b, b^*) = (42, 34)$. Suppose there exists such a design. By symmetry, we assume this design is equal to $d(7, 3) - (123) + \bar{B}_1 + \bar{B}_2 + \cdots + \bar{B}_8$, where $\bar{B}_1, \bar{B}_2, \cdots, B_8$ are blocks not equal to (123). The proof now follows from Theorem 3.2 in the same way as in the case of $(b, b^*) = (28, 27)$.

We now prove the sufficiency by construction. For each feasible b^* a BIB design with minimum possible b is exhibited in Table 1. Examples 4.1 and 4.2 below explain the way the designs in Table 1 were obtained through the method of trade off. Table 1 does not contain all the designs claimed in the theorem. To see that the missing designs exist, note that if there exists a BIB design with b blocks which contains a sub BIB design with 7 blocks then one can construct a design with $b + 7$ blocks and the same support size by simply adding a copy of the sub design to the design. All the designs in Table 1, except when $b = b^* = 21$, contain the BIB design

124	167	256	457
135	237	346	

TABLE 1
BIB *designs with* $v = 7$, $k = 3$ *and all possible support size*, b^*.

b^*	7	11	13	14	15	17	18	19	20	21	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
b	7	14	14	14	21	21	21	21	21	21	28	28	28	35	28	28	35	28	35	42	35	42	42	49	35
123	–	–	–	–	–	–	–	–	1	1	3	1	–	–	–	1	1	1	–	1	1	1	1	–	1
124	1	2	1	1	3	2	2	3	1	1	1	1	2	1	1	1	2	1	1	1	1	1	1	2	1
125	–	–	1	–	–	–	1	–	1	1	–	1	1	–	1	–	1	1	1	1	1	1	1	1	1
126	–	–	–	1	–	1	–	–	–	–	–	–	1	2	1	1	–	–	2	2	2	2	2	2	1
127	–	–	–	–	–	–	–	–	–	–	–	1	–	2	1	1	1	1	1	1	–	1	1	2	1
134	–	–	1	1	–	1	1	–	–	–	–	–	–	1	1	–	1	–	1	1	1	1	2	2	1
135	1	2	1	1	2	2	1	1	1	–	1	1	2	1	1	1	2	1	1	1	1	2	1	2	1
136	–	–	–	–	1	–	1	1	1	1	–	2	1	2	1	1	–	1	2	2	1	1	1	1	1
137	–	–	–	–	–	–	–	1	–	1	–	–	1	1	1	1	1	1	1	1	1	1	1	2	1
145	–	–	–	–	–	–	–	–	1	1	–	2	–	3	1	2	1	1	2	3	1	1	1	1	1
146	–	–	–	–	–	–	–	–	–	–	1	–	1	–	–	–	1	1	–	–	1	2	1	1	1
147	–	–	–	–	–	–	–	–	1	1	2	1	1	–	1	1	–	1	1	1	1	1	1	1	1
156	–	–	–	–	–	–	–	1	–	1	2	–	–	–	1	1	1	1	–	–	–	–	1	2	1
157	–	–	–	1	1	1	1	1	–	–	1	–	1	1	–	–	–	–	1	1	2	2	2	1	1
167	1	2	2	1	2	2	2	1	2	1	1	2	1	1	1	1	3	1	1	2	1	1	1	1	1
234	–	–	–	–	–	–	–	–	1	1	–	1	–	1	1	1	1	1	2	2	1	2	1	2	1
235	–	–	–	1	1	1	1	1	–	–	–	–	1	2	1	1	1	–	2	2	1	1	2	2	1
236	–	1	1	–	1	1	1	1	–	–	–	–	1	–	–	–	1	1	–	–	–	–	–	2	1
237	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	1	1	1	1	1	2	2	2	1	1
245	–	–	–	–	–	–	–	–	–	–	2	–	1	1	1	1	–	1	–	–	1	1	1	1	1
246	–	–	–	–	–	–	–	–	1	1	1	2	–	1	1	1	1	1	1	2	1	1	2	1	1
247	–	–	1	1	–	1	1	–	–	–	–	–	1	1	–	–	1	–	1	1	1	1	1	1	1
256	1	1	1	1	1	1	1	1	1	1	1	2	1	2	1	1	2	1	1	1	1	2	1	1	1
257	–	1	–	–	1	1	–	1	1	1	1	1	–	–	–	1	1	1	1	2	1	1	1	2	1
267	–	–	–	–	1	–	1	1	1	1	2	–	1	–	1	1	1	1	1	1	1	1	1	1	1
345	–	–	–	–	–	–	–	1	–	1	1	1	1	–	–	–	–	1	–	–	1	1	1	1	1
346	1	1	1	1	1	1	1	1	1	1	2	1	2	2	2	2	2	1	1	1	1	1	1	1	1
347	–	1	–	–	2	1	1	1	1	–	1	1	1	1	–	1	1	1	1	2	1	1	1	1	1
356	–	–	–	–	–	–	–	–	1	1	1	1	–	1	1	1	1	1	1	2	2	2	2	1	1
357	–	–	1	–	–	–	1	–	1	1	1	1	–	1	1	1	1	1	1	1	–	–	–	1	1
367	–	–	–	1	–	1	–	–	–	–	1	–	–	–	–	–	1	–	1	1	1	2	2	2	1
456	–	1	1	1	2	2	2	1	1	–	–	–	1	–	–	–	1	–	2	2	1	1	1	2	1
457	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	3	1	1	1	1	2	2	2	1
467	–	–	–	–	–	–	–	1	–	1	–	1	–	2	1	1	–	1	1	1	1	1	1	2	1
567	–	–	–	–	–	–	–	–	–	–	–	1	2	2	1	1	–	1	1	1	1	1	1	1	1

Note: for each support size a BIB design with minimum b is contained in the table.

To construct BIB designs with $b^* = 21$ and $b > 21$ the design with $b^* = 21$ and $b = 28$ should be used. We may mention that no BIB design with $b = b^* = 21$ can contain a sub BIB design. □

EXAMPLE 4.1. Let F denote the design (127) + (134) + (156) + (235) + (246) + (367) + (457) and T the trade (127) + (156) + (236) + (357) − (126) − (157) − (237) − (356). Then $d(7, 3)$ plus F minus T is a BIB design with 42 blocks. Since only the two blocks (236) and (357) are missing from the design, the support size is 33. This design is in Table 1.

EXAMPLE 4.2. Let B_i and $\bar{B}_i$, $1 \leqslant i \leqslant 15$, be blocks defined in Example 3.4. Let F denote the design $B_1 + \cdots + B_{14}$ and T the trade $B_1 + \cdots + B_{15} - \bar{B}_1 - \cdots - \bar{B}_{15}$. Then $d(7, 3)$ plus F minus T is a design with $b = \binom{7}{3} + 14 = 49$. Only the block (123) is missing from this design. So the support size is 34. This design is in Table 1, too.

5. Nonexistence of BIB design with $v = 7$, $k = 3$ and $(b, b^*) = (28, 24)$. The proof is by contradiction. Assume that such a design exists and let $C_1, C_2, \cdots, C_{11}$ be the eleven blocks missing from the design. Let B_1, B_2, B_3, and B_4 be the remainders when 24 distinct blocks are removed from the design. Then $C_1 + \cdots + C_{11}$ covers each of the $\binom{7}{2}$ pairs exactly one more time than $B_1 + B_2 + B_3 + B_4$ does. From this observation we want to deduce some necessary conditions on the B blocks and the C blocks.

LEMMA 5.1. *B_1, B_2, B_3 and B_4 are distinct blocks.*

PROOF. Let $B_1 = B_2 = (123)$. We want to derive a contradiction.

There are three C blocks containing the pair (12), three containing (13), and three containing (23). These nine blocks are all distinct, because (123) is not a C block. The remaining two C blocks must cover (45) + (46) + (47) + (56) + (57) + (67), which is impossible. □

COROLLARY 5.1. *$B_1, B_2, B_3, B_4, C_1, \cdots, C_{11}$ are all distinct blocks.*

LEMMA 5.2. *No pair is contained in more than two B blocks.*

PROOF. Assume that the pair (12) is contained in three B blocks. So there are four C blocks containing this pair. But there are simply no seven distinct blocks containing one single pair. □

LEMMA 5.3. *At most one pair is doubly covered by $B_1 + B_2 + B_3 + B_4$.*

PROOF. Assume that there are at least two pairs that are doubly covered by $B_1 + B_2 + B_3 + B_4$. Renaming the varieties if necessary, we may assume that one of the following occurs.

(i) $B_1 = (123)$, $B_2 = (234)$, $B_3 = (134)$.
(ii) $B_1 = (123)$, $B_2 = (234)$, $B_3 = (345)$.
(iii) $B_1 = (123)$, $B_2 = (234)$, $B_3 = (356)$, $B_4 = (357)$.
(iv) $B_1 = (123)$, $B_2 = (234)$, $B_3 = (145)$, $B_4 = (456)$.
(v) $B_1 = (123)$, $B_2 = (234)$, $B_3 = (456)$, $B_4 = (457)$.
(vi) $B_1 = (123)$, $B_2 = (234)$, $B_3 = (456)$, $B_4 = (567)$.

CASE (i). The pairs (13), (23) and (34) are at least triply covered by C blocks. So nine of the C blocks must be (135), (136), (137), (235), (236), (237), (345), (346), and (347). In particular, there are three C blocks containing the pair (35). But (35) can be contained in at most one B block.

CASE (ii). The pairs (23) and (34) are at least triply covered by the C blocks. We may assume that $C_1 = (235)$, $C_2 = (236)$, $C_3 = (237)$, $C_4 = (134)$, $C_5 = (346)$, and $C_6 = (347)$. In particular, the pairs (36) and (37) are doubly covered by these blocks. So B_4 must be (367). But then $C_7 + C_8 + C_9 + C_{10} + C_{11}$ doubly covers (12) and (24) but does not cover (23), (25), (26), or (27). Thus, two of the C blocks must both be equal to (124). A contradiction!

CASE (iii). The pairs (23) and (35) are triply covered by the C blocks. We may assume that $C_1 = (235)$, $C_2 = (236)$, $C_3 = (237)$, $C_4 = (135)$, and $C_5 = (345)$. So $C_6 + \cdots + C_{11}$ doubly covers the pairs (56) and (57), but does not cover the pairs (15), (25), (35), (45). Thus two of the C blocks must both be equal to (567). A contradiction!

CASE (iv). We may assume that $C_1 = (235)$, $C_2 = (236)$, $C_3 = (237)$, $C_4 = (245)$, $C_5 = (345)$, and $C_6 = (457)$. But then (25) must be covered by a B block. This is a contradiction.

Cases (v) and (vi) are similar to the preceding case. The proof of Lemma 5.3 is completed. □

LEMMA 5.4. *No pair is doubly covered by* $B_1 + B_2 + B_3 + B_4$.

PROOF. Assuming that $B_1 = (123)$ and $B_2 = (124)$, we shall derive a contradiction. We may let $C_1 = (125)$, $C_2 = (126)$, $C_3 = (127)$. Claim that (134) is a C block. Assume (134) is not. Since the C blocks at least doubly cover the pairs (13) and (14), four of them are $(13w)$, $(13x)$, $(14y)$, and $(14z)$, where w, x, y, z all belong to the set $\{5, 6, 7\}$. Let $v \in \{w, x\} \cap \{y, z\}$. Then $(1v)$ is at least triply covered by the C blocks. Therefore the pair $(1v)$, as well as the pair (12), is doubly covered by the B blocks, contradicting Lemma 5.3. The claim is justified.

Now we may set $C_4 = (134)$. Similarly, we may set $C_5 = (234)$. Therefore (34) is contained in a B block, and we may set $B_3 = (345)$ without loss of generality. From Lemma 5.2, the block B_4 can not contain both the varieties 1 and 2. Let us assume that B_4 does not contain the variety 1. So the blocks $C_6, \cdots, C_{11}$ must cover the pairs (13), (14) but not the pairs (12), (15), (16), (17). One of these six C blocks must then be (134), which coincides with C_4. A contradiction! The lemma is proved.

From Lemma 5.4, we conclude that the following are the only possible nonisomorphic types of the B blocks.

Type 1. (125), (136), (147), and (234).

Type 2. (125), (136), (234), and (456).

Type 3. (125), (136), (234), and (457).

The lemmas in the sequel will show the impossibility of all these types and therefore conclude the nonexistence of a BIB design with $v = 7$, $k = 3$, $b = 28$, and $b^* = 24$.

LEMMA 5.5. *The eleven C blocks do not include any* BIB(7, 7, 3, 3, 1) *design*.

PROOF. Assume that $C_5, C_6, \cdots, C_{11}$ form a BIB design D. Then $B_1 + B_2 + B_3 + B_4 - C_1 - C_2 - C_3 - C_4$ is a trade of volume 4. Since there is only one type of trade of volume 4, we may take $B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4$ to be (124), (135), (236), (456), (123), (145), (246), (356), respectively. It then suffices to show that every BIB(7, 7, 3, 3, 1) design must contain one of these eight blocks. But this is straightforward to verify, because a BIB(7, 7, 3, 3, 1) design is isomorphic to a projective plane of order two. We leave out the trivial details in the proof. □

LEMMA 5.6. *The B blocks can not be of Type* 1 *or Type* 2.

PROOF. Assume that the B blocks are of either Type 1 or Type 2. Then we can find three blocks which, together with the B blocks, form a BIB design. Let us denote these blocks by C_{12}, C_{13} and C_{14}. It follows that $\{C_1, C_2, \cdots, C_{14}\}$ is a BIB(7, 14, 6, 3, 2) design with support size 14. It is an easy fact that *such a design can be split into two* BIB (7, 7, 3, 3, 1) *designs*. We may then assume that $\{C_6, \cdots, C_{11}, C_j\}$ is a design where $j = 5$ or 12. But, from Lemma 5.5, we know $j \neq 5$, i.e., $j = 12$. This implies that $B_1 + B_2 + B_3 + B_4 + C_{12} - C_1 - C_2 - C_3 - C_4 - C_5$ is a trade of volume 5, but such a trade does not exist according to Theorem 3.1. □

LEMMA 5.7. *The B blocks can not be of Type* 3.

PROOF. Assume that the B blocks are (125), (136), (234), and (457). There are exactly four C blocks containing the variety 7, and they cover the pairs (17), (27), (37), (47), (47), (57), (57), (67). So we may assume that they are $(47w)$, $(47x)$, $(57y)$, and $(57z)$, where $\{w, x, y, z\} = \{1, 2, 3, 6\}$. Similarly, the four C blocks containing the variety 6 are $(16s)$, $(16t)$, $(36u)$, and $(36v)$, where $\{s, t, u, v\} = \{2, 4, 5, 7\}$. Consequently the unique C block covering (67) must contain a variety between 4 and 5 and also a variety between 1 and 3. This is, of course, impossible. □

6. Closing remarks. An obvious way of obtaining a BIB design with repeated blocks is by combining together two smaller designs that overlap on some blocks. This technique has been incorporated in several constructions in the previous sections and has been used by Hedayat and Federer (1974) in other contexts. But a technique based on composition alone does not always serve for the purpose of constructing new designs. For example, since no BIB design based on $v = 8$, $k = 3$ with $b < \binom{8}{3}$ exists, composition techniques do not apply to the construction of designs with $\binom{8}{3}$ blocks. In some cases a design may not contain a subdesign although smaller designs with the same v and k exist. For example, the design with $b = b^* = 21$ in Table 1 has this property. However, in constructing BIB designs with reduced support sizes, the method of trade off is essentially assumption free.

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