The Annals of Statistics
1979, Vol. 7, No. 6, 1256–1263

ON THE ASYMPTOTIC EFFICIENCY OF CONDITIONAL TESTS FOR EXPONENTIAL FAMILIES

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Let P_η, $\eta \in \Theta \times \Gamma \subset \mathbb{R} \times \mathbb{R}^k$, be an exponential family. It is shown that the sequence of tests $(\varphi_n^*)_{n\in\mathbb{N}}$, where φ_n^*, $n \in \mathbb{N}$, is u.m.p. in the class of all tests similar with respect to the nuisance-parameter γ for the hypothesis $\{P^n_{(\theta,\gamma)}: \gamma \in \Gamma\}$ against alternatives $P^n_{(\theta_1,\eta_1)}$, $\theta_1 > \theta$, $\eta_1 \in \Gamma$, is asymptotically efficient in the class Φ_α^* of test-sequences which are asymptotically of level α (continuously in the nuisance-parameter). Here, asymptotic efficiency of $(\varphi_n^*)_{n\in\mathbb{N}}$ means that for all $\gamma \in \Gamma$, $t > 0$, the power of φ_n^* evaluated at local alternatives $P^n_{(\theta+tn^{-1/2},\gamma)}$ asymptotically attains the upper bound given for test-sequences in Φ_α^*.

1. Introduction. The present study is an answer to the problem of determining an exact asymptotic upper bound for the power of tests of Neyman structure for exponential families, where the power is evaluated at local (contiguous) alternatives. Let $(\varphi_n^*)_{n\in\mathbb{N}}$ be the test-sequence such that, for every sample size n, φ_n^* is the optimal test of Neyman structure, which is uniformly most powerful against one-sided alternatives in the class of all tests similar with respect to the nuisance parameter (for the representation of φ_n^* see formula (6) below). It is shown that $(\varphi_n^*)_{n\in\mathbb{N}}$ is asymptotically efficient in the class of all test-sequences, which are asymptotically of level α (continuously in the nuisance parameter).

The starting problem of our research has been the following: when the number of observations is large, do we lose power at local alternatives if we use the best similar test instead of a test which is asymptotically efficient in the class of all test-sequences which are asymptotically of level α? In Theorems 1 and 2 below we show that the answer to this question is "No".

To begin with let us fix some notations: Let $(X, \mathcal{Q}, \nu)$ be a σ-finite measure-space and assume that $P_\eta|\mathcal{Q}$, $\eta = (\theta, \gamma) \in \Theta \times \Gamma$, where $\Theta \subset \mathbb{R}$ and $\Gamma \subset \mathbb{R}^k$ are open, is a family of probability-measures with ν-densities of the form

$$p(x, \eta) = c(\eta)\exp\left[\theta S_1(x) + \Sigma_{j=1}^k \gamma_j S_{j+1}(x)\right]$$

(θ will be the parameter under consideration, whereas γ is a "nuisance-parameter").

Let

(1) $$\Sigma(\eta) = \left(\sigma_{ij}(\eta)\right)_{i,j=1,\cdots,k+1},$$

Received March 1978; revised August 1978.

AMS 1970 *subject classifications.* Primary 62F05, 62F20.

Key words and phrases. Exponential families, similar tests, Neyman structure, asymptotic efficiency, contiguous alternatives.

where $\sigma_{ij}(\eta) = \text{Cov}_\eta(S_i, S_j)$, $i, j = 1, \cdots, k+1$. Let, furthermore,

(2) $$\Lambda(\eta) = (\Lambda_{ij}(\eta))_{i,j=1,\cdots,k+1} = I(\eta)^{-1}$$

where $I(\eta) = (I_{ij}(\eta))_{i,j=1,\cdots,k+1}$ denotes the Fisher information matrix, *which is assumed to be positive definite* (recall that $I_{ij}(\eta) = \int(\partial/\partial\eta_i)\log p(x, \eta) \cdot (\partial/\partial\eta_j)\log p(x, \eta)P_\eta(dx)$, $i, j = 1, \cdots, k+1$. It easily follows that $I(\eta) = \Sigma(\eta)$).

The following partitions of Σ and Λ are needed in the proofs of the theorems: let

$$\Sigma = \begin{pmatrix} \sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda^{(12)} \\ \Lambda^{(21)} & \Lambda^{(22)} \end{pmatrix}$$

where Σ_{22} and $\Lambda^{(22)}$ are $(k \times k)$-matrices, $\Sigma_{21}, \Lambda^{(21)} \in \mathbb{R}^k$, and $\Sigma_{12} = \Sigma_{21}'$, $\Lambda^{(12)} = \Lambda^{(21)'}$ (a' denotes the transposed vector corresponding to $a \in \mathbb{R}^k$). Then we have (see Steck, page 253),

(3) $$\sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \Lambda_{11}^{-1}.$$

Φ denotes the distribution function of the standard-normal distribution, i.e.,

$$\Phi(t) = (2\pi)^{-1/2}\int_{-\infty}^{t} \exp\left[-\tfrac{1}{2}r^2\right] dr$$

and

$$N_\alpha = \Phi^{-1}(\alpha), \qquad \alpha \in (0, 1).$$

For an $\mathcal{Q}^n$-measurable function f: $X^n \to \mathbb{R}$ we write

$$P_\eta^n(f) = \int f(\mathbf{x})P_\eta^n(d\mathbf{x}), \qquad \mathbf{x} = (x_1, \cdots, x_n),$$

where $P_\eta^n|\mathcal{Q}^n$ is the independent product of n identical component $P_\eta|\mathcal{Q}$.

Given a large sample-size n, we are interested in testing the (composite) hypothesis $\{P_{(\theta,\gamma)}^n: \gamma \in \Gamma\}$, where $\theta \in \Theta$ is fixed, against (local) alternatives $P_{(\theta+tn^{-1/2},\gamma)}^n$, $t > 0$, $\gamma \in \Gamma$, at a level $\alpha \in (0, 1)$.

Within the framework of this paper the concept of asymptotic efficiency of test-sequences deals with the following problem: a specified class Φ_0 of test-sequences $(\varphi_n)_{n\in\mathbb{N}}$ is taken into consideration. First one concentrates on determining an upper bound $B(t, \theta, \gamma)$, say, such that for all $(\varphi_n)_{n\in\mathbb{N}} \in \Phi_0$, and all $t > 0$, $\gamma \in \Gamma$,

$$\limsup_{n\to\infty} P_{(\theta+tn^{-1/2},\gamma)}^n(\varphi_n) \leqslant B(t, \theta, \gamma)$$

and then one attempts to prove that there does not exist a smaller upper bound, i.e., one tries to exhibit a test-sequence $(\tilde{\varphi}_n)_{n\in\mathbb{N}} \in \Phi_0$ such that for all $t > 0$, $\gamma \in \Gamma$

$$\lim_{n\to\infty} P_{(\theta+tn^{-1/2},\gamma)}^n(\tilde{\varphi}_n) = B(t, \theta, \gamma).$$

The sequence $(\tilde{\varphi}_n)_{n\in\mathbb{N}}$ then is called *asymptotically efficient in* Φ_0.

The adequate class Φ_0 for our further considerations is the class Φ_α^* of all test-sequences $(\varphi_n)_{n\in\mathbb{N}}$ (depending on θ and α) which are *asymptotically of level* α *continuously in the nuisance-parameter* γ, i.e., for all $\gamma \in \Gamma$,

(4) $$\limsup_{n\to\infty} P_{(\theta,\gamma_n)}^n(\varphi_n) \leqslant \alpha, \qquad \text{whenever} \quad \lim_{n\to\infty}\gamma_n = \gamma.$$

2. The result. It has been shown by Neyman and more generally by Lehmann and Scheffé (1950) that a test similar for the hypothesis $\{P^n_{(\theta,\gamma)}: \gamma \in \Gamma\}$ must have what these latter authors call "Neyman structure", i.e., its conditional expectation, given $\Sigma_{i=1}^n(S_2(x_i), \cdots, S_{k+1}(x_i))' = u$, is constant for all possible values of $u \in \mathbb{R}^k$. Furthermore, for every sample size n, there exists a best similar test φ_n^* (depending on θ and α), where in this case "best" means that φ_n^* is uniformly most powerful against alternatives $P^n_{(\theta_1,\gamma_1)}$ with $\theta_1 > \theta$ and $\gamma_1 \in \Gamma$.

With

$$S_1^{(n)}(\mathbf{x}) = \Sigma_{i=1}^n S_1(x_i) \quad \text{and} \quad S^{(n)}(\mathbf{x}) = \Sigma_{i=1}^n(S_2(x_i), \cdots, S_{k+1}(x_i))' \tag{5}$$

we have

$$\varphi_n^* = 1_{A_n} + \gamma_n 1_{B_n} \tag{6}$$

with

$$A_n = \{\mathbf{x} \in X^n: S_1^{(n)}(\mathbf{x}) > c^\theta_{n,\alpha}(S^{(n)}(\mathbf{x}))\}$$

$$B_n = \{\mathbf{x} \in X^n: S_1^{(n)}(\mathbf{x}) = c^\theta_{n,\alpha}(S^{(n)}(\mathbf{x}))\} \tag{7}$$

$$\gamma_n = \gamma^\theta_{n,\alpha}(S^{(n)}(\mathbf{x})) \in [0, 1],$$

where the functions $c^\theta_{n,\alpha}(u)$ and $\gamma^\theta_{n,\alpha}(u)$ are chosen in such a way that the conditional expectation of φ_n^*, given $S^{(n)} = u$, equals α for every possible value of $u \in \mathbb{R}^k$.

Let

$$\Phi_\alpha^- = \cap_{\gamma\in\Gamma}\{(\varphi_n)_{n\in\mathbb{N}}: P^n_{(\theta,\gamma)}(\varphi_n) = \alpha, n \in \mathbb{N}\}.$$

Obviously, Φ_α^- is a subclass of Φ_α^*, the class of all test-sequences fulfilling (4).

The following Theorem 1 is due to Chibisov (1973) [see also Pfanzagl (1978)]: Both authors give results on more general families of probability-measures, but consider a slightly smaller class of test-sequences than Φ_α^*.

THEOREM 1. *For every sequence $(\varphi_n)_{n\in\mathbb{N}} \in \Phi_\alpha^*$ (see (4)), for all $\gamma \in \Gamma$ and all $t > 0$,*

$$\limsup_{n\to\infty} P^n_{(\theta+tn^{-1/2},\gamma)}(\varphi_n) \leqslant \Phi(N_\alpha + t\Lambda_{11}(\theta,\gamma)^{-1/2})$$

with $\Lambda_{11}(\theta,\gamma)$ given by (2).

Theorem 2 below provides us with a lower bound for the asymptotic power at local alternatives of $(\varphi_n^*)_{n\in\mathbb{N}}$ (where φ_n^*, $n \in \mathbb{N}$, is the optimal test of Neyman structure), which coincides with the upper bound given in Theorem 1.

THEOREM 2. *Assume that either the distribution of $(S_2, \cdots, S_{k+1})'$ is a lattice-distribution or that $\nu|\mathcal{A} = \lambda|\mathfrak{B}^{k+1}$, where λ is the Lebesgue-measure. Assume, furthermore, in the latter case that there exists $r > 0$ such that*

$$\int_{\mathbb{R}^k}|\varphi_\eta(u, v)|^r\lambda(dv) < \infty$$

for sufficiently small $|u|$, *where* $\varphi_\eta(u, v)$, $u \in \mathbb{R}$, $v \in \mathbb{R}^k$, *denotes the characteristic function of* $(S_1, \cdots, S_{k+1})$. *Let* $(\varphi_n^*)_{n\in\mathbb{N}} \in \Phi_\alpha^=$ *be the sequence of tests* φ_n^*, $n \in \mathbb{N}$, *given by* (6). *Then for every* $\gamma \in \Gamma$ *and all* $t > 0$,

$$\liminf_{n\to\infty} P^n_{(\theta+tn^{-1/2},\gamma)}(\varphi_n^*) \geqslant \Phi\big(N_\alpha + t\Lambda_{11}(\theta,\gamma)^{-1/2}\big),$$

where $\Lambda_{11}(\theta,\gamma)$ *is given according to* (2).

Hence, if we combine Theorems 1 and 2, we see that the upper bound given in Theorem 1 is exact in the sense that it is impossible to prove a smaller one and that, for the test sequence $(\varphi_n^*)_{n\in\mathbb{N}}$ in Theorem 2 and for all $t > 0$, $\gamma \in \Gamma$,

$$\lim_{n\to\infty} P^n_{(\theta+tn^{-1/2},\gamma)}(\varphi_n^*) = \Phi\big(N_\alpha + t\Lambda_{11}(\theta,\gamma)^{-1/2}\big),$$

i.e., $(\varphi_n^*)_{n\in\mathbb{N}} \in \Phi_\alpha^=$ is *asymptotically efficient in the larger class* Φ_α^*.

3. Proofs of the theorems.

PROOF OF THEOREM 1. Since the basic idea of this result, i.e., to choose as "least favourable distribution" on the hypothesis the probability measure concentrated on $P^n_{(\theta,\gamma+tn^{-1/2}\Sigma_{22}^{-1}\Sigma_{21})}$, is due to Chibisov (1973, Theorem 9.1) we shall only indicate the main steps of the proof.

Let

$$A_n = \{\mathbf{x} \in X^n : l_n(\mathbf{x}) \geqslant -N_\alpha t\Lambda_{11}^{-1/2} - \tfrac{1}{2}t^2\Lambda_{11}^{-1}\},$$

where

$$l_n(\mathbf{x}) = \Sigma_{i=1}^n \log\frac{p(x_i, \theta + tn^{-1/2}, \gamma)}{p(x_i,\theta,\gamma_n)} \qquad \text{with} \quad \gamma_n = \gamma + tn^{-1/2}\Sigma_{22}^{-1}\Sigma_{21}$$

and write $P_n = P^n_{(\theta,\gamma_n)}$ and $P_n' = P^n_{(\theta+tn^{-1/2},\gamma)}$.

From Witting-Nölle, A 6.4, page 183, we immediately obtain

$$\mathcal{L}(l_n|P_n) \Rightarrow N\big(-\tfrac{1}{2}t^2\Lambda_{11}^{-1}, t^2\Lambda_{11}^{-1}\big). \tag{8}$$

This implies $\lim_{n\to\infty} P_n(A_n) = \alpha$. Hence, by Satz 2.15 in Witting-Nölle, page 56, for all $(\varphi_n)_{n\in\mathbb{N}} \in \Phi_\alpha^*$,

$$\liminf_{n\to\infty}\big(P_n'(A_n) - P_n'(\varphi_n)\big) \geqslant 0. \tag{9}$$

By Korollar 2.22 in Witting-Nölle, page 63, (8) implies

$$\mathcal{L}(l_n|P_n') \Rightarrow N\big(\tfrac{1}{2}t^2\Lambda_{11}^{-1}, t^2\Lambda_{11}^{-1}\big).$$

Hence,

$$\lim_{n\to\infty} P_n'(A_n) = \Phi\big(N_\alpha + t\Lambda_{11}^{-1/2}\big). \tag{10}$$

Now (9) and (10) give the desired result.

PROOF OF THEOREM 2. (i) Let

$$\bar{S}_1^{(n)}(\mathbf{x}) = n^{-1/2}\Sigma_{i=1}^n\big(S_1(x_i) - E_\eta S_1\big) \tag{11}$$

$$\bar{S}^{(n)}(\mathbf{x}) = n^{-1/2}\Sigma_{i=1}^n\big(S_2(x_i) - E_\eta S_2, \cdots, S_{k+1}(x_i) - E_\eta S_{k+1}\big)'$$

and, for $u \in \{\bar{S}^{(n)}(\mathbf{x}): \mathbf{x} \in X^n\} \subset \mathbb{R}^k$,

$$(12) \qquad \bar{c}_n(u) = \bar{c}_{n,\alpha}(u, \eta) = n^{-1/2} c^{\theta}_{n,\alpha}\big(n^{1/2}u + n(E_\eta S_2, \cdots, E_\eta S_{k+1})'\big) - n^{1/2}E_\eta S_1,$$

where $c^{\theta}_{n,\alpha}$ given according to (7) is the cut-off point of the test φ_n^*.

To simplify further notations let

$$P_n = P^n_{(\theta,\gamma)} \quad \text{and} \quad P'_n = P^n_{(\theta + tn^{-1/2}, \gamma)}.$$

Let

$$T_n = \begin{pmatrix} \bar{S}_1^{(n)} \\ \bar{S}^{(n)} \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 1 \\ -\Sigma_{22}^{-1}\Sigma_{21} \end{pmatrix}.$$

In Part (iii) of the proof we shall show that

$$(13) \qquad \bar{S}_1^{(n)} - \bar{c}_n(\bar{S}^{(n)}) - d'T_n \to \Lambda_{11}^{-1/2}N_\alpha \quad \text{in } P'_n\text{-probability}.$$

Hence, Slutzky's theorem implies that $\mathfrak{L}(\bar{S}_1^{(n)} - \bar{c}_n(\bar{S}^{(n)})|P'_n)$ has the same weak limit as $\mathfrak{L}(d'T_n + \Lambda_{11}^{-1/2}N_\alpha|P'_n)$. Therefore, in order to obtain a result on the asymptotic behavior of $P'_n(\varphi_n^*)$, it suffices to look at the weak limit of $\mathfrak{L}(d'T_n|P'_n)$ (see (6), (7), (11) and (12)).

(ii) From the expansion

$$\log c(\theta + tn^{-1/2}, \gamma) = \log c(\eta) - tn^{-1/2}E_\eta S_1 - \tfrac{1}{2}t^2 n^{-1}\sigma_{11}(\eta) + o(n^{-1})$$

(see, e.g., Witting-Nölle, A 6.4, page 183) we immediately obtain

$$(14) \qquad l_n - h'T_n \to -\tfrac{1}{2}h'\Sigma h \quad \text{in } P_n\text{-probability}$$

where $h = (t, 0)' \in \mathbb{R}^{k+1}$ and

$$l_n(\mathbf{x}) = \Sigma_{i=1}^n \log \frac{p(x_i, \theta + tn^{-1/2}, \gamma)}{p(x_i, \theta, \gamma)}.$$

Since $\mathfrak{L}(T_n|P_n) \Rightarrow N(0, \Sigma)$, (14) implies by Theorem 7.2 in Roussas, page 38, that

$$(15) \qquad \mathfrak{L}(T_n|P'_n) \Rightarrow N(\Sigma h, \Sigma).$$

(Observe that the contiguity-condition in Roussas' theorem is superfluous because of his assumptions (7.3) and (7.4), i.e., these conditions yield that $\{P'_n\}$ is contiguous to $\{P_n\}$ (use, e.g. Satz 2.20 in Witting-Nölle, page 61)).

Now (15) implies

$$(16) \qquad \mathfrak{L}(d'T_n|P'_n) \Rightarrow N(d'\Sigma h, d'\Sigma d) = N(t\Lambda_{11}^{-1}, \Lambda_{11}^{-1})$$

where the last equality follows from (3).

From (6), (7), (11), (12), (13) together with the lines following (13), and (16) we finally obtain our result

$$\begin{aligned}\liminf_{n\to\infty} P_n'(\varphi_n^*) &\geqslant \lim_{n\to\infty} P_n'\{\mathbf{x} \in X^n: \bar{S}_1^{(n)}(\mathbf{x}) - \bar{c}_n(\bar{S}^{(n)}(\mathbf{x})) > 0\} \\ &= \lim_{n\to\infty} P_n'\{\mathbf{x} \in X^n: d'T_n(\mathbf{x}) > -\Lambda_{11}^{-1/2} N_\alpha\} \\ &= \Phi(N_\alpha + t\Lambda_{11}^{-1/2}).\end{aligned}$$

The main device in proving this result is the stochastic expansion (13) of the test-statistic $\bar{S}_1^{(n)} - \bar{c}_n(\bar{S}^{(n)})$. Since the expansion is of the form

$$a(\eta)e_1\Lambda(\eta)\Big(n^{-1/2}\Sigma_{i=1}^n \frac{\partial}{\partial \eta_1}\log p(x_i, \eta), \cdots, n^{-1/2}\Sigma_{i=1}^n \frac{\partial}{\partial \eta_{k+1}}\log p(x_i, \eta)\Big)',$$

where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^{k+1}$, the result immediately follows from Pfanzagl (1978). We have preferred to give a short outline of the proof, as our derivation shows that one can prove the result under slightly weaker conditions by using contiguity arguments. (Nevertheless in our special situation Pfanzagl's assumptions obviously are fulfilled).

(iii) It now remains to prove (13). From (6), (7), (11), (12), and the paragraph following (7) we obtain for all $u \in A_n(\eta) := \{\bar{S}^{(n)}(\mathbf{x}): \mathbf{x} \in X^n\} \subset \mathbb{R}^k$,

$$P_\eta^n(\bar{S}_1^{(n)} > \bar{c}_{n,\alpha}(u, \eta) | \bar{S}^{(n)} = u) \leqslant \alpha$$

and

$$P_\eta^n(\bar{S}_1^{(n)} \geqslant \bar{c}_{n,\alpha}(u, \eta) | \bar{S}^{(n)} = u) \geqslant \alpha.$$

Hence, by Theorem 2.4 in Steck (1957, page 256) for every $R > 0$,

$$\lim_{n\to\infty}\sup_{u\in K_n(\eta, R)}|\alpha - P(X > \bar{c}_{n,\alpha}(u, \eta) | Y = u)| = 0, \tag{17}$$

where $K_n(\eta, R) = \{u \in A_n(\eta): \|u\| \leqslant R\}$ and where the joint distribution of (X, Y) is a $(k + 1)$-dimensional normal distribution with mean-vector zero and covariance-matrix $\Sigma(\eta)$. Hence,

$$P(X > \bar{c}_{n,\alpha}(u, \eta) | Y = u) = \Phi(d_{n,\alpha}(u, \eta)) \tag{18}$$

with

$$d_{n,\alpha}(u, \eta) = \Lambda_{11}(\eta)^{1/2}[\Sigma_{12}(\eta)\Sigma_{22}(\eta)^{-1}u - \bar{c}_{n,\alpha}(u, \eta)] \tag{19}$$

(here we have used the fact that $\Lambda_{11}^{-1} = \sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ (see (3))).

From (17) and (18) we obtain that for every $R > 0$,

$$\lim_{n\to\infty}\sup_{u\in K_n(\eta, R)}|\alpha - \Phi(d_{n,\alpha}(u, \eta))| = 0.$$

This immediately implies that for every $R > 0$,

$$\lim_{n\to\infty}\sup_{u\in K_n(\eta, R)}|N_\alpha - d_{n,\alpha}(u, \eta)| = 0. \tag{20}$$

Now given positive functions f_n, $n \in \mathbb{N}$, such that for all $R > 0$, $\lim_{n\to\infty} f_n(R) = 0$, there exists a sequence $(R_n)_{n\in\mathbb{N}}$ of positive numbers with $\lim_{n\to\infty} R_n = \infty$ such that $\lim_{n\to\infty} f_n(R_n) = 0$. (Hint: for every $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that for every $n \geqslant N_k$, $f_n(k) < 1/k$. W.l.o.g. we may assume that $N_k < N_{k+1}$, $k \in \mathbb{N}$. Let $k(n) = 1$, if $n < N_1$, and $k(n) = \max\{k\colon N_k \leqslant n\}$, if $n \geqslant N_1$. With $R_n = k(n)$, $n \in \mathbb{N}$, we then obtain $\lim_{n\to\infty} R_n = \infty$ and, for all $n \geqslant N_1$, $f_n(R_n) < 1/R_n$.)

Therefore, there exist constants $R_n(\eta)$, $n \in \mathbb{N}$, with $\lim_{n\to\infty} R_n(\eta) = \infty$ such that (20) holds true with R replaced by $R_n(\eta)$. Since (15) implies

$$\lim_{n\to\infty} P_n'\{\mathbf{x} \in X^n\colon \|\bar{S}^{(n)}(\mathbf{x})\| > R_n(\eta)\} = 0$$

(recall that $\bar{S}^{(n)}$ represents the last k components of T_n), (13) follows from (20) (with R replaced by $R_n(\eta)$) and (19).

4. Concluding remark. Having shown that the sequence $(\varphi_n^*)_{n\in\mathbb{N}}$ of optimal tests of Neyman structure is asymptotically efficient in the class Φ_α^*, i.e., *first order efficient*, the problem remains to be investigated whether $(\varphi_n^*)_{n\in\mathbb{N}}$ is *second order efficient* in the class $\Phi_\alpha^{**} \subset \Phi_\alpha^*$ of all test-sequences $(\varphi_n)_{n\in\mathbb{N}}$ with the property that, for all $\gamma \in \Gamma$,

$$\limsup_{n\to\infty} n^{1/2}\big(P^n_{(\theta,\gamma_n)}(\varphi_n) - \alpha\big) \leqslant 0,$$

whenever $\lim_{n\to\infty}\gamma_n = \gamma$. (For the definitions of first and second order efficiency see Definition 7.1 in Pfanzagl (1978)).

Here our main problem is whether it is possible to prove an asymptotic expansion for the conditional distribution of $\bar{S}_1^{(n)}$ given $\bar{S}^{(n)}$, i.e., whether in Theorem 2.4 of Steck (used in part (iii) of the proof of Theorem 2) the accuracy of the normal approximation can be improved by adding the $n^{-1/2}$-term to the limiting distribution.

Acknowledgment. The author is indebted to Professor J. Pfanzagl for drawing his attention to the considered problem. Furthermore, he wishes to thank Professor H. Witting for suggestions on the proofs, which led to considerable improvement in presentation.

Note added in proof. In the meantimė these problems have been solved by the present authors in two papers to appear in **Journal of Multivariate Analysis** (1979).

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