The Annals of Statistics
1979, Vol. 7, No. 6, 1246–1255

APPROXIMATE AND LOCAL BAHADUR EFFICIENCY OF LINEAR RANK TESTS IN THE TWO-SAMPLE PROBLEM.

BY ERHARD KREMER

University of Hamburg

For linear rank tests in the two-sample case the concept of approximate Bahadur efficiency (BE) is developed, and as the main result of this paper the equality of the approximate and exact local BE is shown. According to a result of Wieand, local approximate BE equals Pitman efficiency under rather general conditions and as a consequence these three approaches to efficiency generally coincide for the class of linear rank tests.

1. Introduction. In 1960/67 Bahadur introduced a concept of approximate and of exact relative efficiency for the asymptotic comparison of two tests. Since in the beginning the theory of large deviations (cf. Sethuraman (1970)), forming the base for the exact concept, was not deeply enough explored, in most examples the easier approximate concept was initially applied. But examples showed that for alternatives far from the null hypothesis the approximate and exact efficiency, though often coinciding locally (cf. Abrahamson (1965, 1967), Bahadur (1967)), differ to a great extent. Therefore several authors advised to regard the approximate results with caution (cf. Abrahamson (1965), Bahadur (1967), Gleser (1966)). The approximate concept received new attention, when in 1976 Wieand extended certain results of Bahadur (1960a). He proved that the local approximate BE often equals the limiting Pitman efficiency, a result which enabled him to compute the limiting Pitman efficiency for certain nonparametric tests, mainly goodness of fit tests.

Now it will be shown that for the class of linear rank tests the approximate concept is also useful for treating the exact BE near the null hypothesis by proving that the approximate local efficiency and the exact local efficiency coincide in general for all linear rank statistics in the two-sample case (Corollary 4). For proof, first the existence and the value of the approximate slope will be derived (Theorem 1). Then it is shown as the main result of this paper, that the approximate and exact slopes of a linear rank statistic are equivalent when approaching the null hypothesis (Theorem 3). Applying this result, an explicit formula for the exact local efficiency in some subclasses of alternatives is derived (Corollary 5), and the usefulness of the approximate approach for proving local optimality of special two-sample tests demonstrated.

Received June 1977; revised June 1978.

AMS 1970 *subject classifications.* Primary 62G20.

Key words and phrases. Bahadur efficiency, linear rank statistics, equivalence of exact and approximate slopes, local efficiency, local optimality.

2. Preliminaries. Let $\theta \in \Theta$ be the parameter for the distribution of a random variable (rv) X and consider the problem of testing the hypothesis $\theta \in H$ against $\theta \in K$ $(H + K \subset \Theta)$. Usually the approximate BE of two asymptotic tests $\varphi_i = \{\varphi_n^{(i)}\}(i = 1, 2)$ can be computed by using the fact that the corresponding sequences of test statistics are standard sequences (for the general definition of approximate BE see Bahadur (1967) pages 311/312). Bahadur (1960) defined a sequence $\{S_n\}$ of real-valued test statistics to be a *standard sequence*, if there exist

(A) a continuous distribution function (df) F such that

$$\lim_{n\to\infty} P_\theta(S_n < t) = F(t), \forall t, \forall \theta \in H,$$

(B) a constant h, $0 < h < \infty$, such that

$$2 \cdot \ln[1 - F(t)] = -h \cdot t^2 \cdot [1 + o(1)], \qquad \text{as } t \to \infty,$$

(C) a function τ on K, $0 < \tau < \infty$, such that the stochastic limit of

$$n^{-1/2} \cdot S_n \quad \text{equals } \tau(\theta) \text{ for each } \theta \in K.$$

Then $c^a(\theta) = h \cdot [\tau(\theta)]^2$ is called *approximate slope* and for two tests φ_i based on $\{S_n^{(i)}\}$, $i = 1, 2$, $e^a_{1,2}(\theta) = c_1^a(\theta)/c_2^a(\theta)$ *approximate* BE of φ_1 relative to φ_2 at θ.

Finally we report the main results on exact BE of linear rank tests used in this paper. Let $R_n = (R_{n1}, \cdots, R_{nn})$, $n = n_1 + n_2$ be the vector of the ranks of the pooled sample $X = (X_{11}, \cdots, X_{1n_1}, X_{21}, \cdots, X_{2n_2})$, where the X_{ij} are independent real rv's with continuous df $F_i (i = 1, 2)$. Write $\mathfrak{F}$ for the set of continuous df's and define $\Theta = \mathfrak{F} \times \mathfrak{F}$. Then the two-sample problem may be described by the hypothesis

$$H = \{(F, F) : F \in \mathfrak{F}\}$$

and the alternative

$$K = \left\{(F_1, F_2) \in \Theta : F_1 \underset{\neq}{<} F_2\right\}.$$

Let Θ be furnished with a metric d, generating the topology of convergence in distribution $\to_{\mathfrak{D}}$ in both components of $\mathfrak{F} \times \mathfrak{F}$. In the following sections we consider simple linear rank statistics

$$T_n = \Sigma_{i=1}^{n_1} B_n(R_{ni})$$

such that the function $b_n(u) = B_n(1 + [n \cdot u])$ is related to a score-generating function $b/(0, 1)$ by

$$\lim_{n\to\infty} \int (b_n - b)^2 \, d\lambda_1 = 0, \qquad 0 < \int b^2 \, d\lambda_1 < \infty, \tag{2.1}$$

and the asymptotic sample size ratio determined by

$$\lim_{n\to\infty}(n_1/n) = s, \qquad \text{for some } s \in (0, 1). \tag{2.2}$$

Here λ_1 denotes the Lebesgue-measure on (0, 1).

Woodworth (1970) shows that the stochastic limit $t(\theta, s) = P_\theta - \lim_{n\to\infty}(T_n/n)$ exists and equals

$$t(\theta, s) = s \cdot \int bG \, dF_1 \tag{2.3}$$

for $\theta = (F_1, F_2) \in K$ and $G = s \cdot F_1 + (1 - s) \cdot F_2$. Now write

(2.4) $$I_{b,s}(t) = r \cdot t + s \cdot \ln z - \int \ln((1 - s) + s \cdot \exp(r \cdot b) \cdot z)\, d\lambda_1$$

for $t \in (\underline{t}(b, s), \bar{t}(b, s))$ with

$$\underline{t}(b, s) = s \cdot \int b \, d\lambda_1$$

$$\bar{t}(b, s) = \sup\{s \cdot \int b \cdot g \, d\lambda_1 | g \in [0, s^{-1}], \int g \, d\lambda_1 = 1\};$$

where z and r are the unique solutions of the integral equations:

(2.5) $$V(z, r) = \int \frac{\exp(r \cdot b) \cdot z}{(1 - s) + s \cdot \exp(r \cdot b) \cdot z} \, d\lambda_1 = 1$$

(2.6) $$W(z, r) = \int s \cdot b \cdot \frac{\exp(r \cdot b) \cdot z}{(1 - s) + s \cdot \exp(r \cdot b) \cdot z} \, d\lambda_1 = t$$

(cf. Woodworth (1970) page 259).

Let $\varphi_i = \{\varphi_n^{(i)}\}(i = 1, 2)$ be asymptotic upper rank tests based on linear rank statistics, satisfying the conditions (2.1)–(2.2) for some score-generating functions b_i and $t_i(\theta, s) \in (\underline{t}(b_i, s), \bar{t}(b_i, s))$. Then the *exact* BE of φ_1 relative to φ_2 at θ equals (cf. Woodworth (1970) page 263)

$$e_{1,2}(\theta, s) = c_1(\theta, s)/c_2(\theta, s)$$

with the *exact slopes* of tests φ_i

(2.7) $$c_i(\theta, s) = 2 \cdot I_{b_i, s}(t_i(\theta, s)), \qquad i = 1, 2.$$

3. Main results and proofs. We begin by proving the existence of the approximate slope

THEOREM 1. *Let φ be a rank test based on a linear rank statistic satisfying $t(\theta, s) > \underline{t}(b, s)$. Then the approximate slope of φ at $\theta = (F_1, F_2) \in K$ exists and is given by*

(3.1) $$c^a(\theta, s) = (s \cdot (1 - s))^{-1} \cdot \left(\frac{t(\theta, s) - s \cdot \mu(b)}{\sigma(b)}\right)^2$$

with

$$\mu(b) = \int b d\lambda_1, \quad \sigma^2(b) = \int (b - \mu(b))^2 \, d\lambda_1.$$

PROOF. First we normalize the scores of the tests statistic according to

$$\tilde{B}_n(i) := \frac{n}{(n_1 \cdot n_2)^{1/2}} \cdot \frac{B_n(i) - \overline{B_n}.}{\left[(n - 1)^{-1} \cdot \sum_{i=1}^n (B_n(i) - \overline{B_n}.)^2\right]^{1/2}},$$

$$\overline{B_n}. = n^{-1} \cdot \sum_{i=1}^n B_n(i).$$

Then Theorem 2.1 in Behnen (1972) implies that

$$\tilde{T}_n = n^{-1/2} \cdot \sum_{i=1}^{n_1} \tilde{B}_n(R_{ni}),$$ being

equivalent to T_n, has asymptotic standard normal distribution for each $\theta \in H$ and according to (2.3) the stochastic limit $\tilde{t}(\theta, s)$ of $n^{-1/2} \cdot \tilde{T}_n$ under θ equals

$$\tilde{t}(\theta, s) = (s \cdot (1 - s))^{-1/2} \cdot (t(\theta, s) - \underline{t}(b, s))/\sigma(b).$$

Therefore (A)–(C) are satisfied for the sequence $\{\tilde{T}_n\}$, $h = 1$ and $\tau = \tilde{t}$. □

EXAMPLE. Theorem 1 yields for the median test $(b(u) = \text{sign}(u - 1/2))$ the approximate slope $c_M^a(\theta, s) = (s/(1 - s)) \cdot [2 \cdot F_1(G^{-1}(1/2)) - 1]^2$, for the Wilcoxon test $(b(u) = u - 1/2)$ $c_W^a(\theta, s) = 3 \cdot s \cdot (1 - s) \cdot (2 \cdot \int F_2\, dF_1 - 1)^2$ and for the normal scores test ($b(u) = \Phi^{-1}(u)$, Φ denotes the standard normal df) $c_N^a(\theta, s) = (s/(1 - s)) \cdot (\int \Phi^{-1}(G)\, dF_1)^2$. For the subclass of normal shift alternatives with $F_1(y) = \Phi((y - \mu)/\sigma)$, $F_2(y) = \Phi(y/\sigma)$ and fixed $\sigma > 0$ we get the approximate efficiency curves presented in figure 1.

Comparison with figure 1 in Woodworth (1970) shows that also for linear rank tests the approximate efficiency yields incorrect results for alternatives far from the null hypothesis. The equality of the limit of the exact and approximate efficiency for $\mu/\sigma \to 0$ will be proved in general in the following Theorem 3. We need for this:

LEMMA 2. *Suppose the score-generating function b is nondecreasing and $\{\theta_j\}$ is a sequence of alternatives $\theta_j \in K$ satisfying*

$$\theta_j \to_d \theta_0, \quad \text{for some } \theta_0 \in H. \tag{3.2}$$

Then

$$\lim_{j \to \infty} t(\theta_j, s) = \underline{t}(b, s).$$

PROOF. Write $\theta_j = (F_{1j}, F_{2j})$, $G_j = s \cdot F_{1j} + (1 - s) \cdot F_{2j}$. Since G_j is continuous, we have for the distributions $\bar{P}_{ij}$ corresponding to the dfs $\bar{F}_{ij} = F_{ij}(G_j^{-1})$, $i = 1, 2$: $s \cdot \bar{P}_{1j} + (1 - s) \cdot \bar{P}_{2j} = \lambda_1$. So there exists a λ_1-density $\bar{f}_{1j}$ of $\bar{F}_{1j}$ with:

$$s \cdot \bar{f}_{1j} \leqslant 1, \tag{3.3}$$

and from (2.3)

$$t(\theta_j, s) = s \cdot \int b \cdot \bar{f}_{1j}\, d\lambda_1 \tag{3.4}$$

follows. For fixed $\varepsilon > 0$ we can choose $\beta > 0$ such that for $M = \{|b| \leqq \beta\}$:

$$\int_{CM} |b|\, d\lambda_1 \leqslant \varepsilon/(2 \cdot (1 + s)). \tag{3.5}$$

Since (3.2) entails: $\bar{F}_{1j} \to_{\mathfrak{D}} \lambda_1$, and as $b \cdot I_M$ is bounded, λ_1-a.e. continuous, we have:

$$\left|\int_M b\, d\bar{F}_{1j} - \int_M b\, d\lambda_1\right| \leqslant \varepsilon/(2 \cdot s), \forall j \geqslant N, \quad \text{say.} \tag{3.6}$$

Now (3.3)–(3.6) imply:

$$|t(\theta_j, s) - \underline{t}(b, s)| \leqslant s \cdot \left|\int_M b\, d\bar{F}_{1j} - \int_M b\, d\lambda_1\right| + s \cdot \int_{CM} |b|\, d\lambda_1 + \int_{CM} |b| \cdot (s \cdot \bar{f}_{1j})\, d\lambda_1 \leqslant \varepsilon, \quad \forall j \geqslant N.$$

□

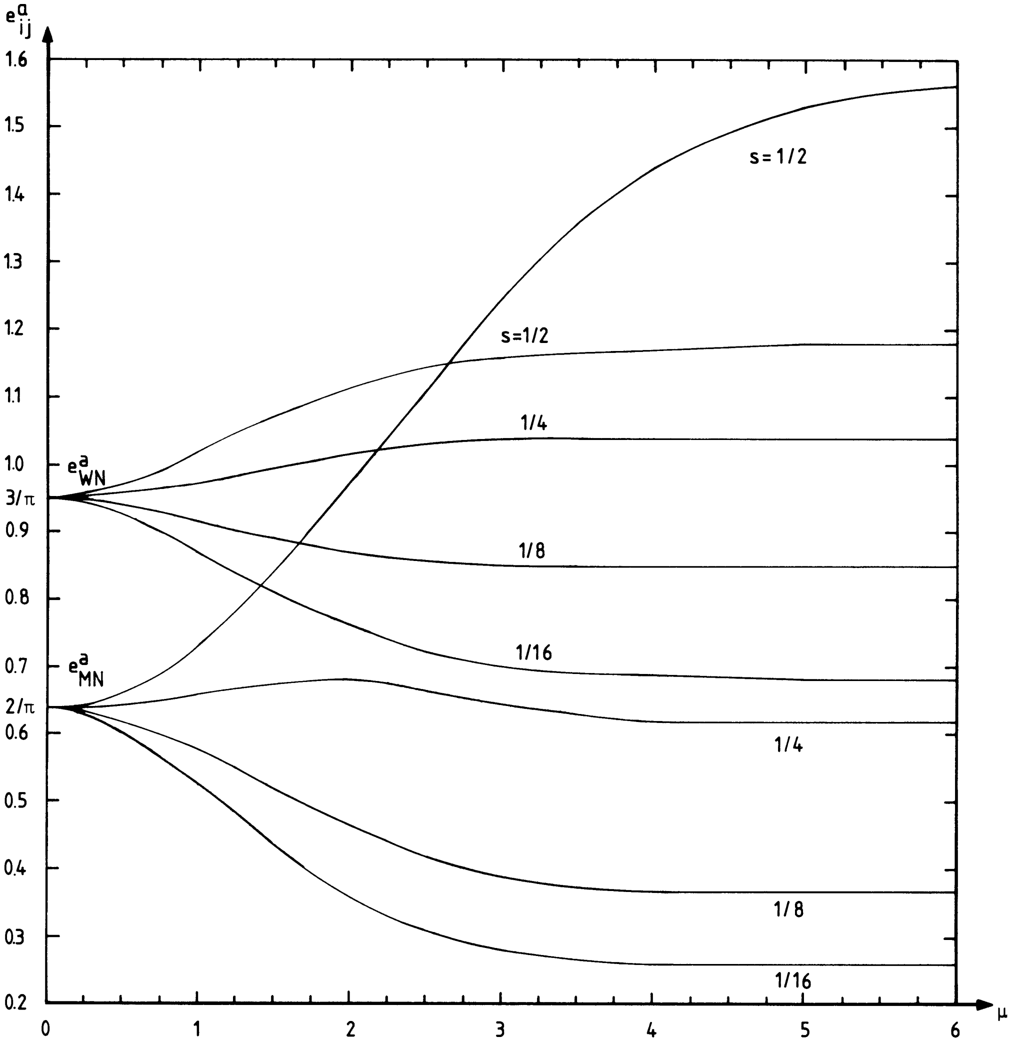

FIG. 1

e^a_{MN}: approximate efficiency of the median test relative to the normal scores test for normal shift alternatives

e^a_{WN}: approximate efficiency of the Wilcoxon test relative to the normal scores test for normal shift alternatives

Now the main result of this paper is as follows

THEOREM 3. *Let in addition to the conditions of Lemma 2 b^3 be λ_1-integrable and $t(\theta_j, s) \in (\underline{t}(b, s), \bar{t}(b, s))$, $\forall j$. Then the approximate and exact slopes $c^a(\theta_j, s)$, $c(\theta_j, s)$ of the test φ based on $\{T_n\}$ are equivalent in the sense*:

$$\lim_{j\to\infty}\big(c(\theta_j, s)/c^a(\theta_j, s)\big) = 1.$$

PROOF. Without loss of generality assume $\mu(b) = 0$ and let $z = z(r)$ be a solution of (2.5). Then $W(r) = W(z(r), r)$ is strictly monotone increasing and continuous in $r \geqslant 0$ (cf. Woodworth (1970) Lemma 4, 5). Consequently the unique solution $r = r(t) \geqslant 0$ of (2.6) satisfies

(3.7) $$\lim_{t \to 0^+} r(t) = 0,$$

yielding after some manipulation (compare (2.5))

(3.8) $$\lim_{t \to 0^+} z(t) = 1, \qquad \text{for } z(t) = z(r(t)).$$

In order to get an asymptotic expression of $I_{b,s}(t)$ for $t \to 0^+$, we expand the formulas (2.4)–(2.6) in a two-dimensional Taylor series in (z, r) at the point $(z_0, r_0) = (1, 0)$. This results in the following asymptotic equations:

(3.9)
$$V(z, r) = 1 + (z - 1)\cdot(1 - s) + (r^2/2)\cdot[(1 - s)\cdot(1 - 2s)\cdot\sigma^2(b) + o(1)] + r\cdot(z - 1)\cdot o(1) - (z - 1)^2\cdot(s\cdot(1 - s) + o(1)) = 1$$

(3.10)
$$W(z, r) = s\cdot(1 - s)\cdot\sigma^2(b)\cdot r + (r^2/2)\cdot[s\cdot(1 - s)\cdot(1 - 2s)\cdot\eta(b) + o(1)] + r\cdot(z - 1)\cdot[s\cdot(1 - s)\cdot(1 - 2s)\cdot\sigma^2(b) + o(1)] + ((z - 1)^2/2)\cdot o(1) = t, \qquad \text{for } z \to 1, r \to 0,$$

with $\eta(b) = \int b^3\, d\lambda_1$, and

(3.11) $$I_{b,s}(t) = r\cdot t + (r^2/2)\cdot[o(1) - s\cdot(1 - s)\cdot\sigma^2(b)] + r\cdot(z - 1)\cdot o(1) - ((z - 1)^2/2)\cdot[s\cdot(1 - s) + o(1)].$$

Then (3.7)–(3.9) entail for $t \to 0^+$:

$$\left(\frac{z - 1}{r}\right)\cdot(1 - s) - \frac{(z - 1)^2}{r}\cdot(s\cdot(1 - s) + o(1)) = o(1),$$

from which after some routine calculations $z - 1 = o(r)$ for $t \to 0^+$ follows. By substituting this into (3.10) we get $s\cdot(1 - s)\cdot\sigma^2(b)\cdot r + o(r) = t$. So we have shown $r(t) = t/(s\cdot(1 - s)\cdot\sigma^2(b)) + o(t)$, $z(t) = 1 + o(t)$, for $t \to 0^+$. Then (3.11) may be rewritten as

(3.12) $$I_{b,s}(t) = (2\cdot s\cdot(1 - s))^{-1}\cdot\left(\frac{t}{\sigma(b)}\right)^2 + o(t^2), \qquad \text{for } t \to 0^+.$$

According to Lemma 2 we have: $\lim_{j\to\infty} t(\theta_j, s) = 0$, which yields with (3.12) and (2.7):

$$c(\theta_j, s) = (s\cdot(1 - s))^{-1}\cdot\left(\frac{t(\theta_j, s)}{\sigma(b)}\right)^2 + o(t^2(\theta_j, s)), j \to \infty.$$

Since we have from Theorem 1, that the first term of the right-hand side equals the approximate slope $c^a(\theta_j, s)$, the theorem follows at once. $\square$

REMARK 1. The expression (3.12) corresponds to an expansion of Woodworth (1970 page 262). But our conditions of Theorem 3 are more suitable for application than the assumption of Woodworth (p. 261), i.e. z can be developed in a Taylor series in r. In the special case of the Wilcoxon test (3.12) is already developed in a paper of Hoadley ((1965) pages 72–75).

In analogy to Bahadur (1960a, 1967) for a sequence $\{\theta_j\}$ of K with (3.2) the value

$$E_{1,2}(\{\theta_j\}, s) = \liminf_{j\to\infty} e_{1,2}(\theta_j, s)$$

with the exact efficiency $e_{1,2}$ shall be called *exact* and with the approximate efficiency $e^a_{1,2}$ the limit

$$E^a_{1,2}(\{\theta_j\}, s) = \liminf_{j\to\infty} e^a_{1,2}(\theta_j, s)$$

approximate local BE *under the sequence* $\{\theta_j\}$.
Theorem 3 immediately implies:

COROLLARY 4. *Under conditions of Theorem* 3 *for* φ_i, $i = 1, 2$:

$$E_{1,2}(\{\theta_j\}, s) = E^a_{1,2}(\{\theta_j\}, s).$$

REMARK 2. Now a theorem of Wieand (1976, page 1005) about the equality of approximate local BE and Pitman efficiency can be applied to show that the concepts of Pitman efficiency, approximate and exact local BE generally coincide for linear rank tests. According to a lemma of Wieand (1976, page 1007) this equality holds, if in addition to our conditions the distribution of a suitable standardisation of the linear rank statistic converges to a normal distribution under fixed alternatives and the rate of convergence is uniform in a neighborhood of the null-hypothesis. Sufficient conditions for uniform asymptotic normality of linear rank statistics are given, e.g., by Chernoff, Savage (1958), Hájek (1968) and Pyke, Shorack (1968).

4. Applications. Corollary 4 enables one to compute the exact local efficiency using the approximate approach. Under some regularity conditions an explicit formula for the exact efficiency is derived in

COROLLARY 5. *Let score-generating functions* $b_i (i = 1, 2)$ *be given, satisfying the conditions of Theorem* 3 *and being continuously differentiable in* (0, 1). *For* $F \in \mathfrak{F}$ *denote by* $K_F = \{(F_\Delta, F) : \Delta \in (0, \bar{\Delta})\}$ *some subclass of* K *with* $t_i((F_\Delta, F), s) \in (\underline{t}(b_i, s), \bar{t}(b_i, s))$, $\forall \Delta \in (0, \bar{\Delta})$, *and*

$$F_\Delta \to_{\mathfrak{D}} F, \quad \textit{for } \Delta \to 0. \tag{4.1}$$

(4.2) *The derivative* $f_\Delta = \partial F_\Delta / \partial \Delta$ *exists and satisfies for some function* f:

$$\lim_{j\to\infty} f_{\Delta_j} = f \quad F - \text{a.e.}, \qquad \forall \{\Delta_j\} \textit{ with } \lim_{j\to\infty} \Delta_j = 0.$$

(4.3) *There are F-integrable functions* $h_i (i = 1, 2)$ *such that for the derivative* b_i' *of* b_i *and* $G_\Delta = s \cdot F_\Delta + (1 - s) \cdot F$:

$$|b_i'(G_\Delta) \cdot f_\Delta| \leq h_i \quad F - \text{a.e.}, \qquad \forall \Delta \in (0, \bar{\Delta}), \; i = 1, 2.$$

$$(4.4) \qquad \int b_i'(F) \cdot f \, dF \neq 0, \qquad \text{for at least one } i.$$

Then the exact local BE of φ_1 *relative to* φ_2 *equals*

$$(4.5) \quad E_{1,2}\big(\{(F_{\Delta_j}, F)\}, s\big) = \left[\frac{\int b_1'(F) \cdot f \, dF}{\int b_2'(F) \cdot f \, dF} \right]^2 \cdot \frac{\sigma^2(b_2)}{\sigma^2(b_1)}, \; \forall \{\Delta_j\} \text{ with } \lim_{j \to \infty} \Delta_j = 0.$$

PROOF. Here (2.3) is simply: $t_i((F_\Delta, F), s) = \mu(b_i) - (1 - s) \cdot \int b_i(G_\Delta) dF$. From Lemma 2 we know $\lim_{j \to \infty} t_i((F_{\Delta_j}, F), s) = s \cdot \mu(b_i)$ and (4.1)–(4.3) entail

$$\lim_{j \to \infty} \left(\frac{\partial t_i((F_\Delta, F), s)}{\partial \Delta} \right) \Big|_{\Delta = \Delta_j} = s \cdot (s - 1) \cdot \int b_i'(F) \cdot f \, dF, \; i = 1, 2.$$

Using l'Hôpital's rule, we get from (3.1), (4.4) and Corollary 4 the above statement. □

REMARK 3. Under a stronger set of regularity conditions, Chernoff and Savage (1958) derived (4.5) as an expression for Pitman efficiency. This correspondence is a direct consequence of our remark 2.

Except for simplifying the computation of the exact local BE one can use Theorem 3 for proving the optimality of tests with regard to the local BE. The test φ_1 shall be called *local B-optimal* (cf. Bahadur (1960b, 1967)) for $\tilde{K} \subset K$, if for each other test φ_2 and each sequence $\{\theta_j\}$ of $\tilde{K}$ with (3.2) $E_{1,2}(\{\theta_j\}, s) \geq 1$, $\forall s \in (0, 1)$ holds. Let $\theta_j = (F_{1j}, F_{2j})$ and $\tilde{\theta}_j = (G_j, G_j)$ with $G_j = s \cdot F_{1j} + (1 - s) \cdot F_{2j}$. For proving the local B-optimality of a test φ it is sufficient to show (see Bahadur and Raghavachari (1970, 1972)): $\lim_{j \to \infty}[c(\theta_j, s)/(2 \cdot K^*(\theta_j, \tilde{\theta}_j))] = 1$ and under the conditions of Theorem 3:

$$(4.6) \qquad \lim_{j \to \infty}\left[c^a(\theta_j, s) / \left(2 \cdot K^*(\theta_j, \tilde{\theta}_j)\right)\right] = 1.$$

Here K^* denotes the Kullback-Leibler-information number (cf. Bahadur and Raghavachari (1970, 1972)) for the two-sample case, i.e.

$$K^*(\theta_j, \tilde{\theta}_j) = s \cdot \int \ln(f_{1j}) \, dF_{1j} + (1 - s) \cdot \int \ln(f_{2j}) \, dF_{2j},$$

where f_{ij} are the densities of $F_{ij} (i = 1, 2)$ w.r. to G_j.

EXAMPLE. For the subclass $\tilde{K} \subset K$ of Lehmann alternatives $\theta = (F_\Delta, F)$ with $F \in \mathfrak{F}$ and $F_\Delta = (1 - \Delta) \cdot F + \Delta \cdot F^2$ one obtains the approximate slope of the Wilcoxon test:

$$c_W^a(\Delta, s) = s \cdot (1 - s) \cdot \Delta^2 / 3,$$

and the Kullback-Leibler-Information number with $\tilde{\theta} = (G_\Delta, G_\Delta)$:

$$K^*(\theta, \tilde{\theta}) = (1 - s)/2 + (s/(4 \cdot \Delta)) \cdot \left[(1 + \Delta)^2 \cdot \ln(1 + \Delta) - (1 - \Delta)^2 \cdot \ln(1 - \Delta)\right] + (4 \cdot s \cdot \Delta)^{-1} \cdot \left[(1 - s \cdot \Delta)^2 \cdot \ln(1 - s \cdot \Delta) - (1 + s \cdot \Delta)^2 \cdot \ln(1 + s \cdot \Delta)\right]$$

Using l'Hôspital's rule we get by direct calculation (4.6), i.e. the local B-optimality of the Wilcoxon test for the subclass $\tilde{K}$.

Similarly the local B-optimality of the normal scores test for the subclass of normal shift alternatives and of the median test for double-exponential shift alternatives can be derived. Since in these special cases the uniform asymptotic normality of the test statistics can be proved, we get from Remark 2, that the local B-optimality in the above examples is nothing else than the optimality in the sense of Pitman efficiency.

Acknowledgment. This article is part of the author's Diplom-thesis, written at the University of Giessen under the direction of G. Neuhaus. My thanks go to him for helpful comments and his interest during the preparation of this paper. I would also like to thank the referee for his careful reading of the manuscript and him as well as an associate editor for helpful suggestions, especially concerning the relation between Pitman and Bahadur efficiency.

REFERENCES

[1] Abrahamson, I. G. (1965). On the stochastic comparison of tests of hypothesis. Ph.D. dissertation, Univ. of Chicago.

[2] Abrahamson, I. G. (1967). The exact Bahadur efficiencies for the Kolmogoroff-Smirnov and Kuiper one- and two-sample statistics. *Ann. Math. Statist.* **38** 1475–1490.

[3] Bahadur, R. R. (1960a). Stochastic comparison of tests. *Ann. Math. Statist.* **31** 276–295.

[4] Bahadur, R. R. (1960b). Asymptotic efficiency of tests and estimates. *Sankhyā* **22** 229–252.

[5] Bahadur, R. R. (1967). Rates of convergence of estimates and test statistics. *Ann. Math. Statist.* **38** 303–324.

[6] Bahadur, R. R. and Raghavachari, M. (1972). Some asymptotic properties of likelihood ratios on general sample spaces. *Proc. of Sixth Berkeley Symp. Math. Statist.* 129–152. Univ. California Press.

[7] Behnen, K. (1972). A characterisation of certain rank-order-tests with bounds for the asymptotic relative efficiency. *Ann. Math. Statist.* **43** 1839–1851.

[8] Chernoff, H. and Savage, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *Ann. Math. Statist.* **29** 972–994.

[9] Gleser, L. J. (1966). The comparison of multivariate tests of hypothesis by means of Bahadur efficiency. *Sankhyā Ser. A* **28** parts 2 and 3, 157–174.

[10] Hájek, J. (1968). Asymptotic normality of simple linear rank statistics under alternatives. *Ann. Math. Statist.* **39** 325–346.

[11] Hoadley, A. B. (1965). The theory of large deviations with statistical applications. Ph.D. dissertation, Univ. of California at Berkeley.

[12] Pyke, R. and Shorack, G. R. (1968). Weak convergence of a two-sample empirical process and a new approach to Chernoff-Savage theorems. *Ann. Math. Statist.* **39** 755–771.

[13] Raghavachari, M. (1970). On a theorem of Bahadur on the rate of convergence of tests statistics. *Ann. Math. Statist.* **41** 1695–1699.

[14] SETHURAMAN, J. (1970). Probabilities of deviations. In *Contributions to Probability and Statistics. Essays in Honor of S. N. Roy* 655–672.

[15] WIEAND, H. S. (1976). A condition under which the Pitman and Bahadur approaches to efficiency coincide. *Ann. Statist.* **4** 1003–1011.

[16] WOODWORTH, G. G. (1970). Large deviations and Bahadur efficiency of linear rank statistics. *Ann. Math. Statist.* **41** 251–283.

INSTITUT FÜR MATHEMATISCHE STOCHASTIK
UNIVERSITÄT HAMBURG
BUNDESSTRASSE 55
2000 HAMBURG 13
WEST GERMANY

STATEMENT OF OWNERSHIP, MANAGEMENT AND CIRCULATION
(Required by 39 U S C 3685)

1 TITLE OF PUBLICATION: Annals of Statistics (USPS [illegible])
A. PUBLICATION NO: [illegible]
2 DATE OF FILING: October 1, 1979
3 FREQUENCY OF ISSUE: [illegible]
A. NO. OF ISSUES PUBLISHED ANNUALLY: [illegible]
B. ANNUAL SUBSCRIPTION PRICE: 37.00
4 LOCATION OF KNOWN OFFICE OF PUBLICATION: 3401 Investment Blvd. Suite 6, Hayward (Alameda) California 94545
5 LOCATION OF THE HEADQUARTERS OR GENERAL BUSINESS OFFICES OF THE PUBLISHERS *(Not printers)*: 3401 Investment Blvd. Suite 6, Hayward (Alameda) California 94545

6 NAMES AND COMPLETE ADDRESSES OF PUBLISHER EDITOR, AND MANAGING EDITOR

PUBLISHER *(Name and Address)*: Institute of Mathematical Statistics, 3401 Investment Blvd. # 6, Hayward, CA 94545
EDITOR *(Name and Address)*: Rupert G. Miller, Jr. Dept. of Statistics, Stanford University, Stanford, CA 94305
MANAGING EDITOR *(Name and Address)*: Donald Truax, Mathematics Dept. University of Oregon, Eugene, OR 97403

7 OWNER *(If owned by a corporation, its name and address must be stated and also immediately thereunder the names and addresses of stock holders owning or holding 1 percent or more of total amount of stock. If not owned by a corporation, the names and addresses of the individual owners must be given. If owned by a partnership or other unincorporated firm, its name and address, as well as that of each individual must be giving. If the publication is published by a nonprofit organization, its name and address must be stated.)*

NAME	ADDRESS
Institute of Mathematical Statistics	3401 Investment Bl. # 6
(Unincorporatied non-profit society)	Hayward, CA 94545

8 KNOWN BONDHOLDERS, MORTGAGEES AND OTHER SECURITY HOLDERS OWNING OR HOLDING 1 PERCENT OR MORE OF TOTAL AMOUNT OF BONDS, MORTGAGES OR OTHER SECURITIES *(If there are none, so state)*

NAME	ADDRESS
None	

9 FOR COMPLETION BY NONPROFIT ORGANIZATIONS AUTHORIZED TO MAIL AT SPECIAL RATES *(Section 132 122, PSM)*
The purpose, function, and nonprofit status of this organization and the exempt status for Federal income tax purposes *(Check one)*

[XX] HAVE NOT CHANGED DURING PRECEDING 12 MONTHS
[ ] HAVE CHANGED DURING PRECEDING 12 MONTHS
(If changed, publisher must submit explanation of change with this statement.)

10 EXTENT AND NATURE OF CIRCULATION	AVERAGE NO COPIES EACH ISSUE DURING PRECEDING 12 MONTHS	ACTUAL NO COPIES OF SINGLE ISSUE PUBLISHED NEAREST TO FILING DATE
A TOTAL NO COPIES PRINTED *(Net Press Run)*	6438	6628
B PAID CIRCULATION 1 SALES THROUGH DEALERS AND CARRIERS, STREET VENDORS AND COUNTER SALES	None	None
2. MAIL SUBSCRIPTIONS	4520	4776
C TOTAL PAID CIRCULATION *(Sum of 10B1 and 10B2)*	4520	4776
D FREE DISTRIBUTION BY MAIL, CARRIER OR OTHER MEANS SAMPLES COMPLIMENTARY, AND OTHER FREE COPIES	12	12
E TOTAL DISTRIBUTION *(Sum of C and D)*	4532	4788
F COPIES NOT DISTRIBUTED 1 OFFICE USE LEFT OVER, UNACCOUNTED, SPOILED AFTER PRINTING	1906	1840
2 RETURNS FROM NEWS AGENTS	None	None
G TOTAL *(Sum of E, F1 and 2—should equal net press run shown in A)*	6438	6628

11 I certify that the statements made by me above are correct and complete
SIGNATURE AND TITLE OF EDITOR, PUBLISHER, BUSINESS MANAGER, OR OWNER: [signature] Treasurer, IMS

12 FOR COMPLETION BY PUBLISHERS MAILING AT THE REGULAR RATES *(Section 132 121, Postal Service Manual)*

39 U S C 3626 provides in pertinent part No person who would have been entitled to mail matter under former section 4359 of this title shall mail such matter at the rates provided under this subsection unless he files annually with the Postal Service a written request for permission to mail matter at such rates

In accordance with the provisions of this statute I hereby request permission to mail the publication named in Item 1 at the phased postage rates presently authorized by 39 U S C 3626

SIGNATURE AND TITLE OF EDITOR, PUBLISHER, BUSINESS MANAGER, OR OWNER: [signature] Treasurer, IMS

PS Form Aug 1978 3526 *(Page 1)* (See instructions on reverse)