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ASYMPTOTIC DEFICIENCIES OF ONE-SAMPLE RANK TESTS UNDER RESTRICTED ADAPTATION[1]

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In this paper we consider adaptive one-sample rank tests of the following type: the score function J of the test is estimated from the sample under the restriction that $J \in \mathcal{J}$, for some given one-parameter family $\mathcal{J} = \{J_r, r \in I \subset R^1\}$. Using deficiencies, we compare the performance of such tests to that of rank tests with fixed scores. Conditions on the estimator S of the parameter r and on J_r are given, under which the deficiency tends to a finite limit, which is obtained. For a particular class of estimators which are related to the sample kurtosis, explicit results are obtained.

1. Introduction. Let $X_1, \cdots, X_N$ be independent identically distributed (i.i.d.) random variables (rv's) with common absolutely continuous distribution function (df) $F(x - \theta)$, where F is such that $F(x) + F(-x) = 1$ for all x, i.e., the distribution of X_1 is symmetric about θ. Then we want to test H_0: $\theta = 0$ against H_1: $\theta > 0$.

Widely used tests for this one-sample problem are linear rank tests. These are distribution-free, i.e., the distribution of their test statistic T does not depend on F under H_0. Let $0 < Z_1 < \cdots < Z_N$ be the order statistics of $|X_1|, \cdots, |X_N|$. If $|X_{R_j}| = Z_j$, define

$$(1.1) \qquad V_j = 1 \quad \text{if} \quad X_{R_j} > 0, \qquad = 0 \quad \text{otherwise.}$$

Let X, Z and V denote the corresponding vectors, and let $a = (a_1, \cdots, a_N)$ be a vector of scores. Then the rank statistic T is defined as

$$(1.2) \qquad T = \Sigma a_j V_j$$

(Σ always means $\Sigma_{j=1}^{N}$, unless stated otherwise). We shall restrict attention to rank tests with smooth scores

$$(1.3) \qquad a_j = EJ(U_{j:N}),$$

$j = 1, \cdots, N$, where J is a continuous function on (0, 1) and $U_{1:N} < \cdots < U_{N:N}$ are order statistics of a sample of size N from the uniform distribution on (0, 1).

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It is well known that for each df F, that satisfies certain mild regularity conditions, there exists a score function J such that the corresponding rank test is asymptotically optimal for testing H_0 against contiguous alternatives (see Hájek and Šidák (1967)). The usefulness of this result is severely limited by the fact that the optimal J depends on the generally unknown F. In fact, it is just this lack of knowledge which stimulates the use of rank tests.

One may try to overcome this difficulty by using the vector of order statistics Z to obtain an estimator $\hat{J}$ for the optimal J. This means that the corresponding statistic $\hat{T} = \Sigma E\hat{J}(U_{j:N})V_j$ adapts itself to each particular sample. For this reason the test $\hat{\chi}$ based on $\hat{T}$, is called an adaptive rank test. Note that this terminology is somewhat misleading: $\hat{\chi}$ still is distribution-free, but it no longer is a rank test as the scores depend on Z, which means that $\hat{\chi}$ is merely a permutation test. Hájek and Šidák (1967) have shown that there exist estimators $\hat{J}$ such that the corresponding test $\hat{\chi}$ is asymptotically optimal for all sufficiently regular df's F. They also mention, however, that the convergence of $\hat{J}$ towards the optimal J seems to be very slow, which makes the practical usefulness of $\hat{\chi}$ doubtful. This feeling is shared by several other authors (see e.g., Stein (1956) and Huber (1964), page 80).

In view of this the following approach seems worthwhile: suppose that F belongs to some known set of df's such that for each F in this set there exists an optimal score function J. Let $\mathcal{J}$ be the set of these J. Then we use an estimator $\hat{J}$ based on Z for the optimal J, under the restriction that $\hat{J} \in \mathcal{J}$. This method is motivated by the feeling that, while unrestricted adaptation needs exorbitant sample sizes, a very moderate type of adaptation might work (see Huber (1972), page 1058). A similar approach in estimation has been used by Hogg (1967) and Jaeckel (1971). By taking sufficiently restricted sets $\mathcal{J}$, the degree of adaptation can be made arbitrarily small. In the present paper we shall consider parametric families of df's $\{F_r, r \in I \subset R^1\}$ (the case of a multi-dimensional parameter can be dealt with in exactly the same way, but it will lead to even more complicated expressions than in the one-dimensional case). Let J_r be optimal for F_r, let $\mathcal{J} = \{J_r, r \in I \subset R^1\}$ and let $S = S(Z)$ be an estimator of r, attaining values in I. Then we consider the adaptive rank test χ_S, which is performed as follows: given $Z = z$, we evaluate the scores $EJ_s(U_{j:N})$ for $s = S(z)$ and reject $H_0 : \theta = 0$ for large values of

$$T_s = \Sigma EJ_s(U_{j:N})V_j. \tag{1.4}$$

Hence χ_S is a permutation test and therefore it is distribution-free.

We shall compare the performance of χ_S to that of the fixed-scores rank test χ_r, based on $T_r = \Sigma EJ_r(U_{j:N})V_j$, for $r \in I$. Such a comparison is especially interesting if the true df F is such that $F = F_{\tilde{r}}$ for some $\tilde{r} \in I$, for then $\chi_{\tilde{r}}$ is asymptotically optimal among all rank tests with fixed scores. Obviously, there exist estimators S of r such that χ_S has efficiency 1 with respect to (w.r.t.) $\chi_{\tilde{r}}$ under $F_{\tilde{r}}$. Hence, for an effective comparison of χ_S and $\chi_{\tilde{r}}$, we need more information.

A possible way to obtain this is to study the asymptotic behaviour of the deficiency d_N of χ_S w.r.t. $\chi_{\tilde{r}}$ under $F_{\tilde{r}}$. In the present situation d_N is defined as the additional number of observations which is required for χ_S to attain the same power as $\chi_{\tilde{r}}$, when $\chi_{\tilde{r}}$ is based on N observations (see Hodges and Lehmann (1970)). Note that the fact that χ_S has asymptotically efficiency 1 w.r.t. $\chi_{\tilde{r}}$ under $F_{\tilde{r}}$ merely implies that $d_N = o(N)$ in that case. Hence a study of the asymptotic behaviour of d_N may supply valuable additional information on the rate of convergence of the power of χ_S towards that of $\chi_{\tilde{r}}$ under $F_{\tilde{r}}$.

For the determination of the asymptotic behaviour of d_N, asymptotic expansions to order N^{-1} for the power of χ_S and $\chi_{\tilde{r}}$ are needed. For $\chi_{\tilde{r}}$, being a linear rank test, such an expansion is given by Albers, Bickel and van Zwet (1976). (For further reference we denote this paper as ABZ (1976)). For χ_S this expansion can be derived by analogous methods. This is done in Section 2. Using these results it is shown in Section 3 that under suitable regularity conditions on S the deficiency d_N of χ_S w.r.t. $\chi_{\tilde{r}}$ under $F_{\tilde{r}}$ has a finite limit d, which is obtained. For $r \neq \tilde{r}$, the test χ_S typically has asymptotic relative efficiency (ARE) $e > 1$ w.r.t. χ_r under $F_{\tilde{r}}$. The section is concluded with a few examples.

In order to obtain completely explicit results, it remains to investigate which choices of S are suitable, to verify the conditions from Section 3 for these choices and to determine their asymptotic behaviour. This is done in Section 4 for a particular family of estimators $S_{p,q}$. It is introduced as follows: let p and q be constants such that $0 < p < q$ and let $\kappa(F) = \int|x|^q dF(x)/\{\int|x|^p dF(x)\}^{q/p}$. Then we choose for $S_{p,q}$ the value of $r \in I$ that minimizes

$$|N^{-1}\Sigma Z_j^q/\{N^{-1}\Sigma Z_j^p\}^{q/p} - \kappa(F_r)|. \tag{1.5}$$

Using results of Shapiro, Wilk and Chen (1968), Hogg (1967) and Hogg (1972), it is argued that these $S_{p,q}$ are reasonable estimators. It is also indicated how p and q should be chosen for a given $\mathcal{J}$. Next the asymptotic behaviour of $S_{p,q}$ is determined. Together with the results of Section 3 this leads to an explicit expression for the asymptotic deficiency d of χ_S w.r.t. $\chi_{\tilde{r}}$ under $F_{\tilde{r}}$. As an application the value of d is obtained for the examples from Section 3.

2. Asymptotic expansions for adaptive rank tests. In this section we shall give an expansion to order N^{-1} for the power of the test χ_S (see (1.4)) under contiguous location alternatives $F(x - \theta)$. The derivation of such an expansion is a complicated matter which requires a lot of space. Fortunately, it is closely related to the derivation of a similar expansion for the power of rank tests with fixed scores (like $\chi_{\tilde{r}}$), as was obtained by ABZ (1976). Therefore we shall here only state the result we need in the subsequent sections and indicate how the proof of ABZ (1976) can be adapted to cover the present case (for more details see Albers (1976)).

First we introduce the following notation and conditions. We consider i.i.d. rv's $X_1, \cdots, X_N$ with df $F(x - \theta)$. About $\theta(= \theta_N)$ we shall assume that for some positive constant C and $N = 1, 2, \cdots$

$$0 \leqslant \theta \leqslant CN^{-1/2}. \tag{2.1}$$

As concerns F, we shall suppose that $F \in \mathcal{F}$, determined by

DEFINITION 2.1. $\mathcal{F}$ is the class of df's F on R^1 with positive densities f that are symmetric about zero, four times differentiable and such that for

$$\psi_i = \frac{f^{(i)}}{f},\ \Psi_i(t) = \psi_i(F^{-1}((1+t)/2)), \tag{2.2}$$

$m_1 = 6, m_2 = 3, m_3 = \frac{4}{3}, m_4 = 1,$

$$\limsup_{y\to 0}\int_{-\infty}^{\infty}|\psi_i(x+y)|^{m_i}f(x)dx < \infty, \qquad i = 1, \cdots, 4, \tag{2.3}$$

$$\limsup_{t\to 0,\,1} t(1-t)\left|\frac{\Psi_1''(t)}{\Psi_1'(t)}\right| < \tfrac{3}{2}. \tag{2.4}$$

Next we consider the family of score functions $\mathcal{J}$. In the first place we shall assume that $\mathcal{J} = \{J_r\}$ always has the form

$$\mathcal{J} = \{J_r | J_r = J_0 + rh,\ -D_1 \leqslant r \leqslant D_2\}, \tag{2.5}$$

where J_0 and h are fixed continuous functions on (0, 1) and D_1 and D_2 are positive constants. This restriction has two obvious advantages: the score function $J_S = J_0 + Sh$ of χ_S has a very simple form and the expansion for the power of χ_S is considerably less complicated than in the general case. On the other hand, the loss of generality incurred by the restriction is not as serious as it may seem. To see this, we note that (2.5) implies that $J_S = J_{\tilde{r}} + (S - \tilde{r})h$. But under $F_{\tilde{r}}$, typically $S \to_P \tilde{r}$ and hence for general families $\{J_r\}$ we have a similar approximate result: $J_S \approx J_{\tilde{r}} + (S - \tilde{r})((\partial/\partial x)J_x)_{x=\tilde{r}}$.

The following regularity conditions are imposed on $\mathcal{J}$: for a function ω on (0, 1), let $I(\omega)$ denote $\int_0^1\omega(t)dt$. Then we suppose that J_0, h, D_1 and D_2 in (2.5) are such that

$$I(J_0^4) < \infty, \quad I(h^4) < \infty, \tag{2.6}$$

$$J_0 \text{ and } h \text{ are twice continuously differentiable,} \tag{2.7}$$

$$|J_0'(\tau)| > \max(D_1, D_2)|h'(\tau)| \qquad \text{for some} \quad 0 < \tau < 1, \tag{2.8}$$

$$\limsup_{t\to 0,\,1}\{t(1-t)|J_r''(t)/J_r'(t)|\} < 3/2 \qquad \text{for} \quad -D_1 \leqslant r \leqslant D_2, \tag{2.9}$$

$$\int_t^1 J_r(u)du > 0 \qquad \text{for} \quad 0 \leqslant t < 1 \qquad \text{and} \quad -D_1 \leqslant r \leqslant D_2. \tag{2.10}$$

Just as in the introduction, let F_r be the symmetric df for which J_r is optimal, for each $J_r \in \mathcal{J}$. It is easy to show that under condition (2.10) this definition makes sense, i.e., that there corresponds exactly one symmetric df to each J_r (cf. Hájek and Šidák (1967), page 21, Lemma I 2.4 f). This F_r is determined through its inverse G_r as follows:

$$G_r\left(\frac{1+t}{2}\right) = \int_0^t\left[\int_u^1 J_r(v)dv\right]^{-1}du, \tag{2.11}$$

for $0 \leqslant t < 1$. In analogy to Definition 2.1, we introduce $f_r = F_r'$, $\psi_{ir} = f_r^{(i)}/f_r$ and $\Psi_{ir}(t) = \psi_{ir}(G_r[1 + t]/2)$. Hence $-\Psi_{1r} = J_r$ and χ_r is optimal for F_r.

The final step before presenting the theorem is the introduction of the notation for the results. Let α denote the level of the tests involved, let Φ be the standard normal df, let $\phi = \Phi'$ and let $u_\alpha = \Phi^{-1}(1-\alpha)$, where Φ^{-1} is the inverse of Φ. Define

(2.12)

$$\bar{\pi}_r(\theta) = 1 - \Phi(u_\alpha - N^{1/2}\theta I^{1/2}(J_r^2)) + \frac{N^{-1/2}\theta I^{1/2}(J_r^2)}{72}\phi(u_\alpha - N^{1/2}\theta I^{1/2}(J_r^2))$$
$$\times\{I(J_r^4)I^{-2}(J_r^2)[-6(u_\alpha^2-1) + 3N^{1/2}\theta I^{1/2}(J_r^2)u_\alpha + 5N\theta^2 I(J_r^2)]$$
$$-12N\theta^2 I(\Psi_{2r}^2)I^{-1}(J_r^2) - 9N^{1/2}\theta I^{1/2}(J_r^2)(u_\alpha - N^{1/2}\theta I^{1/2}(J_r^2))$$
$$-36I^{-1}(J_r^2)\textstyle\int_{1/N}^{1-1/N}(J_r'(t))^2 t(1-t)\,dt\},$$

(2.13)

$$\bar{K}_r(S,\theta) = \phi(u_\alpha - N^{1/2}\theta I^{1/2}(J_r^2))\Big\{\frac{N^{1/2}\theta}{2}E_\theta(S-r)^2[I(h^2)I^{-1/2}(J_r^2)$$
$$-I^2(J_r h)I^{-3/2}(J_r^2)] - N^{-1/2}\theta E_\theta[\Sigma\{EJ_r(U_{j:N})I(J_r h)I^{-3/2}(J_r^2)$$
$$-Eh(U_{j:N})I^{-1/2}(J_r^2)\}(S-r)(\psi_{1r}(Z_j) - E_\theta\psi_{1r}(Z_j))]\Big\}.$$

Then we have

THEOREM 2.1. *Let C, ε, D_1 and D_2 be any fixed positive constants and J_0 and h be fixed continuous functions on* (0, 1). *Assume that θ satisfies* (2.1), *$\varepsilon \leqslant \alpha \leqslant 1-\varepsilon$ and that $\mathcal{J}$ satisfies* (2.5)–(2.10). *Then for every fixed $\tilde{r} \in [-D_1, D_2]$ such that $F_{\tilde{r}} \in \mathcal{F}$, there exist positive constants A, $\delta_1, \delta_2, \cdots$ such that $\lim_{N\to\infty}\delta_N = 0$ and for every N the power $\pi_S(\theta)$ of χ_S satisfies under $F_{\tilde{r}}(x-\theta)$*

(2.14)

$$|\pi_S(\theta) - \bar{\pi}_{\tilde{r}}(\theta) + \bar{K}_{\tilde{r}}(S,\theta)| \leqslant \delta_N\{N^{-1} + E_\theta(S-\tilde{r})^2$$
$$+|E_\theta S - \tilde{r}|\} + A\{N^{-3/2}\textstyle\int_{1/N}^{1-1/N}(J_{\tilde{r}}'(t))^2(t(1-t))^{1/2}dt + E_\theta|S-\tilde{r}|^3\}.$$

REMARK 2.1. Apart from (2.8) and (2.10) the conditions of the theorem correspond in an obvious way to similar conditions in Theorems 4.1 and 4.2 of ABZ (1976). To see which condition (2.8) corresponds to, it suffices to note that it ensures that J_r is nonconstant on (0, 1) for all $-D_1 \leqslant r \leqslant D_2$. The purpose of (2.10) has already been explained.

REMARK 2.2. According to the theorem, $\bar{\pi}_{\tilde{r}}(\theta) - \bar{K}_{\tilde{r}}(S,\theta)$ is an expansion for the power $\pi_S(\theta)$ of the adaptive rank test χ_S. To clarify this result we remark that $\bar{\pi}_{\tilde{r}}(\theta)$ is an expansion for the power $\pi_{\tilde{r}}(\theta)$ of $\chi_{\tilde{r}}$, the optimal rank test with fixed scores under $F_{\tilde{r}}$, while $\bar{K}_{\tilde{r}}(S,\theta)$ is an expansion for the shortcoming $\pi_{\tilde{r}}(\theta) - \pi_S(\theta)$. This follows by considering the special case where $S = \tilde{r}$ a.s. Then $\bar{K}_{\tilde{r}}(S,\theta) = 0$ and the expansion for $\pi_S(\theta)$ reduces to $\bar{\pi}_{\tilde{r}}(\theta)$. On the other hand, if $S = \tilde{r}$ a.s., the test χ_S is equivalent to $\chi_{\tilde{r}}$ and therefore $\pi_S(\theta) = \pi_{\tilde{r}}(\theta)$. Hence, in this case, (2.14) reduces to $|\pi_{\tilde{r}}(\theta) - \bar{\pi}_{\tilde{r}}(\theta)| \leqslant \delta_N N^{-1} + AN^{-3/2}\int_{1/N}^{1-1/N}(J_{\tilde{r}}'(t))^2\{t(1-t)\}^{1/2}dt$. Exactly the same result is given in Theorem 4.2 of ABZ (1976) which thus can be considered as a special case of the present theorem.

REMARK 2.3. In Theorem 2.1 we have considered the special case where the underlying df equals $F_{\tilde{r}}$ for some $-D_1 \leqslant \tilde{r} \leqslant D_2$. The case of an arbitrary $F \in \mathcal{F}$ can be dealt with in exactly the same way (see Albers (1976), Theorem 2.3; also cf. Theorem 4.1 of ABZ (1976)).

About the proof we make the following remarks: conditional on $Z = z$, the scores $a_{sj} = EJ_s(U_{j:N})$ are constant and the statistic T_s in (1.4) is indistinguishable from an ordinary rank statistic. Hence a similar approach as used by ABZ (1976) can be applied. First an expansion $\bar{\pi}_S(\theta|Z)$ for the conditonal power $\pi_S(\theta|Z)$ of χ_S is obtained, from which an expansion for $\pi_S(\theta)$ follows by taking expectations. In carrying out this program essentially the same steps are taken as in ABZ (1976). In addition a_{Sj} has to be expanded everywhere around $a_{\tilde{r}j}$. This results in an expansion that consists of the two parts $\bar{\pi}_{\tilde{r}}(\theta)$ and $\bar{K}_{\tilde{r}}(S, \theta)$ (for more details see Albers (1976)). Perhaps it is good to mention at this point that the following shorter method is not correct: replace everywhere in the expansion $\bar{\pi}_{\tilde{r}}(\theta)$ for $\pi_{\tilde{r}}(\theta)$ the variable $\tilde{r}$ by the rv S and evaluate the expectation of the resulting expression to obtain an expansion for $\pi_S(\theta)$. This is not allowed since S and V_j from (1.1) are only independent under $H_0 : \theta = 0$. In fact, the complicated last term of $\bar{K}_r(S, \theta)$ in (2.13) is precisely due to the dependence of S and V_j for $\theta > 0$.

3. Comparison of χ_S and χ_r. In this section we shall use the results of Section 2 for a comparison of the performance of the adaptive rank test χ_S to that of the fixed scores rank test χ_r under $F_{\tilde{r}}$, for $-D_1 \leqslant r \leqslant D_2$. Note that $\chi_{\tilde{r}}$ is the locally most powerful rank test under $F_{\tilde{r}}$. Let $d_N(\tilde{r})$ be the deficiency of χ_S w.r.t. $\chi_{\tilde{r}}$ under $F_{\tilde{r}}$, let $d(\tilde{r})$ be its limit if it exists and let $e(\tilde{r}, r)$ be the ARE of χ_S w.r.t. χ_r under $F_{\tilde{r}}$. Define

$$
\begin{aligned}
(3.1)\qquad d_N^*(r) = {} & NE_\theta(S-r)^2\big[I(h^2)I^{-1}(J_r^2) - I^2(J_r h)I^{-2}(J_r^2)\big] \\
& -2E_\theta\big[\Sigma\{EJ_r(U_{j:N})I(J_r h)I^{-2}(J_r^2) - Eh(U_{j:N})I^{-1}(J_r^2)\} \\
& \times (S-r)(\psi_{1r}(Z_j) - E_\theta\psi_{1r}(Z_j))\big].
\end{aligned}
$$

Then we have

THEOREM 3.1. *Suppose that the conditions of Theorem* 2.1 *are satisfied. Moreover, assume that* $\theta^{-1} = O(N^{1/2})$,

$$(3.2)\qquad J_{\tilde{r}}'(t) = o(\{t(1-t)\}^{-1})$$

near 0 *and* 1 *and that S satisfies under $F_{\tilde{r}}$ for some* $\beta > 1$

$$(3.3)\qquad E_\theta|S - \tilde{r}|^{2\beta} = O(N^{-\beta}),$$

$$(3.4)\qquad E_\theta S - \tilde{r} = O(N^{-1}).$$

Then the following limits exist and are given by

$$d(\tilde{r}) = \lim_{N\to\infty} d_N^*(\tilde{r}), \tag{3.5}$$

$$e(\tilde{r}, r) = I(J_{\tilde{r}}^2)I(J_r^2)I^{-2}(J_{\tilde{r}}J_r). \tag{3.6}$$

The proof of this theorem is contained in Albers (1976). We shall not give it here; it is rather straightforward and, moreover, closely related to the proof of Theorem 6.1 of ABZ (1976). The idea is, of course, that the additional conditions (3.2)–(3.4) ensure that the right-hand side of (2.14) is $o(N^{-1})$: (3.3) and (3.4) take care of all terms involving $(S - \tilde{r})$ and (3.2) disposes of the remaining term (cf. the remarks to this effect following Theorem 4.2 of ABZ (1976)). The result (3.5) then follows from the definition of deficiency and the fact that $\pi_{\tilde{r}}(\theta) - \pi_S(\theta) = \bar{K}_{\tilde{r}}(S, \theta) + o(N^{-1})$. Finally, (3.6) is standard.

To interpret the result, note that (3.6) implies that $e(\tilde{r}, r) > 1$ for $r \neq \tilde{r}$ unless $J_r = BJ_{\tilde{r}}$ a.s. for some constant $B \neq 0$. Hence, χ_S typically has ARE larger than 1 w.r.t. χ_r for $r \neq \tilde{r}$, while its deficiency w.r.t. the optimal test $\chi_{\tilde{r}}$ tends to a finite limit.

It should be noted that the results of Section 2 allow a more general result for $d_N(\tilde{r})$ (see Albers (1976)). In Theorem 3.1 we have isolated the case where $d(\tilde{r})$ exists, which is, of course, the most interesting situation.

We conclude the section with a few examples. Let

$$J^{(1)}(t) = \Phi^{-1}\left(\frac{1+t}{2}\right), \qquad J^{(2)}(t) = t, \qquad J^{(3)}(t) = 1. \tag{3.7}$$

These are the score functions of the absolute normal scores test, Wilcoxon's signed rank test and the sign test, respectively, which are optimal against normal, logistic and double-exponential alternatives, respectively.

First choose $J_0 = J^{(1)}$ and $h = J^{(2)}$ in (2.5). Then (2.6)–(2.9) are satisfied for all $D_1, D_2 > 0$, while (2.10) holds for $0 < D_1 < (8/\pi)^{1/2}$ and every $D_2 > 0$. Hence Theorem 2.1 can be applied for each $\tilde{r} > -(8/\pi)^{1/2}$ for which $F_{\tilde{r}} \in \mathfrak{F}$. The most interesting case is of course $\tilde{r} = 0$, i.e., $F_{\tilde{r}} = \Phi$. Clearly, $\Phi \in \mathfrak{F}$. Moreover, (3.2) also holds in this case and therefore (3.5) and (3.6) can be applied for all S that satisfy (3.3) and (3.4). We find that for the present choice of h the asymptotic deficiency $d(0)$ of χ_S w.r.t. the absolute normal scores test under normal alternatives is the limit of

$$\frac{\pi - 3}{3\pi} NE_\theta S^2 + 2E_\theta\left\{\Sigma\left[\pi^{-1/2}E\Phi^{-1}\left(\frac{1+U_{j:N}}{2}\right) - \frac{j}{N+1}\right]S(Z_j - E_\theta Z_j)\right\}. \tag{3.8}$$

By using Schwarz' inequality and (A2.16) of ABZ (1976), the mixed second term can be bounded in absolute value by $2\{NE_\theta S^2(2\pi + 12(3)^{1/2} - 27)/6\pi\}^{1/2}$ (see Albers (1976)).

Next consider the choice $J_0 = J^{(1)}$, $h = J^{(3)}$. Just as in the first example, all conditions are satisfied (only replace $(8/\pi)^{1/2}$ by $(2/\pi)^{1/2}$). Now we find that $d(0)$

is the limit of

$$(3.9)\quad \left(1-\frac{2}{\pi}\right)NE_\theta S^2 + 2E_\theta\left\{\Sigma\left[\left(\frac{2}{\pi}\right)^{1/2} E\Phi^{-1}\left(\frac{1+U_{j:N}}{2}\right)-1\right]S(Z_j - E_\theta Z_j)\right\},$$

where the absolute value of the second term is bounded by $2\{NE_\theta S^2 (\pi-3)/\pi\}^{1/2}$.

Finally, let $J_0 = J^{(2)}$ and $h = J^{(3)}$. Again all conditions are satisfied (replace $(8/\pi)^{1/2}$ by $1/2$). Now $F_0 = 1/(1+e^{-x})$ and we find that in this case the asymptotic deficiency $d(0)$ of χ_S w.r.t. Wilcoxon's signed rank test under logistic alternatives is the limit of

$$(3.10)\quad \tfrac{3}{4}NE_\theta S^2 + 6E_\theta\left\{\left(\Sigma\frac{3j}{N+1}-2\right)S\left[F_0(Z_j) - E_\theta F_0(Z_j)\right]\right\},$$

where the absolute value of the second term is bounded by $\{3NE_\theta S^2/10\}^{1/2}$.

4. A family of estimators $S_{p,q}$. The results of Sections 2 and 3 seem to be final in the sense that these cannot be simplified any further for general S. Hence, it remains to investigate which choices of S are suitable, to evaluate the required moments of $(S - \tilde{r})$ for these choices and to verify that (3.3) and (3.4) hold. As an example we shall study in this section a particular family of estimators $S_{p,q}$.

In Section 2 it was shown that for each J_r that satisfies (2.10) there exists exactly one df F_r for which J_r is optimal. Obviously, if J_r corresponds to $F_r(x)$, $\sigma^{-1}J_r$ corresponds to $F_r(\sigma^{-1}x)$, for every $\sigma > 0$. As the rank tests based on J_r and $\sigma^{-1}J_r$ are equivalent, it follows that J_r is in fact optimal for the whole scale-parameter family $\{F_r(\sigma^{-1}x), \sigma > 0\}$. Hence the problem of finding $S = S(Z)$ such that J_S is optimal in $\mathcal{F}$ as in (2.5), is equivalent to the problem of finding S such that $F_S(\sigma^{-1}x)$ agrees optimally with the underlying df F, for some $\sigma > 0$. To select this $F_S(\sigma^{-1}x)$, we shall use a statistic of the form

$$(4.1)\quad K_{p,q} = \frac{N^{-1}\Sigma Z_j^q}{\{N^{-1}\Sigma Z_j^p\}^{q/p}},$$

where $Z_1, \cdots, Z_N$ are again the order statistics of $|X_1|, \cdots, |X_N|$ and p and q are positive constants with $p < q$.

To motivate this choice, we quote a result of Hogg (1972). He considers a family of symmetric distributions, defined by densities of the form $f(x, \tau) = c(\tau)\exp(-|x|^\tau)$, $-\infty < x < \infty$, $\tau > 0$ and shows that the most powerful scale invariant test of $\tau = q$ against $\tau = p$ rejects the hypothesis for large values of $K_{p,q}$. Hence, for example, $K_{2,4}$ is optimal for testing normality against a distribution with lighter tails than those of the normal, namely the one with $\tau = 4$. For testing normality against distributions with heavier tails, a choice like $K_{1,2}$ is probably better. This result will be useful in the sequel to obtain an idea which choices of p and q are suitable for a particular family $\mathcal{F}$. Note that $K_{2,4}$ is closely related to the well-known sample kurtosis (s.k.) $N\Sigma(X_j - \bar{X})^4/\{\Sigma(X_j - \bar{X})^2\}^2$. In $K_{2,4}$ the sample mean $\bar{X}$ has been replaced by the known distribution mean under the hypothesis, which is zero. This ensures that $K_{2,4}$, unlike the s.k. itself, only depends on Z. The s.k. has been used before to obtain information on the type of the underlying distribution: see Hogg (1967) and Shapiro, Wilk and Chen (1968).

If F is nondegenerate and possesses moments of a sufficiently high order, $K_{p,q}$ obviously converges in probability to

$$\kappa_{p,q} = \frac{\int_{-\infty}^{\infty}|x|^q dF(x)}{\{\int_{-\infty}^{\infty}|x|^p dF(x)\}^{q/p}}. \tag{4.2}$$

If $F = F_r$ in (4.2) for some $-D_1 \leqslant r \leqslant D_2$, we shall write $\kappa_{p,q,r}$ instead of $\kappa_{p,q}$. Now the idea is to choose $S_{p,q}$ such that the difference between $K_{p,q}$ and $\kappa_{p,q,S_{p,q}}$ is minimal. Clearly, the simplest and most interesting case occurs when $\kappa_{p,q,r}$ is continuous and monotone in r. Then $S_{p,q}$ can be defined as the unique solution of

$$K_{p,q} = \kappa_{p,q,S_{p,q}}, \tag{4.3}$$

for $\kappa_{p,q,-D_1} \leqslant K_{p,q} \leqslant \kappa_{p,q,D_2}$ (for $K_{p,q} < \kappa_{p,q,-D_1}$ let $S_{p,q} = -D_1$ and for $K_{p,q} > \kappa_{p,q,D_2}$ let $S_{p,q} = D_2$). About the evaluation of $S_{p,q}$ we remark that $K_{p,q} = N^{-1}\Sigma|X_i|^q/\{N^{-1}\Sigma|X_i|^p\}^{q/p}$ is readily found from the sample, whereas $\kappa_{p,q,r}$ can be evaluated using

$$\int_{-\infty}^{\infty}|x|^q dF_r(x) = \int_0^1\left\{\int_0^t\left[\int_u^1 J_r(v)dv\right]^{-1}du\right\}^q dt. \tag{4.4}$$

As concerns the regularity conditions under which the above holds, we have

THEOREM 4.1. *Suppose that J_0 and h are continuous, positive on* $(0, 1)$ *and such that $\int_0^1 J_0(t)dt < \infty$, $\int_0^1 h(t)dt < \infty$. Moreover, assume that there exists a decreasing function b on* $[0, 1)$, *such that $D_1 b(0) < 1$ and*

$$\int_u^1 h(v)dv = b(u)\int_u^1 J_0(v)dv. \tag{4.5}$$

Finally, suppose that for positive constants p and q, with $p < q$,

$$0 < \int_{-\infty}^{\infty}|x|^p dF_0(x), \qquad \int_{-\infty}^{\infty}|x|^q dF_0(x) < \infty. \tag{4.6}$$

Then $\kappa_{p,q,r}$ is a continuous and increasing function of r on $[-D_1, D_2]$ and for all J_0, h, D_1, D_2, p and q there exist positive constants c and C such that

$$c \leqslant \kappa_{p,q,r} \leqslant C. \tag{4.7}$$

Again we shall only give a brief outline of the proof; for details see Albers (1976). In the first place we note that under the conditions above, (2.10) is satisfied. Hence, F_r indeed exists. The proof of (4.7) and of the continuity of $\kappa_{p,q,r}$ in r is straightforward. The monotonicity part is nontrivial, but here we can use a result of van Zwet (1964). Let F and $\tilde{F}$ be continuous symmetric df's and let G and $\tilde{G}$ be the corresponding inverse functions. Then van Zwet introduces the order relation $<_s$ defined by: $F <_s \tilde{F}$ if $\tilde{G}F$ is concave-convex on the support of F. An application of his Theorem 2.3.2 and of the consequences of this theorem now immediately shows that $F_r <_s F_{\tilde{r}}$ implies that $\kappa_{p,q,r} \leqslant \kappa_{p,q,\tilde{r}}$. It is easily verified that under (4.5), with b nonincreasing, the relation $F_r <_s F_{\tilde{r}}$ holds for $r \leqslant \tilde{r}$. Hence, $\kappa_{p,q,r}$ is nondecreasing in r. By taking b decreasing this result can be strengthened to the desired one.

Next we shall investigate the asymptotic properties of $S_{p,q}$ in order to verify (3.3) and (3.4) and to obtain an explicit expression for the asymptotic deficiency $d(\tilde{r})$ in (3.5). To simplify the notation we shall omit the indices p and q if no confusion is likely. Hence, in the sequel we shall write K, κ, κ_r and S, rather than $K_{p,q}$, $\kappa_{p,q}$, $\kappa_{p,q,r}$ and $S_{p,q}$. The idea is now the following: in the first place we note that under suitable moment conditions on the underlying df $F_{\tilde{r}}$, the statistic K will satisfy $\kappa_{-D_1} < K < \kappa_{D_2}$ for each $\tilde{r} \in (-D_1, D_2)$ with such large probability that it suffices to consider this case for the determination of the asymptotic behaviour of S to $o(N^{-1})$. Then the equation $K = \kappa_S$ has a unique solution. Obviously, the same holds for

$$K - \kappa_{\tilde{r}} = \kappa_S - \kappa_{\tilde{r}}. \tag{4.8}$$

Under suitable moment conditions, $K \rightarrow_P \kappa_{\tilde{r}}$ and both sides of (4.8) are small with high probability. As, moreover, κ_r is continuous, bounded and increasing, we can expand $\kappa_S - \kappa_{\tilde{r}}$ in powers of $(S - \tilde{r})$. Moreover $K - \kappa_{\tilde{r}}$ can be expanded in powers of $N^{-1}\Sigma\{|X_j|^v - E_0|X_j|^v\}$, where $v = p, q$. Through (4.8) this then leads to an expansion for $(S - \tilde{r})$ from which the desired results in connection with (3.1)–(3.5) can be obtained. The program above is carried out in detail in Section 5 of Albers (1976). Here we shall only present the final result.

Let

$$H_{k,r}(t) = \int_0^t \frac{b^k(u)}{\{1 + rb(u)\}^{k+1}} dG_0\left(\frac{1+u}{2}\right), \tag{4.9}$$

where $k = 0$ or 1, $-D_1 \leqslant r \leqslant D_2$, $0 \leqslant t < 1$ and b as introduced in (4.5). Note that $H_{0,r}(t) = G_r([1 + t]/2)$ with G_r as in (2.11). Moreover, let Ω be the class of all twice continuously differentiable functions ω on (0, 1) for which $\lim\sup_{t\rightarrow 0,1} \{t(1-t)|\omega''(t)/\omega'(t)|\} < 3/2$. Then we have

THEOREM 4.2. *Assume that the conditions of Theorems* 2.1, 3.1 *and* 4.1 *hold, except, of course,* (3.3) *and* (3.4). *Moreover, suppose that* $\tilde{r}$ *is such that*

$$0 < \int |x|^p dF_{\tilde{r}}(x), \qquad \int |x|^{6q} dF_{\tilde{r}}(x) < \infty$$

and $H^p_{0,\bar{r}} \in \Omega$, $H^q_{0,\bar{r}} \in \Omega$ *and* $J_{\bar{r}} \in \Omega$ *for* $\bar{r} = \tilde{r} - I(J_{\tilde{r}}^2)I^{-1}(J_{\tilde{r}}h)$. *Then* (3.3) *and* (3.4) *hold and* (3.1) *and* (3.5) *can be replaced by*

$$\begin{aligned} d(\tilde{r}) = {} & I(J_{\tilde{r}}^2)I^2(J_{\tilde{r}}h)I^{-3}(J_{\tilde{r}}^2)I^{-2}(Q_{\tilde{r}})\{I(W_{\tilde{r}}^2) - I^2(W_{\tilde{r}})\} \\ & + I(J_{\tilde{r}}h)I^{-2}(J_{\tilde{r}}^2)I^{-1}(Q_{\tilde{r}})\{I(W_{\tilde{r}}M_{\tilde{r}}) - I(W_{\tilde{r}})I(M_{\tilde{r}})\}, \end{aligned} \tag{4.10}$$

where $Q_{\tilde{r}} = -\{H^{q-1}_{0,\tilde{r}} - I(H^q_{0,\tilde{r}})I^{-1}(H^p_{0,\tilde{r}})H^{p-1}_{0,\tilde{r}}\}H_{1,\tilde{r}}$, $W_{\tilde{r}} = q^{-1}H^q_{0,\tilde{r}} - p^{-1}I(H^q_{0,\tilde{r}})I^{-1}(H^p_{0,\tilde{r}})H^p_{0,\tilde{r}}$ *and* $M_{\tilde{r}}(t) = \int_0^t J_{\tilde{r}}(t)dJ_{\tilde{r}}(t)$.

To see the relation between (4.10) and (3.1), note that $I(h^2)I^{-1}(J_{\tilde{r}}^2) - I^2(J_{\tilde{r}}h)I^{-2}(J_{\tilde{r}}^2) = I(J_{\tilde{r}}^2)I^2(J_{\tilde{r}}h)I^{-3}(J_{\tilde{r}}^2)$, whereas $EJ_{\tilde{r}}(U_{j:N})I(J_{\tilde{r}}h)I^{-2}(J_{\tilde{r}}^2) - Eh(U_{j:N})I^{-1}(J_{\tilde{r}}^2) = I(J_{\tilde{r}}h)I^{-2}(J_{\tilde{r}}^2)EJ_{\bar{r}}(U_{j:N})$.

As an application of the theorem we reconsider the examples from Section 3. Again we begin by choosing $J_0 = J^{(1)}$ and $h = J^{(2)}$ (cf. (3.7)) and again we let $\tilde{r} = 0$. First we take $(p, q) = (2, 4)$, i.e., we essentially base $S_{p,q}$ on the sample kurtosis. After tedious computations which we shall omit here, we obtain that for this $\mathcal{J}$ and this $S_{p,q}$ the asymptotic deficiency of the adaptive rank test w.r.t. the absolute normal scores test equals $d(0) = 18(\pi - 3) + \frac{3}{2} = 4.0 \cdots$. For $(p, q) = (1, 2)$ instead of $(2, 4)$ we obtain $d(0) = (\pi - 3)^2/(6I^2) + \{(\pi/2 + 2)(2)^{\frac{1}{2}} - 5\}/I = 4.7 \cdots$, where $I = \pi(2)^{1/2}\int_0^\infty\{\Phi(x)(1 - \Phi(x))^2\}/\phi(x)dx - 1$ has been obtained numerically.

As a second and final example take $J_0 = J^{(1)}$ and $h = J^{(3)}$. For $\tilde{r} = 0$ and $(p, q) = (2, 4)$ we find that $d(0) = 27\pi/4 - 12 = 9.2 \cdots$. For $(p, q) = (1, 2)$ we have $d(0) = (\pi - 3)\{(\pi/4 - 1/2)/I^2 + 1/I\} = 6.7 \cdots$, where $I = \pi\int_0^\infty(1 - \Phi(x))^2/\phi(x)dx - 1$ has been obtained numerically.

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