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NOTE ON A RESULT OF SEEGER IN PARTITIONING NORMAL POPULATIONS

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The problem of partitioning a set of normal populations by their locations with respect to a control was investigated by Tong, and Seeger considered an extension of this to a comparison with 2 controls. This note points out that the main result in the latter paper is in error, and provides a table for use in the application of Seeger's procedure.

The problem of partitioning a set of k normal populations with respect to one control is considered by Tong [3]. The partitioning is according to the means, and the $k + 1$ populations are assumed to have a common variance σ^2. Seeger [2] examines the case of partitioning with respect to 2 or more controls, and the reader is referred to the notation and classification procedure in Sections 1 and 2 of [2]. The purpose of this note is to point out that Seeger's use of Tong's Table 1 is in error, and to provide an appropriate table.

Seeger's procedure R, based on samples of size n from each of the $k + 2$ populations is such that the probability of a correct decision is not less than a prescribed value P^*, i.e.,

$$(1) \qquad P(CD \mid \mu, \sigma^2; R) \geqslant P^* \quad \text{for every vector } \mu \text{ in a preference zone.}$$

He requires values of r_1, r_2 $(0 \leqslant r_1 + r_2 \leqslant k)$ which minimize

$$(2) \qquad P(Y_s \leqslant b;\ s = 1, 2, \cdots, 2k - r_1 - r_2) = H^{\Sigma}_{2k-r_1-r_2}(b),$$

where $(Y_1, \cdots, Y_{2k-r_1-r_2})$ has a multivariate normal distribution with mean vector 0 and covariance matrix Σ. For this (r_1, r_2), the solution of

$$(3) \qquad H^{\Sigma}_{2k-r_1-r_2}(b) = P^*$$

for b is then required, and the common sample size n determined from $(a/\sigma)(n/2)^{\frac{1}{2}} = b$ where a is a prescribed constant.

Because of the complicated nature of Σ, Seeger makes use of Tong's Theorem 1.2 from which it follows that $h = h(k, P^*; r_1, r_2)$ as a solution of

$$(4) \qquad H^{\Sigma_1}_{k-r_2}(h) + H^{\Sigma_2}_{k-r_1}(h) = 1 + P^*$$

may be used to obtain a conservative value of n since $h \geqslant b$. Let

$$\Phi_+ = \Phi(y + 2^{\frac{1}{2}}h),\ \Phi_- = \Phi(-y + 2^{\frac{1}{2}}h)$$

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and

$$S(r_1, r_2) = \int_{-\infty}^{\infty} \Phi_-^{r_1} \Phi_+^{k-r_1-r_2}\, d\Phi(y).$$

Then the left hand side of (4) can be written as

$$(5) \qquad T(r_1, r_2) = S(r_1, r_2) + S(r_2, r_1) = \int_{-\infty}^{\infty} \Phi_+^{k-r_1-r_2} (\Phi_-^{r_1} + \Phi_-^{r_2})\, d\Phi(y).$$

The problem is to find integers r_1, r_2 $(0 \leqslant r_1 + r_2 \leqslant k)$ which minimize T for all $h > 0$. Define

$$(6) \qquad \eta(m) = (m/2) \quad \text{for } m \text{ even}; \quad = (m-1)/2 \text{ or } (m+1)/2 \quad \text{for } m \text{ odd}.$$

Now Tong [3] shows that for $0 \leqslant r \leqslant k'$

$$(7) \qquad \beta(r) = \int_{-\infty}^{\infty} \Phi_+^{k'-r} \Phi_-^{r}\, d\Phi(y)$$

is minimized *for given* k' by $r = \eta(k')$. Seeger [2] applies this result to $T(r_1, r_2)$ by identifying $S(r_1, r_2)$ with $\beta(r)$ of (7) where $k' = k - r_2$. He claims $S(r_1, r_2)$ is minimized by $r_1 = \eta(k - r_2)$ and $S(r_2, r_1)$ minimized by $r_2 = \eta(k - r_1)$ and hence the sum is minimized by the simultaneous solution of $r_1 = \eta(k - r_2)$ and $r_2 = \eta(k - r_1)$; i.e., $r_1 = r_2 = [k/3]$. This is incorrect as the minimization is not w.r.t. r_1 and r_2 but only w.r.t. $r_1(r_2)$ for $r_2(r_1)$ constant. Clearly $S(r_1, r_2)$ is an increasing function of r_2 and is minimized by

$$(8) \qquad r_2 = 0 \quad \text{and} \quad r_1 = \eta(k).$$

Likewise $S(r_2, r_1)$ is minimized by

$$(9) \qquad r_1 = 0 \quad \text{and} \quad r_2 = \eta(k).$$

Now (8) and (9) cannot hold simultaneously so the minimum value of T cannot be found by minimizing the two terms in the sum separately.

So that the left-hand side of (4) is not less than $1 + P^*$ for all r_1, r_2 we require

$$(10) \qquad h^* = \sup_{r_1, r_2} h(k, P^*; r_1, r_2).$$

Now, for given $c = r_1 + r_2$, $T(r_1, r_2)$ is minimized when r_1 and r_2 are as close together as possible since $\Phi^r(-y + 2^{\frac{1}{2}}h)$ is a convex function of r for every fixed y, h. So, restricting consideration to values of r_1, r_2 satisfying $|r_1 - r_2| \leqslant 1$, $0 \leqslant r_1 + r_2 \leqslant k$, (4) was solved numerically, first expressing it in the form (5). Values of h^* (to $5D$) were found for $P^* = 0.75, 0.90, 0.95, 0.975, 0.99$ and $k = 2(1)12(2)20$, and the required sample size n is the smallest integer exceeding $2h^{*2}\sigma^2/a^2$. When h^* corresponds to $(r_1, r_2) = (0, 0)$, the problem reduces to finding equicoordinate $50(1 + P^*)$ percentage points of a standardized k-variate normal distribution with correlations $1/2$. Such a solution is given, e.g., by Gupta [1] with values of h^* to $3D$. Table 1 gives values of h^* for the P^* and k above in the cases where $(r_1, r_2) \neq (0, 0)$. [Note that for $P^* = 0.95, 0.975$ and 0.99, $r_1 = r_2 = 0$ for all the k values considered.]

TABLE 1
Values of h^* satisfying (4) *and* (10) *and the corresponding* $(r_1, r_2) \neq (0, 0)$

k	$P^* = 0.75$	k	$P^* = 0.75$	k	$P^* = 0.90$
5	1.80303(1, 0)	11	2.07990(1, 1)		
6	1.86996(1, 0)	12	2.10841(2, 1)	12	2.50229(1, 0)
7	1.92625(1, 1)	14	2.15897(2, 2)	14	2.54792(1, 1)
8	1.97389(1, 1)	16	2.20111(2, 2)	16	2.58670(1, 1)
9	2.01422(1, 1)	18	2.23747(3, 2)	18	2.62015(1, 1)
10	2.04914(1, 1)	20	2.27029(3, 3)	20	2.64976(2, 1)

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