The Annals of Statistics
1979, Vol. 7, No. 4, 882–889

A CHARACTERIZATION OF THE UNIFORM DISTRIBUTION ON THE CIRCLE

BY J. T. KENT[1], K. V. MARDIA AND J. S. RAO[1]

University of Cambridge, University of Leeds and University of California at Santa Barbara

Geary's characterization of the normal distribution asserts that if $n > 2$ i.i.d. observations come from some distribution on the line, then the sample mean and variance are independent if and only if the observations are normally distributed. A similar characterization is established here for the uniform distribution on the circle. Given a sample of $n > 2$ i.i.d. random angles from a distribution defined by a density on the circle satisfying some mild regularity conditions, the sample mean direction and resultant length are independent if and only if the angles come from the uniform distribution.

1. Introduction. If $\theta_1, \cdots, \theta_n$ is an i.i.d. sample of points on the circle, set

$$C = \Sigma_1^n \cos \theta_j, \qquad S = \Sigma_1^n \sin \theta_j, \tag{1.1}$$

and if $C^2 + S^2 > 0$, define the *sample mean direction* $\bar{\theta}$ and *resultant length* $R > 0$ by

$$Re^{i\bar{\theta}} = C + iS. \tag{1.2}$$

Then $\bar{\theta}$ and R play analogous roles to the sample mean and variance on the line.

An important independence property for the sample mean and sample variance is due to Geary (1936). Under certain conditions he showed that the sample mean and variance of $n \geqslant 2$ i.i.d. observations from some distribution on the line are independent if and only if the distribution is normal. Successive refinements of this result have removed the need for regularity conditions (e.g., see Kagan, Linnik and Rao, 1973, page 103).

Due to the analogy between $(\bar{\theta}, R)$ and the sample mean and variance on the line, it is of interest to ask whether there are any distributions on the circle for which $\bar{\theta}$ and R are independent. Clearly $\bar{\theta}$ and R are independent if $\theta_1, \cdots, \theta_n$ come from the uniform distribution. The main result of this paper is that, subject to some regularity conditions, the uniform distribution is the *only* such distribution.

THEOREM 1. *Fix $n \geqslant 2$, let $f(\theta)$ be a density which is continuous* a.e., *and suppose $\theta_1, \cdots, \theta_n$ is an* i.i.d. *sample from f. Then the sample mean direction $\bar{\theta}$ and resultant length R are independent if and only if $f(\theta)$ is the uniform density, that is $f(\theta) = 1/2\pi$* a.e.

Received September 1977; May 1978.

[1]Research carried out when the author was visiting the University of Leeds under a Science Research Council grant. Dr. Kent is now at the University of Leeds.

AMS 1970 *subject classification.* Primary 62E10.

Key words and phrases. Directional data, uniform distribution, independence of mean direction, resultant length.

The proof is based on an analysis of the joint density of R and $\bar{\theta}$ and will occupy the rest of the paper. Let $m = \text{ess inf}\{2\pi f(\theta) : \theta \in [0, 2\pi)\}$. The argument breaks naturally into two parts depending on whether $m > 0$ (Section 5) or $m = 0$ (Section 6). In the former case the proof is based on Holder's inequality. In the latter it is shown that the hypothesis of independence leads to a contradiction; namely, independence implies the existence of points for which the joint density of R and $\bar{\theta}$ is both zero and positive.

2. Distributions of $(R, \bar{\theta})$ and (C, S). For a sample of n observations denote the density of (C, S) by $g_n(C, S)$ and the density of $(R, \bar{\theta})$ by $h_n(R, \bar{\theta})$. A change of variables shows that

$$g_n(C, S) = R^{-1}h_n(R, \bar{\theta}) \tag{2.1}$$

where (C, S) and $(R, \bar{\theta})$ are connected by (1.2). We shall not distinguish notationally between random vectors and their realizations in this paper.

Let $p_n(R)$ and $q_n(\bar{\theta})$ denote the marginal densities of R and $\bar{\theta}$. Then, for fixed $n \geqslant 2$, R and $\bar{\theta}$ are independent if and only if

$$h_n(R, \bar{\theta}) = p_n(R)q_n(\bar{\theta}), \qquad \text{a.e.} \quad (R, \theta). \tag{2.2}$$

To get an equation which holds everywhere it is convenient to define

$$h_n^-(R_0, \bar{\theta}_0) = \text{ess lim inf}_{(R, \bar{\theta}) \to (R_0, \bar{\theta}_0)} h_n(R, \bar{\theta}),$$

and similarly for the other densities. (Recall that essential limits are unchanged if h_n is altered on a set of measure zero.) Then if R and $\bar{\theta}$ are independent,

$$h_n^-(R, \bar{\theta}_0) = p_n^-(R)q_n^-(\bar{\theta}) \qquad \text{for all} \quad R \geqslant 0, \bar{\theta} \in [0, 2\pi). \tag{2.3}$$

Since f is continuous a.e., $f(\theta) = f^-(\theta)$ a.e., so f and f^- are versions of the same density. Hence, without loss of generality, we shall suppose throughout the paper that

$$f(\theta) = f^-(\theta) \qquad \text{everywhere.}$$

It is convenient to define $h_n(R, \bar{\theta})$ to be *that version* of the density of $(R, \bar{\theta})$ obtained by using (2.4)–(2.6) below. We do not need to specify any particular versions for $p_n(R)$ and $q_n(\bar{\theta})$.

For $n = 2$ we can calculate $h_n(R, \bar{\theta})$ explicitly. Suppose $0 < R < 2$ and let $\delta = \cos^{-1} R/2$. In the two-to-one transformation $(\theta_1, \theta_2) \to (R, \bar{\theta})$, (θ_1, θ_2) is defined in terms of $(R, \bar{\theta})$ by $\theta_1 = \bar{\theta} \pm \delta$, $\theta_2 = \bar{\theta} \mp \delta$. Thus we can define

$$\begin{aligned} g_2(C, S) &= R^{-1}h_2(R, \bar{\theta}) \\ &= 2R^{-1}f(\theta_1)f(\theta_2)|\partial(\theta_1, \theta_2)/\partial(R, \bar{\theta})| \\ &= 4f(\bar{\theta} + \delta)f(\bar{\theta} - \delta)/\left[R(4 - R^2)^{\frac{1}{2}}\right], \\ &\qquad (0 < R < 2), \end{aligned} \tag{2.4}$$

and $= 0$ for $R > 2$. (For $R = 0$ or 2, $g_2(C, S)$ is not necessarily well defined, but this ambiguity does not affect the value of the integral in (2.5) given below.)

For $n \geqslant 3$, $g_n(C, S)$ may be defined inductively for *all* (C, S) by the convolution formula

$$(2.5)\quad g_n(C, S) = \int_0^{2\pi} g_{n-1}(C - \cos\theta, S - \sin\theta) f(\theta)\, d\theta,$$
$$= \int_0^{2\pi} \cdots \int_0^{2\pi} g_2\big(C - \Sigma_{j=3}^n \cos\theta_j, S - \Sigma_{j=3}^n \sin\theta_j\big) \Pi_{j=3}^n \big[f(\theta_j)\, d\theta_j\big].$$

Thus, for $n \geqslant 3$ and $R > 0$, $h_n(R, \bar\theta)$ is given by

$$(2.6)\qquad h_n(R, \bar\theta) = R \int_{A(R, \bar\theta)} \frac{4\Pi_1^n f(\theta_j)}{R'(4 - R'^2)^{\frac{1}{2}}} \Pi_3^n\, d\theta_j,$$

where

$$A(R, \bar\theta) = \big\{(\theta_3, \cdots, \theta_n) : 0 < |Re^{i\bar\theta} - \Sigma_3^n e^{i\theta_j}| < 2\big\}.$$

In the integrand, θ_1, θ_2, and R' are functions of $(\theta_3, \cdots, \theta_n, R, \bar\theta)$ defined by

$$(2.7)\qquad R' e^{i\bar\theta'} = Re^{i\bar\theta} - \Sigma_3^n e^{i\theta_j}$$

and

$$(2.8)\qquad \theta_1 = \bar\theta' + \cos^{-1} R'/2, \quad \theta_2 = \bar\theta' - \cos^{-1} R'/2.$$

Thus, for all $n \geqslant 2$, $h_n(R, \bar\theta)$ is well defined for $0 < R < n$. Note also that $g_n(C, S) = R^{-1} h_n(R, \bar\theta) = 0$ for $R > n$.

3. The uniform case. We shall use "~" on top of any of the densities described above when the sample comes from the uniform distribution. Thus $\tilde f(\theta) \equiv 1/2\pi$, $\tilde q_n(\bar\theta) \equiv 1/2\pi$ and $\tilde h_n(R, \bar\theta) = \tilde p_n(R)/2\pi$.

From (2.4)–(2.6) it easily follows that $\tilde g_n(C, S)$ is a continuous function at all points for which $R = (C^2 + S^2)^{\frac{1}{2}}$ is not an integer, and that

$$(3.1)\qquad 0 < \tilde g_n(C, S) < \infty \qquad 0 < R < n,\ R \text{ not an integer}.$$

(In passing we note that, in fact, the only discontinuities of $\tilde g_n(C, S)$ are infinite singularities at $R = 0, 2$ when $n = 2$, at $R = 1$ when $n = 3$ and at $R = 0$ when $n = 4$, and also a jump discontinuity at $R = 3$ when $n = 3$. See Pearson, 1906.)

4. The density of $\bar\theta$ when R and $\bar\theta$ are independent.

LEMMA 4.1. *Fix $n \geqslant 2$. If R and $\bar\theta$ are independent, then the marginal density of $\bar\theta$ is given by*

$$(4.1)\qquad q_n(\bar\theta) = \big[2\pi f(\bar\theta)\big]^n / K, \quad \text{a.e.} \quad (\bar\theta),$$

where

$$(4.2)\qquad K = \int_0^{2\pi} \big[2\pi f(\theta)\big]^n\, d\theta < \infty.$$

PROOF. Let θ_0 be a continuity point of $f(\theta)$. We shall examine the behaviour of $h_n(R_0, \theta_0)$ as $R_0 \to n$. First note that given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$|\theta - \theta_0| < \delta \Rightarrow |f(\theta) - f(\theta_0)| < \varepsilon$. If δ is small, a Taylor expansion of $\cos\theta$ shows $1 - \cos(\theta - \theta_0) < \delta^2/3 \Rightarrow |\theta - \theta_0| < \delta$.

Second, let R_0 satisfy $n - \delta^2/3 < R_0 < n$. If $(\theta_1, \cdots, \theta_n)$ satisfies $\Sigma_1^n \cos\theta_j = R_0\cos\theta_0$, $\Sigma_1^n \sin\theta_j = R_0 \sin\theta_0$, then $\cos(\theta_j - \theta_0) > 1 - \delta^2/3$ for $j = 1, \cdots, n$. Thus $|f(\theta_j) - f(\theta_0)| < \varepsilon$ for $j = 1, \cdots, n$ and further,

$$|\Pi_1^n f(\theta_j) - f(\theta_0)^n| < n\varepsilon[f(\theta_0) + \varepsilon]^{n-1}. \tag{4.3}$$

Using (4.3) in (2.4) and (2.6) yields

$$|R_0^{-1}h_n(R_0, \theta_0) - [2\pi f(\theta_0)]^n R_0^{-1}\tilde{h}_n(R_0, \theta_0)| \leqslant n\varepsilon(2\pi)^n[f(\theta_0) + \varepsilon]^{n-1}R_0^{-1}\tilde{h}_n(R_0, \theta_0).$$

Divide by $R_0^{-1}\tilde{h}_n(R_0, \theta_0) = \tilde{p}_n(R_0)/[2\pi R_0]$, which is positive and finite by (3.1). Since ε is arbitrary, we see that

$$\lim_{R\uparrow n}\frac{h_n(R, \theta_0)}{\tilde{h}_n(R, \theta_0)} = [2\pi f(\theta_0)]^n. \tag{4.4}$$

Now (4.4) holds for almost all θ_0 because f is continuous a.e. Also, the independence hypothesis implies $h_n(R, \theta) = p_n(R)q_n(\theta)$, a.e. (R, θ). Thus

$$\text{ess}\lim_{R\uparrow n}\frac{p_n(R)q_n(\theta)}{\tilde{p}_n(R)/2\pi} = q_n(\theta)\text{ess}\lim_{R\uparrow n}\frac{p_n(R)}{\tilde{p}_n(R)/2\pi} = [2\pi f(\theta)]^n, \quad \text{a.e.} \quad (\theta). \tag{4.5}$$

Therefore, $q_n(\theta)$ is proportional to $[f(\theta)]^n$ a.e., so K is finite and (4.1) holds. □

5. Proof of Theorem 1 when $m > 0$. Suppose $m = \inf 2\pi f(\theta) > 0$. We shall show that a contradiction arises if f is *not* uniform. Note that (2.4) and (2.6) immediately imply that

$$h_n(R, \bar{\theta}) \geqslant m^n\tilde{h}_n(R, \bar{\theta}) \qquad \text{for all} \quad 0 < R < n, \bar{\theta} \in [0, 2\pi). \tag{5.1}$$

By Holder's inequality

$$1 = \textstyle\int_0^{2\pi} 1 f(\theta)\, d\theta < [\int_0^{2\pi} 1^{n/(n-1)}\, d\theta]^{(n-1)/n}[\int_0^{2\pi} f(\theta)^n\, d\theta]^{1/n} = (2\pi)^{1-1/n}[\int f(\theta)^n\, d\theta]^{1/n}.$$

Thus, $K = \int_0^{2\pi}[2\pi f(\theta)]^n\, d\theta > 2\pi$. The inequality is strict because we are supposing f is nonconstant.

Pick $\delta > 0$ such that $(m + \delta)^n/K < m^n/2\pi$. Then let θ_0 be a continuity point such that $2\pi f(\theta_0) < m + \delta$. By Lemma 4.1

$$q_n^-(\theta_0) = [2\pi f(\theta_0)]^n/K < m^n/2\pi. \tag{5.2}$$

However, integrating (5.1) over R yields

$$q_n(\theta) \geqslant m^n\tilde{q}_n(\theta) = m^n/2\pi$$

for almost all θ. Thus, $q_n^-(\theta_0) \geqslant m^n/2\pi$, contradicting (5.2).

6. Proof of Theorem 1 when $m = 0$. The following lemma is useful when $m = 0$.

LEMMA 6.1. *Let $n \geqslant 2$, $\bar{\theta}_0 \in [0, 2\pi)$ and $0 < R_0 < n$. Suppose there exists $(\theta_1^0, \cdots, \theta_n^0)$ such that $0 < |e^{i\theta_1^0} + e^{i\theta_2^0}| < 2$, $R_0 e^{i\bar{\theta}_0} = \Sigma_1^n e^{i\theta_j^0}$, and $f(\theta_j^0) > 0$ for $j = 1, \cdots, n$. Then $h_n^-(R_0, \bar{\theta}_0) > 0$.*

PROOF. The properties of $(\theta_1^0, \cdots, \theta_n^0)$ ensure that there exist numbers $\delta_1 > 0$ and $\varepsilon > 0$ such that $|\theta - \theta_j^0| < \delta_1 \Rightarrow f^-(\theta) > \varepsilon$, for each $j = 1, \cdots, n$. If $n = 2$ we see from (2.4) that

$$R^{-1}h_2(R, \bar{\theta}) > \varepsilon^2$$

for $(R, \bar{\theta})$ in some neighbourhood of $(R_0, \bar{\theta}_0)$, because $\bar{\theta} + \cos^{-1} R/2$ and $\bar{\theta} - \cos^{-1} R/2$ are continuous functions of $(R, \bar{\theta})$, and because $4/[R(4 - R^2)^{\frac{1}{2}}] \geqslant 1$ for $0 < R < 2$. Thus, $h_2^-(R_0, \bar{\theta}_0) > 0$.

For $n \geqslant 3$ it is straightforward to show that there exists $\delta > 0 (\delta < \delta_1)$ such that, if $|R - R_0| < \delta$, $|\bar{\theta} - \bar{\theta}_0| < \delta$ and $|\theta_j - \theta_j^0| < \delta$, $j = 3, \cdots, n$, then $0 < R' < 2$, $(\theta_3, \cdots, \theta_n) \in A(R, \bar{\theta})$, and each of (θ_1, θ_2) lies within δ_1 or (θ_1^0, θ_2^0) or (θ_2^0, θ_1^0), where θ_1, θ_2 and R' are defined by (2.7)–(2.8). Then by (2.6)

$$h_n(R, \bar{\theta}) \geqslant R(2\delta)^{n-2}\varepsilon^n$$

for $|R - R_0| < \delta$, $|\bar{\theta} - \bar{\theta}_0| < \delta$, and so $h_n^-(R_0, \bar{\theta}_0) > 0$. (Note that we do not exclude the possibility that $h_n^-(R_0, \bar{\theta}_0) = \infty$.) □

Suppose $m = \inf 2\pi f(\theta) = 0$. The proof here is a bit trickier. Let θ_0 be a continuity point of f such that $f(\theta_0) > 0$. Set

$$a = \sup\{a' \in (\theta_0, \theta_0 + 2\pi) : \text{ess inf}_{\theta \in (\theta_0, a')} f(\theta) > 0\}$$

and

$$b = \sup\{b' \in [a, a + 2\pi) : \text{ess sup}_{\theta \in (a, b')} f(\theta) = 0\}.$$

Note that $a \neq \theta_0$, but it is possible that $b = a$. Also $f(\theta) > 0$ for $\theta \in [\theta_0, a)$ and $f(\theta) = 0$ for $\theta \in (a, b)$. Since $m = 0$ the arc $[a, b] = I$, measured counterclockwise, is well defined and $b < \theta_0 + 2\pi$. Let $|I|$ denote the length of I. Then the argument depends on whether $|I| = 0$, $0 < |I| < \pi$, or $|I| \geqslant \pi$. We shall show that the hypothesis of independence of R and $\bar{\theta}$ leads to a contradiction in each case by constructing a resultant vector $(R_0, \bar{\theta}_0)$ for which $h_n^-(R_0, \theta_0)$ is positive (or zero), and yet for which the product of the marginal densities $p_n^-(R_0)q_n^-(\bar{\theta}_0)$ is zero (or positive).

CASE 1. Suppose $a = b$. For simplicity rotate the zero direction so that $a = b = 0$ and $-2\pi < \theta_0 < 0$. We proceed by constructing a resultant vector $(R_0, 0)$ for which $h_n^-(R_0, 0) > 0$. By the definition of b, there exist points $\phi > 0$ arbitrarily close to 0 such that $f(\phi) > 0$. Pick such a point $\phi_0 > 0$ close enough to 0 so that $\phi_0 < \pi/6$ and so that ψ lies in $(\theta_0, 0)$, where $-\pi/2 < \psi < 0$ is defined by $\sin \psi = -2 \sin \phi_0$. Then by the definition of a, $f(\psi) > 0$ and $f(-\phi_0) > 0$.

Choose points $(\theta_1^*, \cdots, \theta_n^*)$ from the set $\{\phi_0, -\phi_0, \psi\}$ so that $\Sigma_1^n \sin \theta_j^* = 0$ and let $R_0 = \Sigma_1^n \cos \theta_j^*$. Then $f(\theta_j^*) > 0$ for $j = 1, \cdots, n$, so by Lemma 6.1, $h_n^-(R_0, 0) > 0$. However, the definition of a implies that $f(0) = f^-(0) = 0$ and hence by Lemma 4.1, $q_n^-(0) = 0$, contradicting (2.3).

CASE 2. Suppose $0 < |I| < \pi$. Rotate the zero direction so that $a = -b$, (so $0 < b < \pi/2$ and $\theta_0 < -b$). Then $f(\theta) = 0$ for $-b < \theta < b$. We shall construct a resultant vector $(R_0, \bar{\theta}_0)$, $-b < \bar{\theta}_0 < b$, for which $h_n^-(R_0, \bar{\theta}_0) > 0$.

Arguing as above, choose a point $\phi_0 > b$ satisfying $f(\phi_0) > 0$, which is close enough to b so that $\phi_0 < \pi/2$, $-\phi_0 \in (\theta_0, -b)$ and $\tan \phi_0 < n \tan b$. Set for $1 \leqslant j \leqslant n$,

$$\theta_j^* = \phi_0, \quad j \text{ odd}, \qquad \text{and} \quad \theta_j^* = -\phi_0, \quad j \text{ even},$$

and define $(R_0, \bar{\theta}_0)$ by $R_0 e^{i\bar{\theta}_0} = \Sigma_1^n e^{i\theta_j^*}$. As above $f(\theta_j^*) > 0$ for $j = 1, \cdots, n$, so $h_n^-(R_0, \bar{\theta}_0) > 0$. However $\bar{\theta}_0 \in (-b, b)$, which implies $q_n^-(\bar{\theta}_0) = 0$, again contradicting (2.3).

CASE 3. Suppose $|I| \geqslant \pi$ and rotate the zero direction so that $a = 0$ and $\pi \leqslant b < 2\pi$. We shall construct $(R_0, \bar{\theta}_0)$ for which the joint density $h_n^-(R_0, \bar{\theta}_0)$ vanishes, yet for which each of the marginal densities is strictly positive. From (4.5) we see that $p_n^-(R) > 0$ for $n - \delta \leqslant R < n$, for some number $\delta > 0$. Let us suppose $\delta < \frac{1}{4}$.

From the definition of a, there exists $\varepsilon > 0$ such that $f(\theta) > 0$ for all $-\varepsilon < \theta < a = 0$. Pick a number γ such that

$$\gamma < \min\left(\varepsilon, [2\delta/(n-\delta)]^{\frac{1}{2}}, [\delta/n^3]^{\frac{1}{2}}\right), \tag{6.1}$$

and let R_0 and $\bar{\theta}_0$ be any numbers satisfying

$$n - \delta < R_0 < n - \tfrac{1}{2}\delta, \quad -\gamma < \bar{\theta}_0 < 0. \tag{6.2}$$

Since $f(\theta) = 0$ if $\sin \theta > 0$, it follows from (2.4) and (2.6) that $h_n(R_0, \bar{\theta}_0) = 0$ if there does not exist any $(\theta_1, \cdots, \theta_n)$ satisfying

$$R_0 e^{i\bar{\theta}_0} = \Sigma_1^n e^{i\theta_j}, \tag{6.3}$$

$$\sin \theta_j \leqslant 0 \qquad \text{for} \quad j = 1, \cdots, n. \tag{6.4}$$

Suppose, if possible, that $(\theta_1, \cdots, \theta_n)$ satisfying (6.3)–(6.4) exists. Then

$$-\Sigma_1^n \sin \theta_j = -R_0 \sin \bar{\theta}_0 < n\gamma, \tag{6.5}$$

$$\begin{aligned} \Sigma_1^n \cos \theta_j &= R_0 \cos \bar{\theta}_0 \\ &\geqslant (n - \delta)(1 - \gamma^2/2) \\ &\geqslant n - 2\delta \qquad \text{(from (6.1))}, \end{aligned} \tag{6.6}$$

and

$$\Sigma_1^n \cos \theta_j \leqslant R_0 < n - \delta/2. \tag{6.7}$$

Now (6.4) and (6.5) imply $0 \leqslant -\sin\theta_j \leqslant n\gamma$ for $j = 1, \cdots, n$. Therefore,

$$|\cos\theta_j| = \left(1 - \sin^2\theta_j\right)^{\frac{1}{2}} \geqslant (1 - n^2\gamma^2)^{\frac{1}{2}} \geqslant 1 - n^2\gamma^2/2$$

for $j = 1, \cdots, n$. Thus $|\Sigma_1^n \cos\theta_j - k| \leqslant n^3\gamma^2/2 < \delta/2$ for some integer k. However (6.6) and (6.7) imply that $\Sigma_1^n \cos\theta_j$ is *at least* a distance $\delta/2$ from its nearest integer, n. Therefore, there is *no* $(\theta_1, \cdots, \theta_n)$ satisfying (6.3) and (6.4), so $h_n(R_0, \bar{\theta}_0) = 0$. This statement is valid for all $(R_0, \bar{\theta}_0)$ satisfying (6.2), so

$$h_n^-(R, \bar{\theta}) = 0 \qquad \text{for} \quad n - \delta < R < n - \tfrac{1}{2}\delta, \; -\gamma < \bar{\theta} < 0.$$

However, $q_n^-(\bar{\theta}) > 0$ for $-\gamma < \bar{\theta} < 0$ because $f^-(\bar{\theta}) > 0$, and $p_n^-(R) > 0$ by choice of δ, thus contradicting (2.3).

7. Final remarks. 1. Note that there are two discrete counterexamples to the theorem just given. Firstly, if the distribution is degenerate at $\theta = \theta_0$, then R and $\bar{\theta}$ are independent for all $n \geqslant 2$. Secondly, and a bit less trivially, suppose the distribution gives weight $\frac{1}{2}$ to each of $\theta_0, \theta_0 + \pi$. Then R and $\bar{\theta}$ are independent for all $n \geqslant 2$ on the set $\{R \neq 0\}$. (Since $\bar{\theta}$ is not defined if $R = 0$ and since the set $\{R = 0\}$ has positive probability if n is even, this set must be avoided.) This latter distribution is the uniform distribution on a '0-dimensional' circle.

We conjecture that these two examples are in fact the *only* two nonuniform distributions for which R and $\bar{\theta}$ are independent. However, our result does seem to cover any possible case of practical interest.

2. Theorem 1 can be easily generalized to higher dimensional spheres, at least for densities for which $m > 0$. Let $\Omega_p = \{\mathbf{x} = (x_1, \cdots, x_p)^T : \Sigma x_i^2 = 1\}$ denote the unit sphere in R^p, let $\omega_p = 2\pi^{\frac{1}{2}p}/\Gamma(\frac{1}{2}p)$ be the surface area of Ω_p, and let $\omega_p(d\mathbf{x})$ denote Lebesgue measure on Ω_p. Thus, $\omega_p^{-1}\omega_p(d\mathbf{x})$ represents the uniform density.

Suppose $f(\mathbf{x})$ is a density (with respect to $\omega_p(d\mathbf{x})$) which is continuous a.e. and which satisfies $m = \text{ess inf}\{\omega_p f(\mathbf{x}) : \mathbf{x} \in \Omega_p\} > 0$. For a sample $\mathbf{x}_1, \cdots, \mathbf{x}_n$ from $f(\mathbf{x})(n \geqslant 2)$, define $R > 0$ and $\bar{\mathbf{x}} \in \Omega_p$ by $R\bar{\mathbf{x}} = \Sigma_1^n \mathbf{x}_j$. Then the arguments of Sections 4–5 can be adapted here to conclude that R and $\bar{\mathbf{x}}$ are independent if and only if $f(\mathbf{x})$ is the uniform density. We conjecture that the restriction $m > 0$ is unnecessary but, unfortunately, the techniques of Section 6 do not seem to generalize easily.

Acknowledgment. We are thankful to the referee for several helpful suggestions.

REFERENCES

[1] Geary, R. C. (1936). The distribution of "Student's" ratio for non-normal samples. *J. Roy. Statist. Soc. Ser. B* 3 178–184.

[2] KAGAN, A. M., LINNIK, Y. V. and RAO, C. R. (1973). *Characterization Problems in Mathematical Statistics*. John Wiley and Sons, New York.

[3] MARDIA, K. V. (1972). *Statistics of Directional Data*. Academic Press, London.

[4] MARDIA, K. V. (1975). Statistics of directional data (with discussion). *J. Roy. Statist. Soc. Ser. B* **37** 349–393.

[5] PEARSON, K. (1906). Mathematical contributions to the theory of evolution, XV. In *Drapers' Company Research Memoirs*. Biometric Series III.

DEPARTMENT OF STATISTICS
SCHOOL OF MATHEMATICS
UNIVERSITY OF LEEDS
LEEDS LS2 9JT
ENGLAND

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA 93106