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# AN ALTERNATIVE TO THE FRIEDMAN TEST WITH CERTAIN OPTIMALITY PROPERTIES

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R. L. Anderson proposed a $\chi^2$-type rank statistic for the nonparametric analysis of a randomized blocks design. In this paper the asymptotic distribution of the test statistic is derived under a sequence of alternatives contiguous to the null hypothesis. Using Bahadur's concept of local approximate slope, it is shown that the test is optimal within a large class of rank tests including the tests proposed by Friedman and by Page. The results are extended to BIB designs. Ties are considered.

**1. Introduction.** In 1959 Anderson [1] proposed a nonparametric test for the analysis of a randomized blocks experiment, which was rediscovered recently by Kannemann [5]. The method can be described as follows: consider a randomized blocks experiment consisting of $N$ blocks. Rank the observations in each block separately. Define

$$D_{kj} = \text{number of blocks in which treatment } j \text{ receives rank } k. \tag{1.1}$$

Compute the $\chi^2$-type test statistic

$$A = \frac{p}{N} \Sigma_{k=1}^{p} \Sigma_{j=1}^{p} \left( D_{kj} - \frac{N}{p} \right)^2 \tag{1.2}$$

and reject the hypothesis of no treatment effect if $A$ is too large. (Note that $\Sigma_{k=1}^{p} D_{kj} = \Sigma_{j=1}^{p} D_{kj} = N$, $\Sigma_{k=1}^{p} \Sigma_{j=1}^{p} D_{kj} = Np$, and that under the hypothesis $ED_{kj} = N/p$.)

In this article we derive some asymptotic properties of the Anderson test. In Section 2, it is shown that under the hypothesis $(p-1)/pA$ has an asymptotic $\chi^2_{(p-1)^2}$-distribution. In Section 3 the asymptotic distribution of $A$ as well as some of its competitors is obtained under a contiguous sequence of alternatives. Section 4 extends these results to BIB designs. Section 5 is devoted to efficiency comparisons. It is shown, using Bahadur's concept of local approximate slope, that Anderson's test has an optimality property: it is always as efficient as the optimal test of the Friedman type. The same holds with respect to tests of the Page type and with respect to the locally most powerful test. Section 6 contains some remarks about handling of ties.

**2. Asymptotic distribution under the hypothesis.** Let $X_{ij}$ $(i = 1, \cdots, N;\ j = 1, \cdots, p)$ be the observation under treatment $j$ in the $i$th block. As usual, we

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always assume that observations in different blocks are independent. Anderson derived the asymptotic distribution of $A_N$ under the null hypothesis of no treatment effect. But since our proof is considerably shorter and mainly serves as an introduction of notation, it is included here.

THEOREM 2.1. *For each $1 \leqslant i \leqslant N$ let $(X_{i1}, \cdots, X_{ip})$ have an absolutely continuous distribution symmetric in its $p$ arguments. Then $(p-1)/pA_N$ converges in law to a $\chi^2$-distribution with $(p-1)^2$ df, as $N \to \infty$.*

PROOF. Define $R_{ij}$ = rank of the $j$th observation within the $i$th block and set

$$Z_{kj}^{(i)} = 1 \quad \text{if} \quad R_{ij} = k; \qquad = 0 \quad \text{otherwise.} \tag{2.1}$$

By direct computation, using the fact that all rank permutations are equally likely, one obtains

$$EZ_{kj}^{(i)} = \frac{1}{p} \tag{2.2}$$

and

$$\begin{aligned} \operatorname{Cov}\left(Z_{kj}^{(i)}, Z_{k'j'}^{(i)}\right) &= \frac{p-1}{p^2} && \text{for } j = j', k = k'; \\ &= \frac{1}{p^2(p-1)} && \text{for } j \neq j', k \neq k'; \\ &= -\frac{1}{p^2} && \text{otherwise.} \end{aligned} \tag{2.3}$$

Let

$$Z^{(i)} = \left(Z_{11}^{(i)}, Z_{12}^{(i)}, \cdots, Z_{1p}^{(i)}, Z_{21}^{(i)}, \cdots, Z_{pp}^{(i)}\right)', \qquad 1 \leqslant i \leqslant N;$$

then this result can be written as

$$EZ^{(i)} = \frac{1}{p}1 \tag{2.4}$$

and

$$\Sigma = \operatorname{Cov} Z^{(i)} = \frac{1}{p-1} M_p \otimes M_p, \tag{2.5}$$

where $1' = (1, 1, \cdots, 1)$, $I_p$ = unit matrix of order $p$,

$$M_p = I_p - \frac{1}{p}11' \tag{2.6}$$

and $\otimes$ denotes the Kronecker product of matrices.

Because of the independence of the $Z^{(i)}$, $1 \leqslant i \leqslant N$, the multivariate central limit theorem implies

$$N^{-\frac{1}{2}}\left(\Sigma_{i=1}^{N} Z^{(i)} - \frac{N}{p}1\right) \to_{\mathcal{L}} N(0, \Sigma) \quad \text{as} \quad N \to \infty. \tag{2.7}$$

Since $M_p$ is idempotent, one gets

$$\Sigma^2 = (p-1)^{-2} M_p^2 \otimes M_p^2 = (p-1)^{-1}\Sigma, \tag{2.8}$$

and hence $(p-1)\Sigma$ is idempotent. By a standard result from multivariate analysis (e.g. Rao [8], page 443) it follows that $(p-1)N^{-1}\|\Sigma_{i=1}^{N} Z^{(i)} - (N/p)1\|^2$, where $\|x\|^2 = x'x$, converges to a $\chi_r^2$-distribution with $r = (p-1)\text{tr}\,\Sigma = (\text{tr}\,M_p)^2 = (p-1)^2$. Since $A_N = p/N\|\Sigma_{i=1}^{N} Z^{(i)} - (N/p)1\|^2$ this concludes the proof.

**3. The asymptotic distribution under contiguous alternatives.** In this section we assume that the observations have the joint density

$$Q_N = \Pi_{i=1}^{N}\Pi_{j=1}^{p} f(x_{ij} - \theta_{jN} - \alpha_i), \tag{3.1}$$

where the block effects $\alpha_i$, $1 \leqslant i \leqslant N$, are arbitrary and the treatment effects have the structure

$$\theta_{jN} = N^{-\frac{1}{2}} c_j \tag{3.2}$$

with

$$\Sigma_{j=1}^{p} c_j = 0, \qquad \Sigma_{j=1}^{p} c_j^2 = c_0 > 0. \tag{3.3}$$

Furthermore it is assumed that $f'$ exists and that

$$I(f) = \int \left(\frac{f'(x)}{f(x)}\right)^2 f(x)\, dx < \infty. \tag{3.4}$$

Related to this sequence of alternatives, we consider the sequence

$$P_N = \Pi_{i=1}^{N}\Pi_{j=1}^{p} f(x_{ij} - \alpha_i) \tag{3.5}$$

belonging to the hypothesis. Since we consider rankings within blocks only, our results do not depend on shifts between blocks, therefore, w.l.o.g., we assume $\alpha_i = 0$, $i = 1, \cdots, N$ for the rest of this paper.

For the proof of the theorem of this section it is convenient to introduce the notation

$$\psi(x) = \frac{f'(x)}{f(x)}; \tag{3.6}$$

$$Z_N = \Sigma_{i=1}^{N} Z^{(i)} \qquad \text{with elements} \quad Z_{N,k,j} = \Sigma_{i=1}^{N} Z_{kj}^{(i)}; \tag{3.7}$$

$$T_N = -\Sigma_{i=1}^{N}\Sigma_{j=1}^{p}\theta_{jN}\psi(X_{ij}) = -N^{-\frac{1}{2}}\Sigma_{j=1}^{p} c_j \Sigma_{i=1}^{N}\psi(X_{ij}); \tag{3.8}$$

$$d_j = E\psi(X^{(j)}), \text{ where } X^{(j)} = j\text{th order statistic from} \tag{3.9}$$

a size $p$ sample from $f$, $d = (d_1, \cdots, d_p)'$.

LEMMA 3.1. *Under* $\{P_N\}$ *probability we have* $\operatorname{Cov}(T_N, Z_{N,k,j}) = -N^{\frac{1}{2}}/(p-1)d_k c_j$.

PROOF. Since $E\psi(X) = 0$, and because of the independence of the rankings in different blocks, one gets

$$
\begin{aligned}
&-N^{-\frac{1}{2}}\operatorname{Cov}(T_N, Z_{N,k,j})\\
&= NN^{-\frac{1}{2}}E\Sigma_{l=1}^{p}\theta_{lN}\psi(X_{1l})Z_{kj}^{(1)}\\
&= E\Sigma_{l=1}^{p}c_l\psi(X_{1l})Z_{kj}^{(1)}\\
&= \Sigma_{j'=1}^{p}\Sigma_{l'=1}^{p}\Sigma_{l=1}^{p}E\left[c_l\psi(X_{1l})Z_{kj}^{(1)} \mid R_{1l} = l', R_{1j} = j'\right]\\
&\quad \cdot P(R_{1l} = l', R_{1j} = j')\\
&= \Sigma_{l'=1}^{p}\Sigma_{l=1}^{p}c_l E\left[\psi(X_{1l}) \mid R_{1l} = l', R_{1j} = k\right]\\
&\quad \cdot P(R_{1l} = l', R_{1j} = k)\\
&= \Sigma_{l' \neq k}\Sigma_{l \neq j}c_l E\psi(X^{(l')})\frac{1}{p(p-1)} + c_j E\psi(X^{(k)})\frac{1}{p}\\
&= (-c_j)(-d_k)\left(\frac{1}{p(p-1)} + \frac{1}{p}\right) = \frac{1}{p-1}d_k c_j,
\end{aligned}
$$

since $\Sigma_{j=1}^{p}d_j = \Sigma_{j=1}^{p}c_j = 0$.

THEOREM 3.2. *Under the sequence* $\{Q_N\}$ *the statistic* $S_N = N^{-\frac{1}{2}}(Z_N - (N/p)1)$ *is asymptotically* $N(\mu, \Sigma)$ *distributed, with* $\mu = -1/(p-1)d \otimes c$ *and* $\Sigma = 1/(p-1)M_p \otimes M_p$, *where* $M_p$, $c$, $d$ *are defined by* (2.6), (3.2) *and* (3.9), *respectively.*

PROOF. Set

$$L_N = \log \Pi_{i=1}^{N}\Pi_{j=1}^{p}f(X_{ij} - \theta_{jN}) - \log \Pi_{i=1}^{N}\Pi_{j=1}^{p}f(X_{ij}) = \log\frac{Q_N}{P_N}. \tag{3.10}$$

By Theorem VI, 2.1 of Hájek and Šidák [4]

$$L_N - T_N \to -\tfrac{1}{2}c_o I(f) \quad \text{in} \quad \{P_N\} \quad \text{probability.} \tag{3.11}$$

Let $\lambda_o$, $\lambda_{kj}$, $1 \leqslant k, j \leqslant p$ be arbitrary constants. Define the vectors

$$\lambda = (\lambda_{11}, \lambda_{12}, \cdots, \lambda_{1p}, \lambda_{21}, \cdots, \lambda_{pp})' \quad \text{and} \quad \bar{\lambda} = (\lambda_o, \lambda')'. \tag{3.12}$$

Set

$$U_N = \lambda' S_N. \tag{3.13}$$

We show that under $\{P_N\}$ probability

$$
\begin{aligned}
R_N &= \lambda_o T_N + \Sigma_{k=1}^{p}\Sigma_{j=1}^{p}\lambda_{kj}\left(Z_{N,k,j} - \frac{N}{p}\right)N^{-\frac{1}{2}}\\
&= \lambda_o T_N + U_N \to_{\mathfrak{L}} N(0, \sigma^2),
\end{aligned} \tag{3.14}
$$

with

$$\sigma^2 = \bar{\lambda}'\Gamma\bar{\lambda} \tag{3.15}$$

and

$$\Gamma = \left[\begin{array}{c|c} c_o I(f) & \dfrac{-1}{p-1} d' \otimes c' \\ \hline \dfrac{-1}{p-1} d \otimes c & \dfrac{1}{p-1} M_p \otimes M_p \end{array}\right]. \tag{3.16}$$

Since $R_N$ is the sum of $N$ independent and identically distributed random variables, each summand having expectation 0 and finite variance, asymptotic normality follows from the classical central limit theorem. For the computation of $\Gamma$ we use Lemma 3.1, (2.5) and the fact that $E\psi(X_{ij})^2 = I(f)$ under $\{P_N\}$.

Because of (3.11), (3.14) implies

$$\tilde{R}_N = \lambda_o L_N + U_N = \lambda_o(L_N - T_N) + R_N \to_{\mathcal{E}} N\left(-\tfrac{1}{2}\lambda_o c_o I(f), \bar{\lambda}'\Gamma\bar{\lambda}\right). \tag{3.17}$$

Since $\bar{\lambda}$ is arbitrary, we get (Cramér-Wold technique)

$$(L_N, U_N) \to_{\mathcal{E}} N(\mu, \Delta) \text{ in } \{P_N\} \text{ probability,} \tag{3.18}$$

with $\mu_1 = -\frac{1}{2}c_o I(f)$, $\mu_2 = 0$, $\Delta_{11} = c_o I(f)$, $\Delta_{12} = -1/(p-1)\Sigma_{k=1}^p \Sigma_{j=1}^p \lambda_{kj} c_k d_j$ and $\Delta_{22} = 1/(p-1)\lambda' M_p \otimes M_p \lambda$. By LeCam's third lemma (see Hájek & Šidàk [4], Lemma VI, 1.4)

$$U_N \to_{\mathcal{E}} N(\Delta_{12}, \Delta_{22}) \qquad \text{under } \{Q_N\} \text{ probability.} \tag{3.19}$$

Using the Cramér-Wold technique once again, we obtain

$$S_N \to_{\mathcal{E}} N\left(\frac{-1}{p-1} d \otimes c, \frac{1}{p-1} M_p \otimes M_p\right) \qquad \text{under } \{Q_N\} \text{ probability.} \tag{3.20}$$

COROLLARY 3.3. *Under* $\{Q_N\}$ *the statistic* $(p-1)/pA_N$ *has an asymptotic* $\chi^2_{(p-1)^2, \delta^2}$*-distribution, where the noncentrality parameter* $\delta^2$ *is given by*

$$\delta^2 = \frac{1}{p-1}\Sigma_{i=1}^p c_i^2 \Sigma_{i=1}^p d_i^2. \tag{3.21}$$

PROOF. Obviously $(p-1)/pA_N = (p-1)\|S_N\|^2 = \|(p-1)^{\frac{1}{2}} S_N\|^2$. Since $(p-1)^{\frac{1}{2}}S_N$ has an idempotent covariance matrix of rank $(p-1)^2$ the result follows from the preceding theorem, since $(d \otimes c)'(d \otimes c) = d'd \cdot c'c$.

Let $a_1, \cdots, a_p$ be arbitrary numbers with $\Sigma_{j=1}^p a_j = 0$, $a = (a_1, \cdots, a_p)'$. Define

$$V_j = \Sigma_{k=1}^p a_k \Sigma_{i=1}^N Z_{kj}^{(i)} = \Sigma_{k=1}^p a_k D_{kj}, \qquad 1 \leqslant j \leqslant p. \tag{3.22}$$

Obviously $\bar{V} = 1/p\Sigma_{j=1}^p V_j = N\bar{a} = 0$.

DEFINITION. Tests having rejection region $\Sigma_{j=1}^p V_j^2 > C_{N,\alpha}$ are called Friedman type tests (Friedman [3]).

REMARK. In the case of the classical Friedman test we have $a_k = k - \frac{1}{2}(p + 1)$.

COROLLARY 3.4. *Let $V^{(N)} = N^{-\frac{1}{2}}(V_1, V_2, \cdots, V_p)'$. Then in $\{Q_N\}$ probability $V^{(N)} \to_{\mathcal{L}} N(\mu, \Sigma)$ with $\mu = -1/(p-1)a'd \cdot c$, $\Sigma = 1/(p-1)a'aM_p$ and*

$$T_N = \frac{1}{N}\frac{p-1}{a'a}\Sigma_{j=1}^p V_j^2 \to_{\mathcal{L}} \chi^2_{p-1,\delta^2}, \tag{3.23}$$

*where*

$$\delta^2 = \frac{1}{p-1}(a'd)^2 c'c/a'a. \tag{3.24}$$

PROOF. Evidently

$$V^{(N)} = (a' \otimes I_p)S_N. \tag{3.25}$$

Asymptotic normality, therefore, follows from the preceding theorem, and

$$\mu = (a' \otimes I)\left(\frac{-1}{p-1}\right)(d \otimes c) = \frac{-1}{p-1}a'd \cdot c$$

$$\Sigma = (a' \otimes I)\left(\frac{1}{p-1}\right)(M_p \otimes M_p)(a \otimes I)$$

$$= \frac{1}{p-1}(a'M_p a) \otimes M_p = \frac{1}{p-1}a'a \cdot M_p.$$

The limit distribution of $\{(p-1)/a'a\}^{\frac{1}{2}}V^{(N)}$ therefore, has expectation $-a'd \cdot c/\{a'a(p-1)\}^{\frac{1}{2}}$ and covariance $M_p$, which is idempotent and of rank $p-1$. Hence $(p-1)/a'a\|V^{(N)}\|^2 \to \chi^2_{p-1,\delta^2}$ with $\delta^2 = (a'd)^2 \cdot c'c/(a'a(p-1))$.

Let $b_1, \cdots, b_p$ be arbitrary numbers with $\Sigma_{j=1}^p b_j = 0$, $b = (b_1, \cdots, b_p)'$. Define

$$W_k = \Sigma_{j=1}^p b_j \Sigma_{i=1}^N Z_{kj}^{(i)} = \Sigma_{j=1}^p b_j D_{kj}, \qquad 1 \leqslant k \leqslant p. \tag{3.26}$$

DEFINITION. Tests having rejection region $\Sigma_{k=1}^p W_k^2 > C'_{N,\alpha}$ are called dual Friedman type tests. (The term "dual" has been proposed by K. Kannemann in an unpublished manuscript.)

COROLLARY 3.5. *Let $W^{(N)} = N^{-\frac{1}{2}}(W_1, \cdots, W_p)'$, then, in $\{Q_N\}$ probability $W^{(N)} \to_{\mathcal{L}} N(\mu, \Sigma)$, with $\mu = -1/(p-1)c'b \cdot d$, $\Sigma = 1/(p-1)b'bM_p$ and*

$$\tilde{T}_N = \frac{1}{N}\frac{p-1}{b'b}\Sigma_{k=1}^p W_k^2 \to_{\mathcal{L}} \chi^2_{p-1,\delta^2}, \tag{3.27}$$

*where*

$$\delta^2 = \frac{1}{p-1}(c'b)^2 d'd/b'b. \tag{3.28}$$

PROOF. Similar to proof of Corollary 3.4, using the obvious duality.

Let $a_1, \cdots, a_p; b_1, \cdots, b_p$ as before. Set

$$X_N = \Sigma_{k=1}^p \Sigma_{j=1}^p a_k b_j \Sigma_{i=1}^N Z_{kj}^{(i)} = \Sigma_{k=1}^p \Sigma_{j=1}^p a_k b_j D_{kj}. \tag{3.29}$$

DEFINITION. Tests based on rejection regions $X_N > C''_{N,\alpha}$ are called Page type tests (Page [7]).

REMARK. In the classical Page test we have $a_j = b_j = j - \frac{1}{2}(p + 1)$.

COROLLARY 3.6. *Under* $\{Q_N\}$ *probability*

$$N^{-\frac{1}{2}} X_N \to_{\mathcal{L}} N\left(-\frac{1}{p-1} a'd \cdot b'c, \frac{1}{p-1} a'a \cdot b'b\right).$$

PROOF. From Corollary 3.4

$$N^{-\frac{1}{2}} X_N = b' \cdot V^{(N)} \to_{\mathcal{L}} N\left(-\frac{1}{p-1} a'd \cdot b'c, \frac{1}{p-1} a'ab'M_p b\right).$$

**4. Extension to balanced incomplete block designs.** A BIBD is represented by a design matrix $E = (e_1, e_2, \cdots, e_p)$ of order $N \times p$ consisting of 0's and 1's and such that $E\,1 = b\,1$, $b =$ block size, $e_j'e_j = r =$ number of replicates $(1 \leqslant j \leqslant p)$ and $e_j'e_k = \lambda(1 \leqslant k \neq j \leqslant p)$.
For such designs the relations

$$N \cdot b = p \cdot r \tag{4.1}$$

and

$$\lambda p(p-1) = Nb(b-1) \tag{4.2}$$

hold.

The idea of the Anderson test can be extended to such designs: assign ranks to the observations within blocks, construct a $b \times p$ matrix $D$ with

$$D_{kj} = \text{number of elements } i \in \{1, \cdots, N\} \text{ with } R_{ij} = k \qquad 1 \leqslant k \leqslant p, \quad 1 \leqslant j \leqslant b \tag{4.3}$$

and compute

$$A = \frac{p}{N}\Sigma_{k=1}^{b}\Sigma_{j=1}^{p}\left(D_{kj} - \frac{N}{p}\right)^2. \tag{4.4}$$

Reject the hypothesis of no treatment effect if $A$ is too large.

Define $E_i = \{j\colon e_{ij} = 1\}$. Then the $Q_N$ densities of the observations are assumed to have the form

$$Q_N = \Pi_{i=1}^{N}\Pi_{j\in E_i} f(x_{ij} - \theta_{jN}). \tag{4.5}$$

The corresponding $\{P_N\}$ sequence is given by

$$P_N = \Pi_{i=1}^{N}\Pi_{j\in E_i} f(x_{ij}). \tag{4.6}$$

We are interested in the asymptotic distribution of $A_N$. For this purpose assume that a sequence of $N \times p$ BIBD is given, where $N$ runs through (a subset of) the integers. Furthermore, suppose that $\theta_{jN} = N^{-\frac{1}{2}}c_j$, $\Sigma_{j=1}^{p} c_j = 0$, $\Sigma_{j=1}^{p} c_j^2 = c_o > 0$, $b, p$ constant, whereas $\lambda = \lambda_N = Nb(b-1)/p(p-1)$, $r = r_N = Nb/p$. In connection with (4.4), note that under $\{P_N\}$, $ED_{kj} = r_N/b = N/p$. As in Section 2 set $Z_N = (D_{11}, D_{12}, \cdots, D_{1p}, D_{21}, \cdots, D_{b1}, \cdots, D_{bp})'$.

THEOREM 4.1. *Under the sequence* $\{Q_N\}$ *the statistic* $S_N = N^{-\frac{1}{2}}(Z_N - N/p)$ *is asymptotically* $N(\mu, \Sigma)$*-distributed, with* $\mu = -1/(p-1)d \otimes c$ *and* $\Sigma = 1/(p-1)M_b \otimes M_p$, *where* $d = (d_1, \cdots, d_b)$ *with* $d_j = E\,\psi(X^{(j)})$, $X^{(j)} = j$th *order statistic among b observations from f.*

PROOF. The proof consists of a slight extension of the proof to Theorem 3.2. Therefore only the differences are indicated. Obviously, under $P_N$, $E\,Z_{kj}^{(i)} = e_{ij}/b$ and

$$\operatorname{Cov}(Z_{kj}^{(i)}, Z_{k'j'}^{(i)}) = e_{ij}e_{ij'}\cdot\frac{b-1}{b^2} \quad \text{for } k = k', j = j'$$
$$\cdot\frac{1}{b^2(b-1)} \quad \text{for } k \neq k', j \neq j'$$
$$\cdot -\frac{1}{b^2} \quad \text{otherwise.} \tag{4.7}$$

Using relations (4.1) and (4.2) one arrives at the result

$$\operatorname{Cov}(Z_N) = \frac{N}{p-1} M_b \otimes M_p, \tag{4.8}$$

with $M_k$ defined by (2.6).

Defining

$$T_N = -N^{-\frac{1}{2}}\Sigma_{l=1}^p c_l \Sigma_{i=1}^N e_{il}\psi(X_{il})$$

and proceeding as in the proof of Lemma 3.1, one obtains

$$\begin{aligned} N^{\frac{1}{2}}\operatorname{Cov}(T_N, Z_{N,k,j}) &= -\Sigma_{i=1}^N \Sigma_{l=1}^p e_{il}e_{ij}c_l E\psi(X_{il})Z_{kj}^{(i)} \\ &= -\Sigma_{i=1}^N \Sigma_{l=1}^p e_{il}e_{ij}c_l \Sigma_{m=1}^b E\left[\psi(X_{il}) | R_{il} = m, R_{ij} = k\right] \\ &\quad \cdot P(R_{il} = m, R_{ij} = k) \\ &= -\left(\frac{\lambda}{b(b-1)} + \frac{r}{b}\right)c_j d_k = -\frac{N}{p-1}c_j d_k \text{ by (4.1) and (4.2).} \end{aligned}$$

Hence, under $\{P_N\}$ probability, using Liapounov's theorem,

$$\lambda_o L_N + \lambda' N^{-\frac{1}{2}}\left(Z_N - \frac{N}{p}1\right) \to_{\mathfrak{L}} N(\mu, \bar{\lambda}'\Gamma\bar{\lambda}) \tag{4.9}$$

with

$$\mu = -\lambda_o \frac{1}{2}\frac{b}{p}c_o I(f) \tag{4.10}$$

and

$$\Gamma = \left[\begin{array}{c|c} c_o \frac{b}{p} I(f) & \frac{-1}{p-1} d' \otimes c' \\ \hline \frac{-1}{p-1} d \otimes c & \frac{1}{p-1} M_b \otimes M_p \end{array}\right], \tag{4.11}$$

where now $L_N = \log \Pi_{i=1}^{N} \Pi_{j \in E_i} f(X_{ij} - \theta_{jN})/f(X_{ij})$. By LeCam's third lemma the result follows.

This theorem enables us to derive the asymptotic distribution of the Anderson statistic for BIB designs.

COROLLARY 4.2. *Under* $\{Q_N\}$ *probability* $(p-1)/pA_N$ *converges to a* $\chi^2_{(p-1)(b-1),\,\delta^2}$*-distribution with*

$$\delta^2 = \frac{1}{p-1} \Sigma_{i=1}^{p} c_i^2 \Sigma_{i=1}^{b} d_i^2, \tag{4.12}$$

*where the* $d_i$ *are given by* (3.9), *but from a sample of size* $b$.

PROOF. $(p-1)^{\frac{1}{2}} S_N$ has an idempotent covariance $M_b \otimes M_p$ and rank $(M_b \otimes M_p) = \operatorname{tr} M_b \cdot \operatorname{tr} M_p = (b-1)(p-1)$. Furthermore, $(p-1)\mu'\mu = 1/(p-1)(d \otimes c)'(d \otimes c) = 1/(p-1)d'dc'c$.

REMARK. Friedman type, dual Friedman type, and Page type tests can be constructed for BIB designs just as for complete block designs. The results of Corollaries 3.4.–3.6. hold with the obvious modification that the $d_i$ are defined on the basis of a sample of size $b$.

**5. Efficiencies.** In this section we return to the basic model with block size equal to number of treatments $p$ and additive block effects. Unfortunately it does not seem to be easy to compare the various tests of Section 3 with respect to their Pitman efficiencies, since the test statistics have, under $H$, different limiting distributions. Therefore, we use Bahadur's [2] concept of comparisons of asymptotic slopes and take the local ratio of slopes as our measure of efficiency. It turns out that the local slopes are in all cases given by the noncentrality parameters under contiguous alternatives as derived in Section 3. Let the data have the joint density

$$Q_N = \Pi_{i=1}^{N} \Pi_{j=1}^{p} f(x_{ij} - c_j - \alpha_i) \tag{5.1}$$

with $c = (c_1, \cdots, c_p)'$ fixed, $\Sigma_1^p c_j = 0$, $I(f) < \infty$. Denote the probability measures by $P_{c,N}$. In order to compute the Bahadur slope a test statistic $S_N$ has to satisfy three conditions:

(i) $P_{o,N}(S_N < x) \to F(x)$, with $F(\cdot)$ a continuous cumulative distribution function.

(ii) For some constant $0 < a < \infty$

$$\log(1 - F(x)) = -\frac{ax^2}{2}(1 + o(1)) \quad \text{as} \quad x \to \infty; \tag{5.2}$$

(iii) For some nonnegative function $\tilde{b}(c)$

$$\lim_{N\to\infty} P_{c,N}\left(\left|\frac{S_N}{N^{\frac{1}{2}}} - \tilde{b}(c)\right| > \varepsilon\right) = 0 \quad \text{for every} \quad \varepsilon > 0.$$

Then the asymptotic slope is given by

$$G(c) = a\tilde{b}(c)^2. \tag{5.3}$$

We take

(5.4) $S_N = \left(\frac{p-1}{p} A_N\right)^{\frac{1}{2}}$ for the Anderson test;
$= (T_N)^{\frac{1}{2}}$ as given by (3.23) for the Friedman type test;
$= (\tilde{T}_N)^{\frac{1}{2}}$ as given by (3.27) for the dual Friedman type test;
$= |X_N|\{Na'ab'b/(p-1)\}^{-\frac{1}{2}}$ as given by (3.29) for the Page type test.

Then, from the results of Section 3, it follows that in all cases $S_N$ has an asymptotic $\chi$-distribution. For such distributions Bahadur [2] has shown that (5.2)(ii) is satisfied with $a = 1$. For the local comparison it thus suffices to compute $\tilde{b}(c)$ and the rate at which $\tilde{b}(\Delta c)$ converges to zero as $\Delta \to 0$.

THEOREM 5.1. *Under the assumptions* (5.1)

(5.5) $(p-1)\lim_{\Delta\to 0}\frac{\tilde{b}^2(\Delta c)}{\Delta^2} = \|c\|^2\|d\|^2$ *for the Anderson test*;
$= (a'd)^2\|c\|^2/\|a\|^2$ *for the Friedman type test;*
$= (c'b)^2\|d\|^2/\|b\|^2$ *for the dual Friedman type test;*
$= (a'd)^2(c'b)^2/\|a\|^2.\|b\|^2$ *for the Page type test.*

PROOF. Since the proofs of the various results are very similar we restrict ourselves to the Anderson case. Because of the independence of blocks the strong law of large numbers implies

$$\frac{D_{kj}}{N} \to \pi_{kj}^{(c)} \text{ a.s.} \quad \text{as} \quad N \to \infty, \tag{5.6}$$

where

$$\pi_{kj}^{(c)} = P_c(R_{1j} = k). \tag{5.7}$$

Hence

$$\frac{1}{N}\frac{p-1}{p}A_N = \frac{(p-1)}{N^2}\Sigma_{j=1}^p\Sigma_{k=1}^p\left(D_{kj} - \frac{N}{p}\right)^2 \tag{5.8}$$

$$\to_{\text{a.s.}} (p-1)\Sigma_{j=1}^p\Sigma_{k=1}^p\left(\pi_{kj}^{(c)} - \frac{1}{p}\right)^2 = \tilde{b}(c)^2.$$

Let $S_k^{(j)}$ be the class of all choices of $\{r_1, \cdots, r_{k-1}\}$ from $\{1, 2, \cdots, j-1, j+1, \cdots, p\}$ with $\{r_{k+1}, \cdots, r_p\}$ being the complementary set. Then

$$\pi_{kj}^{(c)} = \Sigma_{r\in S_k^{(j)}}\int\Pi_{l=1}^{k-1}F(x - c_{r_l})\Pi_{l=k+1}^p(1 - F(x - c_{r_l}))dF(x - c_j) \tag{5.9}$$

and therefore

$$(5.10)\qquad \left.\frac{\partial \pi_{kj}^{(c)}}{\partial c_m}\right|_{c=0} = C_k \qquad \text{for } m \neq j,$$
$$= -(p-1)C_k \qquad \text{for } m = j,$$

where

$$(5.11)\qquad C_k = \binom{p-2}{k-1}\int F(x)^{k-1} f(x)(1-F(x))^{p-k-1}\,dF(x) - \binom{p-2}{k-2}\int F(x)^{k-2}(1-F(x))^{p-k} f(x)\,dF(x).$$

On the other hand

$$(5.12)\qquad d_k = E\psi(\chi^{(k)}) = p\binom{p-1}{k-1}\int F(x)^{k-1}(1-F(x))^{p-k}\frac{f'(x)}{f(x)} f(x)\,dx$$
$$= p(p-1)C_k, \text{ by partial integration.}$$

Finally

$$(5.13)\qquad \lim_{\Delta\to 0}\frac{\tilde{b}^2(\Delta c)}{\Delta^2} = (p-1)\Sigma_{k=1}^{p}\Sigma_{j=1}^{p}\left(\left.\frac{\partial \pi_{kj}^{(\tilde{c})}}{\partial \tilde{c}}\right|'_{\tilde{c}=0}\cdot c\right)^2$$
$$= (p-1)\Sigma_{k=1}^{p}\Sigma_{j=1}^{p}\left[d_k c_j\left(\frac{1}{p(p-1)} + \frac{1}{p}\right)\right]^2$$
$$= \frac{1}{p-1}\|d\|^2\|c\|^2.$$

Using the notation $\rho_{u,v} = u'v/\|u\|\,\|v\|$ for $p$-dimensional vectors $u$, $v$ with $u'\cdot 1 = v'\cdot 1 = 0$, we get

COROLLARY 5.2. *The local approximate Bahadur relative efficiencies for shift alternatives are given by*

$$(5.14)\qquad e\,(\textit{Friedman type/Anderson}) = \rho_{a,d}^2$$
$$e\,(\textit{Friedman dual type/Anderson}) = \rho_{b,c}^2$$
$$e\,(\textit{Page type/Anderson}) = \rho_{a,d}^2\cdot\rho_{b,c}^2.$$

PROOF. Obvious.

REMARK 1. With the limitation imposed by the particular concept of efficiency used, the result shows, that the Anderson test is always asymptotically equivalent to the Friedman type test with optimal scores $E\,\psi(\chi^{(j)})$, $(1 \leqslant j \leqslant p)$. However, these will only be known, when the type of $f(\cdot)$ is specified completely. For the Anderson test such knowledge is not required. Similarly with respect to the other tests.

A comparison between the various tests of Section 3 for the BIBD yields the same result as given in (5.14), except that $X^{(j)}$ is now from a size $b$ sample. The optimality property also extends to the comparison with the locally most powerful rank test with respect to certain restricted classes of alternatives, as will be shown in the following theorem. By definition, a rank test for the problem of analyzing a randomized blocks experiment is to be a test which depends on the observations only through the ranks $R_{ij}(1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant p)$ within blocks.

THEOREM 5.3. *Consider testing*

$$H : P_N = \Pi_{i=1}^{N}\Pi_{j=1}^{p} f(x_{ij} - \alpha_i)$$

*against*

$$K : Q_N(c) = \Pi_{i=1}^{N}\Pi_{j=1}^{p} f(x_{ij} - \alpha_i - c_j)$$

*with* $c \neq 0$, $\Sigma_{j=1}^{p} c_j = 0$, *and a differentiable* $f > 0$. *For any* $c$ *the Anderson test is asymptotically as efficient as the locally most powerful rank test for testing* $H$ vs. $K_c = \{Q_N(\Delta c) : \Delta > 0\}$.

PROOF. Using Hoeffding's result (see e.g. Lehmann [6], page 254) it can be shown, that the locally most powerful test is equivalent to the Page test with optimal scores and optimal regression constants. Details are omitted here.

REMARK 2. It should be pointed out that the locally most powerful rank test requires knowledge of the constants $c_j$ and $d_k = E\,\psi(X^{(k)})$, whereas the Anderson test does not.

Finally, comparing the Anderson test for the BIBD with that of the complete block design, one obtains

(5.15) $e$ (BIBD, complete block design/Anderson)

$$= \frac{\frac{1}{b-1}\,\mathcal{E}\,\Sigma_{j=1}^{b} d_{j,b}^2}{\frac{1}{p-1}\Sigma_{j=1}^{p} d_{j,p}^2},$$

where $d_{j,n} = E\,\psi(X^{j)})$, $X^{(j)} = j^{th}$ order statistic from a size $n$ sample, and $\mathcal{E} = (b-1)p/b(p-1) =$ efficiency factor. To the extent that $1/(b-1)\Sigma_{j=1}^{b} d_{j,b}^2 \approx 1/(p-1)\Sigma_{j=1}^{p} d_{j,p}^2$ (both of these expressions approximate $I(f)$), the efficiency is given by $\mathcal{E}$, as in the classical normal-theory case.

REMARK 3. It should be noted that similar efficiency comparisons can be carried out for alternatives more general than location shifts: let $X_{ij}$ be distributed according to $F(x - \alpha_i, \theta)$ with $\theta = \theta_o$ under $H$ and $\theta = \theta_o + c$, $c \neq 0$ under $K$. Then, by the same reasoning as above, it can be shown that the efficiency of the Anderson test is the maximum of the efficiencies of Friedman type tests, or dual Friedman type tests, or Page type tests.

**6. Ties.** Kannemann [5] suggested that when $t > 1$ observations $X_{ij_1}, \cdots, X_{ij_t}$ are tied, $Z_{kj}^{(i)}$ should be defined as $1/t$ for these $j$'s and the corresponding $k$'s. When computing a Friedman type test statistic, this method corresponds to taking average scores for tied observations (= midrank method in the case of the classical Friedman test). Whereas, for these Friedman type statistics the conditional asymptotic distribution, given the tie structure, is still the $\chi^2_{p-1}$ distribution, if the variance is adjusted properly, the situation is not so neat when the same procedure is used for the Anderson test. This can easily be seen from the following

LEMMA 6.1. *Assume that the order statistics of the* $i$th *block satisfy*

$$X^{(1)} = X^{(2)} = \cdots = X^{(t_1)} < X^{(t_1+1)} = \cdots = X^{(t_1+t_2)} < \cdots = X^{(t_1+t_2+\cdots t_g)}. \tag{6.1}$$

*Define*

$$Z_{kj}^{(i)} = \frac{1}{t_l} \quad \text{if } X_{ij} = X^{(t_1+t_2+\cdots+t_l)} \text{ and } t_1 + \cdots + t_{l-1} < k \leqslant t_1 + \cdots + t_l \tag{6.2}$$
$$= 0 \quad \text{otherwise.}$$

*Then, under the hypothesis, the conditional covariance, given the tie structure, has the form*

$$\operatorname{Cov} Z^{(i)} = \frac{1}{p(p-1)}\left[\left(\begin{array}{c|c|c|c} \frac{p}{t_1}U_{t_1} & 0 & & 0 \\ \hline 0 & \frac{p}{t_2}U_{t_2} & & 0 \\ \hline\hline 0 & 0 & & \frac{p}{t_g}U_{t_g} \end{array}\right) - 11'\right] \otimes M_p \tag{6.3}$$

*where* $U_{t_i} = t_i \times t_i$ *matrix whose elements are ones.*

PROOF. By straightforward computation one obtains

$$EZ_{kj}^{(i)} = \frac{1}{p} \tag{6.4}$$

as in the unconditional case, and

$$\begin{aligned} EZ_{kj}^{(i)}Z_{k'j'}^{(i)} &= \frac{1}{t_l p} \quad \text{for } j = j' \text{ and } t_1 + \cdots + t_{l-1} < k, k' \leqslant t_1 + \cdots + t_l; \\ &= 0 \quad \text{for } j = j' \text{otherwise;} \\ &= \frac{t_l - 1}{t_l p(p-1)} \quad \text{for } j \neq j', \text{ and } t_1 + \cdots + t_{l-1} < k, k' \leqslant t_1 + \cdots + t_l; \\ &= \frac{1}{p(p-1)} \quad \text{for } j \neq j', \text{ otherwise.} \end{aligned} \tag{6.5}$$

From these results (6.3) follows immediately.

Given the tie structures of the blocks $1, \cdots, N$, the covariance matrix of the $Z_N$ vector is the sum of matrices of the type (6.3). It is obvious that in general such a sum is not idempotent (up to a fixed constant) nor can its rank be determined easily. Apart from assigning ranks randomly within ties, a method which might waste a considerable amount of information, the most reasonable procedure might well be to diagonalize numerically the sum of the conditional covariance matrices of type (6.3) and then to construct the corresponding statistic with a conditional asymptotic $\chi^2$-distribution.

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