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INVARIANCE PRINCIPLES FOR THE COUPON COLLECTOR'S PROBLEM: A MARTINGALE APPROACH[1]

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For the coupon collector's problem, invariance principles for the partial sequence of bonus sums after n coupons as well as the waiting times to obtain the bonus sum $t(\geqslant 0)$ are studied through a construction of a triangular array of martingales related to these sequences and verifying the invariance principles for these martingales.

1. Introduction. Consider a sequence $\{\Omega_N, N \geqslant 1\}$ of *coupon collector's situations*

$$\Omega_N = \{(a_N(1), p_N(1)), \cdots, (a_N(N), p_N(N))\}, \qquad N \geqslant 1, \tag{1.1}$$

where $a_N(s)$ and $p_N(s)(>0)$ are real numbers and $\sum_{s=1}^{N} p_N(s) = 1$. Consider also a (double) sequence $\{I_{Nk}, k \geqslant 1\}$ of (row-wise) independent and identically distributed random variables (i.i.d.rv), where, for each $N(\geqslant 1)$,

$$P\{I_{Nk} = s\} = p_N(s) \quad \text{for} \quad s = 1, \cdots, N. \tag{1.2}$$

Let then

$$\begin{aligned} Y_{Nk} &= a_N(I_{Nk}), \quad \text{if} \quad I_{Nk} \notin (I_{N1}, \cdots, I_{Nk-1}), \\ &= 0 \quad \text{otherwise, for} \quad k \geqslant 1; \end{aligned} \tag{1.3}$$

$$Z_{Nn} = \sum_{k=1}^{n} Y_{Nk}, \quad n \geqslant 1 \quad \text{and} \quad Y_{N0} = Z_{N0} = 0. \tag{1.4}$$

Z_{Nn} is termed the *bonus sum after n coupons in the collector's situation* Ω_N. If the $a_N(s)$ are all nonnegative, Z_{Nn} is nondecreasing in $n(\geqslant 0)$, and for every $t \geqslant 0$, let

$$U_N(t) = \min\{k : Z_{Nk} \geqslant t\}. \tag{1.5}$$

Then, $U_N(t)$ is termed the *waiting time to obtain the bonus sum t in the coupon collector's situation* Ω_N.

Asymptotic normality of multi-dimensional marginal distributions of $\{Z_{Nn}\}$ and $\{U_N(t)\}$ has been studied by Rosén (1969, 1970) and Holst (1972a, b, 1973), among others. The object of the present investigation is to propose and formulate an alternative approach to this problem based on the weak convergence of a suitably constructed martingale sequence associated with the Z_{Nn}. The basic regularity conditions are outlined in Section 2. Section 3 deals with the asymptotic normality of Z_{Nn} through the proposed martingale approach. Section 4 is devoted to some

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general remarks concerning the applicability of this martingale approach for studying invariance principles for the partial sequence $\{Z_{Nk}, k \leqslant n\}$ as well as the corresponding sequence of waiting times.

2. Preliminary notions. Note that by (1.2)–(1.4), for every $N(\geqslant 1)$,

$$EY_{Nk} = \Sigma_{s=1}^N a_N(s)p_N(s)[1 - p_N(s)]^{k-1}, \quad k \geqslant 1, EY_{N0} = 0; \tag{2.1}$$

$$\phi_{Nn}^* = EZ_{Nn} = \Sigma_{s=1}^N a_N(s)\{1 - [1 - p_N(s)]^n\}, n \geqslant 1, EZ_{N0} = 0. \tag{2.2}$$

Let us denote by

$$\phi_{Nn} = \Sigma_{s=1}^N a_N(s)[1 - e^{-np_N(s)}], \qquad n \geqslant 0, \tag{2.3}$$

$$d_{Nn}^2 = \Sigma_{s=1}^N a_N^2(s)e^{-np_N(s)}(1 - e^{-np_N(s)}) - n(\Sigma_{s=1}^N a_N(s)p_N(s)e^{-np_N(s)})^2, \qquad n \geqslant 0; \tag{2.4}$$

$$A_{Nr} = N^{-1}\Sigma_{s=1}^N |a_N(s)|^r, \qquad \text{for} \quad r = 1, 2, 3, 4. \tag{2.5}$$

We assume that

$$\sup_N\{\max_{1\leqslant s\leqslant N} Np_N(s)\} \leqslant M_1 < \infty; \tag{2.6}$$

$$\lim_{N\to\infty}\{\max_{1\leqslant s\leqslant N}|a_N(s)|/N^{\frac{1}{2}}A_{N2}^{\frac{1}{2}}\} = 0; \tag{2.7}$$

$$\liminf_{N\to\infty}[(\Sigma_{s=1}^N a_N^2(s)p_N(s))/A_{N2}] \geqslant M_2 > 0. \tag{2.8}$$

Note that $xe^{-x} \leqslant e^{-1}$, $\forall x > 0$ and for $0 \leqslant x \leqslant 1$, $0 \leqslant e^{-nx} - (1 - x)^n \leqslant nx^2e^{-nx}$. Hence from (2.2) and (2.3), we have

$$\begin{aligned} |\phi_{Nn}^* - \phi_{Nn}| &= |\Sigma_{s=1}^N a_N(s)[e^{-np_N(s)} - \{1 - p_N(s)\}^n]| \\ &\leqslant \Sigma_{s=1}^N |a_N(s)| p_N(s)\{np_N(s)e^{-np_N(s)}\} \leqslant e^{-1}M_1A_{N1}, \quad \forall n \geqslant 0, N \geqslant 1. \end{aligned} \tag{2.9}$$

In fact, if the $a_N(s)$ are all nonnegative then $\phi_{Nn}^* \geqslant \phi_{Nn}$. Also, noting that $e^{-x}(1 - e^{-x}) \leqslant x$, $\forall x \geqslant 0$, we obtain from (2.4) that

$$\begin{aligned} d_{Nn}^2 &\leqslant \Sigma_{s=1}^N a_N^2(s)e^{-np_N(s)}[1 - e^{-np_N(s)}] \\ &\leqslant n\Sigma_{s=1}^N a_N^2(s)p_N(s) \leqslant nM_1A_{N2} = 0(nA_{N2}), \quad \forall n \geqslant 1, N \geqslant 1. \end{aligned} \tag{2.10}$$

Further, using the facts that for $0 < x \leqslant 1$, $(1 - e^{-nx}) = (1 - e^{-x})\Sigma_{k=0}^{n-1}e^{-kx} > x(1 - \frac{1}{2}x)\Sigma_{k=0}^{n-1}e^{-kx}$, $[\Sigma_{s=1}^N a_N(s)p_N(s)e^{-np_N(s)}]^2 \leqslant \Sigma_{s=1}^N a_N^2(s)p_N(s)e^{-2np_N(s)}$ (by the Schwarz inequality) and for $0 \leqslant k \leqslant (n - 1)/2$ and $0 \leqslant p \leqslant N^{-1}M_1$, $e^{-(n+k)p} - e^{-2np} \geqslant e^{-(3n-1)p/2}[1 - e^{-\frac{1}{2}(n+1)p}] \geqslant e^{-(3n-1)M_1/2N}[1 - e^{-\frac{1}{2}(n+1)p}]$, we obtain that for $N \geqslant M_1$,

$$\begin{aligned} d_{Nn}^2 \geqslant (1 - \tfrac{1}{2}N^{-1}M_1)\Sigma_{k=0}^{n-1}\{&\Sigma_{s=1}^N a_N^2(s)p_N(s)e^{-(n+k)p_N(s)} \\ &- (\Sigma_{s=1}^N a_N(s)p_N(s)e^{-np_N(s)})^2\} \end{aligned} \tag{2.11}$$

$$\geq \left(1 - \tfrac{1}{2}N^{-1}M_1\right)\Sigma_{k=0}^{n-1}\left\{\Sigma_{s=1}^{N} a_N^2(s)p_N(s)e^{-(n+k)p_N(s)}\left[1 - e^{-(n-k)p_N(s)}\right]\right\}$$

$$\geq \left(1 - \tfrac{1}{2}N^{-1}M_1\right)e^{-(3n-1)M_1/2N}\left[(n+1)/2\right]\Sigma_{s=1}^{N} a_N^2(s)p_N(s)\left[1 - e^{-\frac{1}{2}(n+1)p_N(s)}\right].$$

Now, by (2.6), (2.7) and (2.8), $A_{N2}^{-1}\Sigma_{\{s: p_N(s)>\varepsilon/N\}} a_N^2(s)p_N(s) \geq A_{N2}^{-1}\Sigma_{s=1}^{N} a_N^2(s)p_N(s) - \varepsilon\ \forall \varepsilon > 0$, and noting that for $p_N(s) > \varepsilon/N$ and $n/N > \eta > 0$, $1 - \exp(-\frac{1}{2}(n+1)p_N(s)) \geq c(\varepsilon, \eta) > 0$, we obtain from (2.11) that if

$$0 < \liminf_{N\to\infty} N^{-1}n \leq \limsup_{N\to\infty} N^{-1}n < \infty, \tag{2.12}$$

then $\liminf_{N\to\infty}(d_{Nn}^2/nA_{N2}) > 0$. Thus, under (2.6)–(2.8) and (2.12),

$$0 < \liminf_{N\to\infty}\left(d_{Nn}^2/nA_{N2}\right) \leq \limsup_{N\to\infty}\left(d_{Nn}^2/nA_{N2}\right) < \infty. \tag{2.13}$$

We are primarily concerned with the limiting behavior of the partial sequence $d_{Nn}^{-1}(Z_{Nk} - \phi_{Nk}^*;\ k \leq n)$. Since $d_{Nn}^{-1}a_{N(s)}$, $s = 1, \cdots, N$ remain invariant under any scalar multiplication, we may set (without any loss of generality) that

$$A_{N2} = N^{-1}\Sigma_{s=1}^{N} a_N^2(s) \sim 1. \tag{2.14}$$

Then, by (2.9), (2.13)–(2.14) and the fact that $A_{N1}^2 \leq A_{N2}$, we have $d_{Nn}^{-1}\{\max_{1\leq k\leq n}|\phi_{Nk} - \phi_{Nk}^*|\} \to 0$, so that we may equivalently consider the partial sequence $d_{Nn}^{-1}(Z_{nk} - \phi_{Nk};\ k \leq n)$. In the remainder of this section we consider a basic lemma to be used repeatedly afterwards. Let $Q_{Nk} = p_N(I_{Nk})$, $k \geq 1$ and let $g_{Nu}(Y_{Nk}, Q_{Nk})$, $u = 1, \cdots, p(\geq 2)$ be such that

(2.15)

$$g_{Nu}(0, p) = 0,\ \max_{1\leq u\leq p}\left\{\max_{1\leq s\leq N}|g_{Nu}(a_N(s), p_N(s))|\right\} \leq M_{N,3},\ \sup_N M_{N,3} < \infty$$

and

$$\max_{1\leq u\leq p}\left\{\Sigma_{s=1}^{N}|g_{Nu}(a_N(s), p_N(s))|\right\} \leq M_{N,4},\quad \sup_N N^{-1}M_{N,4} < \infty. \tag{2.16}$$

Note that by (2.15) and (2.16), for some $M_{N,5} \leq M_{N,3}M_{N,4}$,

$$\max_{1\leq u\leq u'\leq p}\left\{\Sigma_{s=1}^{N}|g_{Nu}(a_N(s), p_N(s))g_{Nu'}(a_N(s), p_N(s))|\right\} \leq M_{N,5}. \tag{2.17}$$

LEMMA 2.1. *Under* (2.6), (2.15) *and* (2.16) *for every* $0 = \nu_0 < \nu_1 < \cdots < \nu_p \leq n$,

$$E\Pi_{u=1}^{p} g_{Nu}\left(Y_{N\nu_u}, Q_{N\nu_u}\right) = \Pi_{u=1}^{p} Eg_{Nu}\left(Y_{N\nu_u}, Q_{N\nu_u}\right) + O\left(N^{-1}M_{N,4}^{p-1}\left[M_{N,3} \vee N^{-1}M_{N,4}^2\right]\right), \tag{2.18}$$

$$\operatorname{Cov}\left[g_{N1}\left(Y_{N\nu_1}, Q_{N\nu_1}\right), g_{N2}\left(Y_{N\nu_2}, Q_{N\nu_2}\right)\right] = O\left(N^{-2}\left[M_{N,5} \vee N^{-1}M_{N,4}\right]\right), \tag{2.19}$$

$$V\left[g_{N1}\left(Y_{N\nu_1}, Q_{N\nu_1}\right)\right] = O\left(\left[N^{-1}M_{N,5}\right] \wedge \left[\left(N^{-1}M_{N,4}\right)^2\right]\right). \tag{2.20}$$

PROOF. We shall prove (2.18) and (2.19); the proof of (2.20) follows on similar lines. Note that

$$E\prod_{u=1}^{p} g_{Nu}(Y_{N\nu_u}, q_{N\nu_u}) = \sum_{1\leqslant s_1\neq\cdots\neq s_p\leqslant N}\prod_{u=1}^{p}\{g_{Nu}(a_N(s_u), p_N(s_u)) \\ [1-\sum_{k=u}^{p} p_N(s_k)]^{\nu_u-\nu_{u-1}-1}p_N(s_u)\}$$

$$(2.21)\quad = \sum_{1\leqslant s_1\neq\cdots\neq s_p\leqslant N}\prod_{u=1}^{p}\{g_{Nu}(a_N(s_u), p_N(s_u))p_N(s_u)e^{-\nu_u p_N(s_u)}[1+O(N^{-1})]\}$$

(by (2.6))

$$= \sum_{1\leqslant s_1\neq\cdots\neq s_p\leqslant N}\prod_{u=1}^{p}\{g_{Nu}(a_N(s_u), p_N(s_u))p_N(s_u)e^{-\nu_u p_N(s_u)}\} + O(N^{-p-1}M_{N,4}^{p}),$$

by (2.6) and (2.16). Similarly, for each $u(=1,\cdots,p)$,

$$(2.22)\quad Eg_{Nu}(Y_{N\nu_u}, Q_{N\nu_u}) = \sum_{s=1}^{N} g_{Nu}(a_N(s), p_N(s))p_N(s)e^{-\nu_u p_N(s)} + 0(N^{-2}M_{N,4}),$$

where, by (2.6) and (2.16), the first term on the right-hand side of (2.22) is $0(N^{-1}M_{N,4})$. The product of the p factors of the first term in (2.22) involves N^p terms whereas (2.21) involves $N^{[p]} = N\cdots(N-p+1)$ terms; by (2.15) and (2.16), the contribution of these $N^p - N^{[p]}$ terms is $0(N^{-p}.M_{N,3}.M_{N,4}^{p-1})$. Hence, the proof of (2.18) follows from (2.21)–(2.22). For $p=2$, $N^2 - N^{[2]} = N$ and by (2.6) and (2.17), $\sum_{s=1}^{N} g_{N1}(a_N(s), p_N(s))g_{N2}(a_N(s), p_N(s))p_N^2(s)e^{-(\nu_1+\nu_2)p_N(s)} = 0(N^{-2})$, so that (2.19) follows on parallel lines. □

3. Asymptotic normality of bonus sums. The main result of this section is the following

THEOREM 3.1. *Under* (2.6)–(2.8) *and* (2.12), $d_{Nn}^{-1}(Z_{Nn} - \phi_{Nn})$ *has asymptotically a standard normal distribution.*

PROOF. Unlike the earlier proofs of this result [due to Baum and Billingsley (1965), Rosén (1969, 1970) and Holst (1972a, b)], our proof rests on a construction of a (triangular array of) martingales related to $\{Z_{Nn}\}$. Let $\mathcal{B}_{Nk}$ be the sigma-field generated by $\{I_{Nj}, j\leqslant k\}$, $k\geqslant 1$ and let $\mathcal{B}_{N0}$ be the trivial sigma-field. Then, for every N, $\mathcal{B}_{Nk}$ is nondecreasing. For every N, $n(\geqslant 1)$, we define

$$(3.2)\quad X_{Nk}^{(n)} = Y_{Nk}(1+Q_{Nk})^{k-1}e^{-nQ_{Nk}}, \\ Q_{Nk} = p_N(I_{Nk}),\ k\geqslant 1;\ X_{N0}^{(n)} = 0,$$

$$(3.3)\quad \xi_{Nk}^{(n)} = \sum_{s=1}^{N} a_N(s)p_N(s)e^{-np_N(s)}[1+p_N(s)]^{k-1},\quad k\geqslant 1,\ \xi_{N0}^{(n)} = 0,$$

and consider the sequence

$$(3.4)\quad \tilde{X}_{Nk}^{(n)} = X_{Nk}^{(n)} - E(X_{Nk}^{(n)}|\mathcal{B}_{Nk-1}) \\ = (X_{Nk}^{(n)} - \xi_{Nk}^{(n)}) + \sum_{\nu=0}^{k-1} X_{N\nu}^{(n)}Q_{N\nu}(1+Q_{N\nu})^{k-\nu}, \\ k\geqslant 1;\ \tilde{X}_{N0}^{(n)} = 0.$$

Then, on denoting by

$$(3.5)\quad \tilde{S}_{Nk}^{(n)} = \sum_{j=0}^{k}\tilde{X}_{Nj}^{(n)},\quad k\geqslant 0 \quad\text{and}\quad \tilde{\xi}_{Nk}^{(n)} = \sum_{j=0}^{k}\xi_{Nj}^{(n)},\quad k\geqslant 0,$$

we obtain from (3.2)–(3.5) that

$$\tilde{\xi}_{Nk}^{(n)} = \Sigma_{s=1}^{N} a_N(s) e^{-np_N(s)}\{[1 + p_N(s)]^k - 1\}, \qquad k \geq 0; \tag{3.6}$$

$$\tilde{S}_{Nk}^{(n)} = \Sigma_{i=1}^{k} Y_{Ni} e^{-nQ_{Ni}}[(1 + Q_{Ni})^k - Q_{Ni}(1 + Q_{Ni})^{i-1}] - \tilde{\xi}_{Nk}^{(n)}, \qquad k \geq 0; \tag{3.7}$$

$$E(\tilde{S}_{Nk}^{(n)} | \mathfrak{B}_{Nk-1}) = \tilde{S}_{Nk-1}^{(n)}, \qquad \forall k \geq 1, \tag{3.8}$$

so that for every N, $n(\geq 1)$, $\{\tilde{S}_{Nk}^{(n)}, \mathfrak{B}_{Nk}; k \geq 0\}$ is a martingale. From (2.6) and (3.6), it readily follows that

$$|\tilde{\xi}_{Nn}^{(n)} - \phi_{Nn}| = O(1) \qquad \text{for every } N, n. \tag{3.9}$$

Also, note that $\Sigma_{i=1}^{n} | Y_{Ni} Q_{Ni} e^{-nQ_{Ni}}(1 + Q_{Ni})^{i-1}| \leq \Sigma_{i=1}^{n} | Y_{Ni}| Q_{Ni} \leq \Sigma_{s=1}^{N} |a_N(s)| p_N(s) \leq M_1^{\frac{1}{2}} A_{N_2}^{\frac{1}{2}} \sim M_1^{\frac{1}{2}}$, by (2.6) and (2.14), while for $x \in (0, 1)$, $1 \geq e^{-nx}(1 + x)^n \geq 1 - nx^2$, so that under (2.6) and (2.12), we have from (3.7) that $|\tilde{S}_{Nn}^{(n)} - Z_{Nn} + \phi_{Nn}|$ is bounded, with probability one. Thus, by (2.13), (2.14) and the above, we conclude that for every $\varepsilon > 0$, there exists an $\mathfrak{N}_0(\varepsilon)$, such that, under (2.6)–(2.8) and (2.12),

$$P\{d_{Nn}^{-1} |\tilde{S}_{Nn}^{(n)} - Z_{Nn} + \phi_{Nn}| > \varepsilon\} = 0, \qquad \forall N \geq \mathfrak{N}_0(\varepsilon). \tag{3.10}$$

Consequently, it suffices to show that under (2.6)–(2.8) and (2.12),

$$\mathfrak{L}(d_{Nn}^{-1} \tilde{S}_{Nn}^{(n)}) \to \mathfrak{N}(0, 1). \tag{3.11}$$

Now, for the martingale-difference array $\{d_{Nn}^{-1} \tilde{X}_{Nk}^{(n)}; k \leq n\}$ by (3.4)

$$|\tilde{X}_{Nk}^{(n)}| \leq |Y_{Nk}| + |\xi_{Nk}^{(n)}| + \Sigma_{\nu=1}^{k-1} | Y_{N\nu}| Q_{N\nu}, \qquad 1 \leq k \leq n, \tag{3.12}$$

where $|\xi_{Nk}^{(n)}| \leq \Sigma_{s=1}^{N} |a_N(s)| p_N(s) \leq M_1^{\frac{1}{2}} A_{N_2}^{\frac{1}{2}} \sim M_1$. Also, $|Y_{Nk}| \leq \max_{1 \leq s \leq N} |a_N(s)| = o(N^{\frac{1}{2}}) = o(d_{Nn})$, by (2.7), (2.13) and (2.14). Finally, $(\Sigma_{\nu=1}^{k-1} | Y_{N\nu}| Q_{N\nu}) \leq \Sigma_{s=1}^{N} |a_N(s)| p_N(s) \leq M_1^{\frac{1}{2}} A_{N_2}^{\frac{1}{2}} \sim M_1^{\frac{1}{2}}$, $\forall k \geq 1$. Hence, for every $\varepsilon > 0$, there exists an $\mathfrak{N}_O(\varepsilon)$, such that

$$P\{\max_{1 \leq k \leq n} d_{Nn}^{-1} |\tilde{X}_{Nk}^{(n)}| > \varepsilon\} = 0, \qquad \forall N \geq N_O(\varepsilon); \tag{3.13}$$

the above equation also insures that for $N \geq N_O(\varepsilon)$,

$$d_{Nn}^{-2}\{\Sigma_{k=1}^{n} E([\tilde{X}_{Nk}^{(n)}]^2 I(|\tilde{X}_{Nk}^{(n)}| > \varepsilon\, d_{Nn}) | \mathfrak{B}_{Nk-1})\} = 0 \qquad \text{w.p. 1.} \tag{3.14}$$

Hence, to prove (3.11), we make use of the dependent central limit theorem of Dvoretzky (1972), which for a martingale sequence, satisfying (3.14), demands only an extra condition that

$$d_{Nn}^{-2}\{\Sigma_{k=1}^{n}[l_{Nk}^{(n)}]\} \to 1, \qquad \text{in probability,} \tag{3.15}$$

where

$$\begin{aligned} l_{Nk}^{(n)} &= E([\tilde{X}_{Nk}^{(n)}]^2 | \mathfrak{B}_{Nk-1}) = V(\tilde{X}_{Nk}^{(n)} | \mathfrak{B}_{Nk-1}) \\ &= \Sigma_{s=1}^{N} a_N^2(s) p_N(s) e^{-2np_N(s)}[1 + p_N(s)]^{2(k-1)} \\ &\quad - \Sigma_{\nu=1}^{k-1} Y_{N\nu}^2 Q_{N\nu} e^{-2nQ_{N\nu}}(1 + Q_{N\nu})^{2(k-1)} \\ &\quad - (\xi_{Nk}^{(n)} - \Sigma_{\nu=1}^{k-1} Y_{N\nu} Q_{N\nu} e^{-nQ_{N\nu}}(1 + Q_{N\nu})^{k-1})^2, \qquad k \geq 1. \end{aligned} \tag{3.16}$$

By steps similar to those after (3.12), it follows that the $l_{NK}^{(n)}$ are all bounded with probability 1, while, by (2.13)–(2.14), $d_{Nn}^{-2} = 0(n^{-1})$. Hence, to prove (3.15), it suffices to show that

(3.17)

$$d_{Nn}^{-2}\Sigma_{k=1}^{n}E(l_{Nk}^{(n)}) \to 1 \quad \text{and} \quad \max_{1 \leqslant k < q \leqslant n}|\text{Cov}(l_{Nk}^{(n)}, l_{Nq}^{(n)})| \to 0 \quad \text{as} \quad n \to \infty.$$

For this, we note that for $r = 1, 2$ and $1 \leqslant \nu < k \leqslant n$,

$$\text{(3.18)} \qquad EY_{N\nu}^{r}Q_{N\nu}e^{-rnQ_{N\nu}}(1 + Q_{N\nu})^{r(k-1)}$$
$$= \Sigma_{s=1}^{N}a_N^r(s)p_N^2(s)[1 - p_N(s)]^{\nu-1}e^{-rnp_N(s)}[1 + p_N(s)]^{r(k-1)};$$

$$\text{(3.19)} \qquad EY_{N\nu_1}Q_{N\nu_1}e^{-nQ_{N\nu_1}}(1 + Q_{N\nu_1})^{k-1}Y_{N\nu_2}Q_{N\nu_2}(1 + Q_{N\nu_2})^{k-1}e^{-nQ_{N\nu_2}}$$
$$= \Sigma_{s \neq s'=1}^{N}a_N(s)a_N(s')p_N^2(s)p_N^2(s')\{\exp(-np_N(s) - np_N(s'))\}(1 + p_N(s))^{k-1}\cdot$$
$$(1 + p_N(s'))^{k-1}[1 - p_N(s) - p_N(s')]^{\nu_1-1}[1 - p_N(s')]^{\nu_2-\nu_1-1}, \qquad (\nu_2 > \nu_1).$$

Using (3.18) and (3.19) for the last two terms in (3.16), summing over $k(1 \leqslant k \leqslant n)$ and using the approximation that $[1 + p_N(s)]^n e^{-np_N(s)} = 1 + 0(N^{-2}n)$ (by (2.6)), the first assertion in (3.17) follows by direct steps. For the second assertion, we define $g_{Nu}(Y_{N\nu}, Q_{N\nu})$ as $Y_{N\nu}^2Q_{N\nu}e^{-2nQ_{N\nu}}(1 + Q_{N\nu})^{2(k-1)}$ (or $Y_{N\nu}Q_{N\nu}e^{-nQ_{N\nu}}(1 + Q_{N\nu})^{k-1}$), and note that both (2.15) and (2.16) hold with $M_{N,3} = 0(1)$ (or $0(N^{-\frac{1}{2}})$) and $M_{N,4} \sim M_1$ (or $M_1^{\frac{1}{2}}$), and hence, the result follows by repeated use of (2.18)–(2.20) for the individual terms in the expansion of $l_{Nk}^{(n)}l_{Nq}^{(n)}$ in (3.16). □

We may remark that, intuitively, one may attempt to work with the alternative construction: $\tilde{Y}_{Nk} = Y_{Nk} - E(Y_{Nk}|\mathfrak{B}_{Nk-1})$, $k \geqslant 1$, $\tilde{Y}_{N0} = 0$. Then, one would have

$$\text{(3.20)} \qquad \tilde{Z}_{Nn} = \Sigma_{k=1}^{n}\tilde{Y}_{Nk} = \Sigma_{k=1}^{n}\{Y_{Nk} - \Sigma_{s=1}^{N}a_N(s)p_N(s) + \Sigma_{\nu=1}^{k-1}Y_{N\nu}Q_{N\nu}\}$$
$$= (Z_{Nn} - \phi_{Nn}^*) - \Sigma_{\nu=1}^{n}(n - k)[Y_{N\nu}Q_{N\nu} - EY_{N\nu}Q_{N\nu}].$$

Whereas the asymptotic normality of $d_{Nn}^{-1}\tilde{Z}_{Nn}$ may be proved along the same lines as in $\tilde{S}_{Nn}^{(n)}$, the second term on the right hand side of (3.20) is not generally $o_p(n^{\frac{1}{2}})$, so that this particular construction may not be very helpful for the desired normality of $d_{Nn}^{-1}(Z_{Nn} - \phi_{Nn})$.

4. Invariance principles and the martingale approach. For an arbitrary positive integer b and $\{n_{1N} < \cdots < n_{bN}\}$ satisfying $0 < \liminf_{n\to\infty}N^{-1}n_{1N} < \limsup_{n\to\infty}N^{-1}n_{bN} < \infty$, Rosén (1969, 1970) has studied the asymptotic multinormality of the standardized from of $\{Z_{Nn_{1N}}, \cdots, Z_{Nn_{bN}}\}$. We may remark that the martingale approach considered in Section 3 remains applicable in this case too. Let us define $\tilde{X}_{Nk}^{(n)}$ and $\tilde{S}_{Nk}^{(n)}$, $k \geqslant 0$, as in Section 3. Let then

$$\text{(4.1)} \qquad \hat{X}_{Nk}^{(n)} = \tilde{X}_{Nk}^{(n)}, \quad 0 \leqslant k \leqslant n;$$
$$= 0, \quad k > n;$$

and $\hat{S}_{Nk}^{(n)} = \Sigma_{\nu=0}^{k}\hat{X}_{N\nu}^{(n)}$, $k \geqslant 0$. Note that $\hat{S}_{Nk}^{(n)} = \tilde{S}_{Nk}^{(n)}$ for $k \leqslant n$ and $\hat{S}_{Nk}^{(n)} = \tilde{S}_{Nn}^{(n)}$ for

$k > n$. Moreover, (3.10) holds for each n_j ($j = 1, \cdots, b$), and hence, we may equivalently consider the case of an arbitrary linear compound (where $\boldsymbol{\lambda} \neq \mathbf{0}$)

(4.2)
$$\left(\Sigma_{j=1}^{b}\lambda_j N^{-\frac{1}{2}}\hat{S}_{Nn_{j_N}}^{(n_{jN})} = N^{-\frac{1}{2}}\Sigma_{k=1}^{n_{bN}}\Sigma_{j=1}^{b}\lambda_j\hat{X}_{Nk}^{(n_{jN})}\right) = N^{-\frac{1}{2}}\Sigma_{k=1}^{n_{bN}}\hat{X}_{Nk}^{*} \qquad \text{say,}$$

(where $X_{Nk}^{*} = \Sigma_{j=1}^{b}\lambda_j\hat{X}_{Nk}^{(n_{jN})}$, $k \geqslant 1$) and establish the asymptotic normality of the statistic in (4.2). For this, we note that by (3.2), (3.4) and (4.1), $E(X_{Nk}^{*}|\mathfrak{B}_{Nk-1}) = 0$, for all $k \geqslant 1$. As such, the same martingale central limit theorem (as in Section 3) can be applied to establish the desired asymptotic normality.

Let us now consider the case of $U_N(t)$, defined by (1.5) where the $a_N(s)$ are nonnegative. Let $[x]$ denote the largest integer $\leqslant x$. Then, we have, by definition,

$$P\{U_N(t) > x\} = P\{Z_{N[x]} < t\}, \qquad \text{for all} \quad x, t > 0. \tag{4.3}$$

As such, with the aid of (4.3), one can "invert" the results concerning Z_{Nn} in Theorem 3.1 and obtain the asymptotic normality of $U_N(t)$. A similar treatment holds for the multidimensional case.

Finally, we like to stress the importance of the proposed martingale approach in the study of invariance principles for the bonus sums (or the waiting times). For an arbitrary T ($0 < T < \infty$), let $J = [0, T]$, and, for every N, consider the sample process $W_N = \{W_N(x), x \in J\}$, by letting

$$W_N(x) = N^{-\frac{1}{2}}(Z_{N[Nx]} - \phi_{N[Nx]}), \qquad x \in J. \tag{4.4}$$

Then, W_N belongs to the space $D[J]$, endowed with the Skorokhod J_1-topology. The convergence of the finite-dimensional distributions (f.d.d.) of $\{W_N\}$ to those of some Gaussian functions follows from the results of the earlier part of this section; see also Rosén (1969, 1970). Hence, to establish the weak convergence of $\{W_N\}$, we need to show that $\{W_N\}$ is *tight*.

THEOREM 4.1. *Under* (2.6)–(2.8) *and* (2.14), $\{W_N\}$ *is tight.*

PROOF. By (1.4) and (4.4), $W_N(0) = 0$ with probability 1, $\forall N$. Hence, to prove the theorem, it suffices to show that for every $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta: 0 < \delta < T$ and an integer N_0, such that for every $x \in J$ and $N \geqslant N_0$,

$$P\{\sup[|W_N(y) - W_N(x)| : x \geqslant y \geqslant (x - \delta) \vee 0] > \varepsilon\} < \eta\delta/T. \tag{4.5}$$

Suppose that in (4.4), we replace $Z_{N[Nx]} - \phi_{N[Nx]}$ by $\tilde{S}_{N[Nx]}^{[Nx]}$, $x \in J$, and denote the resulting process by $\tilde{W}_N$. Then proceeding as in (3.9)–(3.10), it follows that for every $\varepsilon > 0$,

$$\lim\sup_{N\to\infty} p\{\sup_{x\in J}|W_N(x) - \tilde{W}_N(x)| > \varepsilon\} = 0. \tag{4.6}$$

Hence, it suffices to prove (4.5) with $\tilde{W}_N$ replacing W_N. Towards this note that

$$N^{-\frac{1}{2}}(\tilde{S}_{Nn}^{(n)} - \tilde{S}_{Nk}^{(k)}) = N^{-\frac{1}{2}}(\tilde{S}_{Nn}^{(n)} - \tilde{S}_{Nk}^{(n)}) + N^{-\frac{1}{2}}(\tilde{S}_{Nk}^{(n)} - \tilde{S}_{Nk}^{(k)}), \quad \forall k \geqslant 0. \tag{4.7}$$

Since $\{S_{Nk}^{(n)}, \mathfrak{B}_{Nk}; k \geqslant 1\}$ is a martingale, by (3.12), (3.14) and (3.17) [insuring that $(\Sigma_{k=1}^{[Nx]}[l_{Nk}^{(n)} - El_{Nk}^{(n)}])/d_{N[Nx]}^2 \to_p 1$, $\forall x \in J$], we are in a position to use Theorem 2 of

Scott (1973) [under Condition (B)] which insures the weak convergence of the above martingale sequence (implying its tightness), and hence, under (2.13) and (2.14), we obtain that for every $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta : 0 < \delta < T$ and an N_0, such that for $N \geqslant N_0$, $n = [Nx]$, $x \in J$,

$$(4.8) \qquad P\left\{\max_{n-\delta N \leqslant k \leqslant n} N^{-\frac{1}{2}}|\tilde{S}_{Nn}^{(n)} - \tilde{S}_{Nk}^{(n)}| > \varepsilon/2\right\} < \eta\delta/2T.$$

Also, if we choose $\delta(>0)$ so small that $\delta M_1 < 1$, then, for $[n-k] \leqslant \delta N$, $(n-k)\{\max_{1 \leqslant s \leqslant N} p_N(s)\} \leqslant \delta M_1 < 1$, by (2.6). Hence, for $n \geqslant k \geqslant (n - \delta N) \wedge 0$, we obtain from (3.7) that

$$(4.9) \qquad N^{-\frac{1}{2}}(\tilde{S}_{Nk}^{(k)} - \tilde{S}_{Nk}^{(n)}) = \Sigma_{i=1}^{k}[g_{n,k,i}(Y_{Ni}, Q_{Ni}) - Eg_{n,k,i}(Y_{Ni}, Q_{Ni})],$$

where

(4.10)

$$g_{n,k,i}(a, b) = N^{-\frac{1}{2}}a(1 - e^{-(n-k)b})[e^{-kb}(1+b)^k - be^{-kb}(1+b)^{i-1}],$$
$$1 \leqslant i \leqslant k \leqslant n \leqslant NT.$$

Note that by (2.6)–(2.7), for every $n \leqslant NT$, (2.15)–(2.17) hold with $M_{N,3} = O(N^{-1}(n-k)N^{-\frac{1}{2}}(\max_{1 \leqslant s < N}|a_N(s)|)) = O((n-k)/N)$, $M_{N,4} = O(N^{\frac{1}{2}}(n-k))$ and $M_{N,5} = O([(N-n)/N]^2)$. Hence, by (2.19)–(2.20) and (4.9)–(4.10), we obtain that

$$(4.11) \qquad E\left\{\left[N^{-\frac{1}{2}}(\tilde{S}_{Nk}^{(k)} - \tilde{S}_{Nk}^{(n)})\right]^2\right\} \leqslant M^*[(n-k)/N]^2,$$
$$k : NT \geqslant n \geqslant k \geqslant (n - \delta N) \vee 0,$$

where $M^*(<\infty)$ does not depend on δ. By (4.11) and Theorem 12.2 of Billingsley (1968, page 94), we conclude that for every $n \leqslant NT$, $T < \infty$,

$$(4.12) \qquad P\left\{\max_{n-\delta N \leqslant k \leqslant n} N^{-\frac{1}{2}}|\tilde{S}_{Nk}^{(k)} - \tilde{S}_{Nk}^{(n)}| > \varepsilon/2\right\} < K^*\varepsilon^{-2}\delta^2, \qquad K^* < \infty,$$

and K^* does not depend on ε and δ. For every $\varepsilon > 0$, $\eta > 0$ and $T < \infty$, we choose $\delta(>0)$ so small that $\delta < \eta\varepsilon^2/2K^*T$, so that the right-hand side of (4.12) is $\leqslant \frac{1}{2}\eta\delta/T$. From (4.8) and (4.12), we obtain that

$$(4.13) \qquad P\left\{\max_{n-\delta N \leqslant k \leqslant n} N^{-\frac{1}{2}}|\tilde{S}_{Nn}^{(n)} - \tilde{S}_{Nk}^{(k)}| > \varepsilon\right\} \leqslant \eta\delta/T,$$
$$\forall NT \geqslant n \geqslant k \geqslant (n - \delta N) \vee 0,$$

and this completes the proof of (4.5) (for $\tilde{W}_N$). □

We conclude this section with the remark that the weak convergence result for the bonus sum process can be transmitted into the weak convergence result for the corresponding waiting time process. Since, given Theorem 4.1 and the convergence of f.d.d.'s (studied earlier), such a transmission follows directly from the results of Vervaat (1972), the details are omitted.

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